Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A reverse Mulholland-type inequality in the whole plane

Jianquan Liao* and Bicheng Yang

*Correspondence: lamth@163.com
Department of Mathematics, Guangdong University of Education, Guangzhou, P.R. China

Abstract

We present a new reverse Mulholland-type inequality in the whole plane with a best possible constant factor by introducing multiparameters, applying weight coefficients, and using the Hermite–Hadamard inequality. Moreover, we consider equivalent forms and some particular cases.

MSC: 26D15; 47A07

Keywords: Mulholland-type inequality; Parameter; Weight coefficient; Equivalent form; Reverse

1 Introduction

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, the Hardy–Hilbert inequality is provided as follows (see [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where $\frac{\pi}{\sin(\pi/p)}$ is the best possible constant factor. By Theorem 343 in [1] (replacing $\frac{a_m}{m}$ and $\frac{b_n}{n}$ by a_m and b_n, respectively), it yields the following Mulholland inequality with the same best value:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{a_m^p}{m}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{b_n^q}{n}\right)^{\frac{1}{q}}. \tag{2}$$

Inequalities (1) and (2) play an important role in analysis and its applications (see [1, 2]).

In 2007, Yang [3] published a Hilbert-type integral inequality in the whole plane. Various extensions of (1)–(2) and Yang's work have been presented since then [4–12]. Recently, Yang and Chen [13] presented the following extension of (1) in the whole plane:

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{a_m b_n}{(|m-\xi|+|n-\eta|)^{\lambda}} < 2B(\lambda_1,\lambda_2)\left[\sum_{|m|=1}^{\infty}|m-\xi|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}|n-\eta|^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{3}$$

where the constant factor $2B(\lambda_1,\lambda_2)$ $(0<\lambda_1,\lambda_2\le 1,\lambda_1+\lambda_2=\lambda,\xi,\eta\in[0,\frac{1}{2}])$ is the best possible. In addition, Xin et al. [14] also carried out a similar result, and Zhong et al. [15] gave the reverse Mulholland's inequality in the whole plane.

In this paper, we present a new reverse Mulholland-type inequality in the whole plane with a best possible constant factor, which is similar to the results of [13], via introducing multiparameters, applying weight coefficients, and using the Hermite–Hadamard inequality. Moreover, we consider equivalent forms and some particular cases.

2 An example and two lemmas

We further assume that $\lambda_1,\lambda_2>0,\lambda_1+\lambda_2=\lambda\le 1,\xi,\eta\in[0,\frac{1}{2}],\alpha,\beta\in[\arccos\frac{1}{3},\frac{\pi}{2}]$, and

$$k_\gamma(\lambda_1):=\frac{2\pi^2\csc^2\gamma}{\lambda^2\sin^2(\frac{\pi\lambda_1}{\lambda})}\quad(\gamma=\alpha,\beta). \tag{4}$$

Remark 1 Since $\alpha,\beta\in[\arccos\frac{1}{3},\frac{\pi}{2}],\xi,\eta\in[0,\frac{1}{2}]$, it follows that

$$\left(\frac{3}{2}\pm\eta\right)(1\mp\cos\beta)\ge 1\quad\text{and}\quad\left(\frac{3}{2}\pm\xi\right)(1\mp\cos\alpha)\ge 1.$$

Example 1 We set $g(u):=\frac{\ln u}{u-1}$ $(u>0),g(1):=\lim_{u\to 1}g(u)=1$. Then we have $g(u)>0,g'(u)<0,g''(u)>0$ $(u>0)$. By Tailor's formula we find

$$g(u)=\frac{\ln[1+(u-1)]}{u-1}=\sum_{k=0}^{\infty}(-1)^k\frac{(u-1)^k}{k+1}=\sum_{k=0}^{\infty}\frac{(-1)^k k!}{k+1}\frac{(u-1)^k}{k!}\quad(-1<u-1\le 1),$$

and then $g^{(k)}(1)=\frac{(-1)^k k!}{k+1}$ $(k=0,1,2,\ldots)$. Hence, $g^{(0)}(1)=g(1)=1,g'(1)=-\frac{1}{2},g''(1)=\frac{2}{3}$. It is evident that $g(u)>0$. We obtain $g'(u)=\frac{h(u)}{u(u-1)^2},h(u):=u-1-u\ln u$. Since

$$h'(u)=-\ln u>0\quad(0<u<1);\qquad h'(u)=-\ln u<0\quad(u>1),$$

it follows that $h_{\max}=h(1)=0$ and $h(u)<0$ $(u\ne 1)$. Then we have $g'(u)<0$ $(u\ne 1)$. Since $g'(1)=-\frac{1}{2}<0$, it follows that $g'(u)<0$ $(u>0)$. We find

$$g''(u)=\frac{J(u)}{u^2(u-1)^3},\quad J(u):=-(u-1)^2-2u(u-1)+2u^2\ln u,$$

$J'(u)=-4(u-1)+4u\ln u$, and

$$J''(u)=4\ln u<0\quad(0<u<1);\qquad J''(u)=4\ln u>0\quad(u>1).$$

It follows that $J'_{\min}=J'(1)=0,J'(u)>0$ $(u\ne 1)$, and $J(u)$ is strictly increasing. Since $J(1)=0$, we have

$$J(u)<0\quad(0<u<1);\qquad J(u)>0\quad(u>1),$$

and $g''(u)>0$ $(u\ne 1)$. Since $g''(1)=\frac{2}{3}>0$, we find $g''(u)>0$ $(u>0)$.

For $0 < \lambda \le 1, 0 < \lambda_2 < 1$, setting $G(u) := g(u^\lambda)u^{\lambda_2 - 1}$ $(u > 0)$, we have $G(u) > 0$,

$$\begin{aligned} G'(u) &= \lambda g'\left(u^\lambda\right)u^{\lambda+\lambda_2-2} + (\lambda_2 - 1)g\left(u^\lambda\right)u^{\lambda_2-2} < 0, \quad \text{and} \\ G''(u) &= \lambda^2 g''\left(u^\lambda\right)u^{2\lambda+\lambda_2-3} + \lambda(\lambda + \lambda_2 - 2)g'\left(u^\lambda\right)u^{\lambda+\lambda_2-3} \\ &\quad + \lambda(\lambda_2 - 1)g'\left(u^\lambda\right)u^{\lambda+\lambda_2-3} + (\lambda_2 - 1)(\lambda_2 - 2)g\left(u^\lambda\right)u^{\lambda_2-3} > 0. \end{aligned}$$

We set $F(x,y) := \frac{\ln(x/y)}{x^\lambda - y^\lambda}\left(\frac{y}{x}\right)^{\lambda_2-1}(x,y > 0)$. Since $F(x,y) = \frac{1}{x^\lambda}G(\frac{y}{x})$, we have

$$F(x,y) > 0, \qquad \frac{\partial}{\partial y}F(x,y) < 0, \qquad \frac{\partial^2}{\partial y^2}F(x,y) > 0.$$

Hence, for $x, y > 1$, we have

$$\frac{1}{y}F(\ln x, \ln y) > 0, \qquad \frac{\partial}{\partial y}\left(\frac{1}{y}F(\ln x, \ln y)\right) < 0, \qquad \frac{\partial^2}{\partial y^2}\left(\frac{1}{y}F(\ln x, \ln y)\right) > 0.$$

Lemma 1 *If $f(u) > 0, f'(u) < 0, f''(u) > 0$ $(u > \frac{3}{2})$ and $\int_{\frac{3}{2}}^\infty f(u)\,du < \infty$, then we have the following Hermite–Hadamard inequality* (*see* [16]):

$$\int_k^{k+1} f(u)\,du < f(k) < \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(u)\,du \quad \left(k \in \mathbf{N}\backslash\{1\}\right),$$

and then

$$\int_2^\infty f(u)\,du < \sum_{k=2}^\infty f(k) < \int_{\frac{3}{2}}^\infty f(u)\,du. \tag{5}$$

For $|x|, |y| \ge \frac{3}{2}$, define

$$A_{\xi,\alpha}(x) := |x - \xi| + (x - \xi)\cos\alpha,$$

$A_{\eta,\beta}(y) = |y - \eta| + (y - \eta)\cos\beta$, *and*

$$k(x,y) := \frac{\ln(\ln A_{\xi,\alpha}(x)/\ln A_{\eta,\beta}(y))}{\ln^\lambda A_{\xi,\alpha}(x) - \ln^\lambda A_{\eta,\beta}(y)}. \tag{6}$$

We define two weight coefficients as follows:

$$\omega(\lambda_2, m) := \sum_{|n|=2}^\infty \frac{k(m,n)}{A_{\eta,\beta}(n)} \cdot \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2} A_{\eta,\beta}(n)}, \quad |m| \in \mathbf{N}\backslash\{1\}, \tag{7}$$

$$\varpi(\lambda_1, n) := \sum_{|m|=2}^\infty \frac{k(m,n)}{A_{\xi,\alpha}(m)} \cdot \frac{\ln^{\lambda_2} A_{\eta,\beta}(n)}{\ln^{1-\lambda_1} A_{\xi,\alpha}(m)}, \quad |n| \in \mathbf{N}\backslash\{1\}, \tag{8}$$

where $\sum_{|j|=2}^\infty \cdots = \sum_{j=-2}^{-\infty} \cdots + \sum_{j=2}^\infty \cdots$ $(j = m, n)$.

Lemma 2 *We have the inequalities*

$$k_\beta(\lambda_1)\bigl(1-\theta(\lambda_2,m)\bigr)<\omega(\lambda_2,m)<k_\beta(\lambda_1),\quad |m|\in\mathbf{N}\backslash\{1\}, \tag{9}$$

where

$$\begin{aligned}\theta(\lambda_2,m)&:=\left[\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\right]^2\int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}}\frac{\ln u}{u^\lambda-1}u^{\lambda_2-1}\,du\\&=O\left(\frac{1}{\ln^{\lambda_2/2}A_{\xi,\alpha}(m)}\right)\in(0,1).\end{aligned} \tag{10}$$

Proof For $|m|\in\mathbf{N}\backslash\{1\}$, let

$$\begin{aligned}k^{(1)}(m,y)&:=\frac{\ln\ln A_\xi(m)-\ln\ln[(y-\eta)(\cos\beta-1)]}{\ln^\lambda A_\xi(m)-\ln^\lambda[(y-\eta)(\cos\beta-1)]},\quad y<-\frac{3}{2},\\k^{(2)}(m,y)&:=\frac{\ln\ln A_\xi(m)-\ln\ln[(y-\eta)(\cos\beta+1)]}{\ln^\lambda A_\xi(m)-\ln^\lambda[(y-\eta)(\cos\beta+1)]},\quad y>\frac{3}{2}.\end{aligned}$$

Then the equality

$$k^{(1)}(m,-y)=\frac{\ln\ln A_\xi(m)-\ln\ln[(y+\eta)(1-\cos\beta)]}{\ln^\lambda A_\xi(m)-\ln^\lambda[(y+\eta)(1-\cos\beta)]},\quad y>\frac{3}{2},$$

yields

$$\begin{aligned}\omega(\lambda_2,m)&=\sum_{n=-2}^{-\infty}\frac{k^{(1)}(m,n)\ln^{\lambda_1}A_{\xi,\alpha}(m)}{(n-\eta)(\cos\beta-1)\ln^{1-\lambda_2}[(n-\eta)(\cos\beta-1)]}\\&\quad+\sum_{n=2}^{\infty}\frac{k^{(2)}(m,n)\ln^{\lambda_1}A_{\xi,\alpha}(m)}{(n-\eta)(1+\cos\beta)\ln^{1-\lambda_2}[(n-\eta)(1+\cos\beta)]}\\&=\frac{\ln^{\lambda_1}A_{\xi,\alpha}(m)}{1-\cos\beta}\sum_{n=2}^{\infty}\frac{k^{(1)}(m,-n)}{(n+\eta)\ln^{1-\lambda_2}[(n+\eta)(1-\cos\beta)]}\\&\quad+\frac{\ln^{\lambda_1}A_{\xi,\alpha}(m)}{1+\cos\beta}\sum_{n=2}^{\infty}\frac{k^{(2)}(m,n)}{(n-\eta)\ln^{1-\lambda_2}[(n-\eta)(1+\cos\beta)]}.\end{aligned} \tag{11}$$

Since $0<\lambda\le1, 0<\lambda_2<1$, by Example 1 we find that, for $y>\frac{3}{2}$,

$$\begin{aligned}&\frac{k^{(i)}(m,(-1)^iy)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]}>0,\\&\frac{d}{dy}\frac{k^{(i)}(m,(-1)^iy)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]}<0,\\&\frac{d^2}{dy^2}\frac{k^{(i)}(m,(-1)^iy)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]}>0\quad(i=1,2),\end{aligned}$$

from which it follows that

$$\frac{k^{(i)}(m,(-1)^iy)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]}\quad(i=1,2)$$

are strictly decreasing and convex in $(\frac{3}{2}, \infty)$. Then (5) and (11) yield that

$$\omega(\lambda_2, m) < \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1-\cos\beta} \int_{\frac{3}{2}}^{\infty} \frac{k^{(1)}(m,-y)}{(y+\eta)\ln^{1-\lambda_2}[(y+\eta)(1-\cos\beta)]}\,dy + \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1+\cos\beta} \int_{\frac{3}{2}}^{\infty} \frac{k^{(2)}(m,y)}{(y-\eta)\ln^{1-\lambda_2}[(y-\eta)(1+\cos\beta)]}\,dy.$$

Setting $u = \frac{\ln[(y+\eta)(1-\cos\beta)]}{\ln A_{\xi,\alpha}(m)}$ $(u = \frac{\ln[(y-\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)})$ in the first (second) integral, from Remark 1 we obtain

$$\begin{aligned}\omega(\lambda_2, m) &< \left(\frac{1}{1-\cos\beta} + \frac{1}{1+\cos\beta}\right)\int_0^{\infty} \frac{\ln u}{u^{\lambda}-1} u^{\lambda_2-1}\,du \\ &= \frac{2\csc^2\beta}{\lambda^2}\int_0^{\infty}\frac{\ln v}{v-1} v^{(\lambda_2/\lambda)-1}\,dv = \frac{2\pi^2\csc^2\beta}{\lambda^2\sin^2(\frac{\pi\lambda_1}{\lambda})} = k_\beta(\lambda_1),\end{aligned}$$

by simplifications. Similarly, (5) and (11) also yield that

$$\begin{aligned}\omega(\lambda_2, m) &> \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1-\cos\beta} \int_{2}^{\infty} \frac{k^{(1)}(m,-y)}{(y+\eta)\ln^{1-\lambda_2}[(y+\eta)(1-\cos\beta)]}\,dy \\ &\quad + \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1+\cos\beta} \int_{2}^{\infty} \frac{k^{(2)}(m,y)}{(y-\eta)\ln^{1-\lambda_2}[(y-\eta)(1+\cos\beta)]}\,dy \\ &\geq \left(\frac{1}{1-\cos\beta} + \frac{1}{1+\cos\beta}\right)\int_{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}}^{\infty} \frac{\ln u}{u^{\lambda}-1} u^{\lambda_2-1}\,du \\ &= k_\beta(\lambda_1) - 2\csc^2\beta \int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}} \frac{\ln u}{u^{\lambda}-1} u^{\lambda_2-1}\,du \\ &= k_\beta(\lambda_1)\big(1-\theta(\lambda_2,m)\big) > 0,\end{aligned}$$

where $\theta(\lambda_2, m)(< 1)$ is defined in (10). Since

$$\frac{\ln u}{u^{\lambda}-1} u^{\lambda_2/2} \to 0 \quad \left(u \to 0^+\right); \qquad \frac{\ln u}{u^{\lambda}-1} u^{\lambda_2/2} \to \frac{1}{\lambda} \quad (u \to 1),$$

there exists a positive constant C such that $\frac{\ln u}{u^{\lambda}-1} u^{\lambda_2/2} \leq C$ $(0 < u \leq 1)$. Then for $A_{\xi,\alpha}(m) \geq (2+\eta)(1+\cos\beta)$, we have

$$\begin{aligned}0 < \theta(\lambda_2, m) &\leq C\left[\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\right]^2 \int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}} u^{\frac{\lambda_2}{2}-1}\,du \\ &= \frac{2C}{\lambda_2}\left[\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\right]^2 \left\{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}\right\}^{\frac{\lambda}{2}}. \qquad (12)\end{aligned}$$

Hence, (9) and (10) are valid. □

Similarly, we have the following:

Lemma 3 *For* $0 < \lambda \leq 1, 0 < \lambda_1 < 1$, *we have the inequalities*

$$k_\alpha(\lambda_1)\big(1-\tilde{\theta}(\lambda_1, n)\big) < \varpi(\lambda_1, n) < k_\alpha(\lambda_1), \quad |n| \in \mathbf{N}\backslash\{1\}, \qquad (13)$$

where

$$\tilde{\theta}(\lambda_1, n) := \left[\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\right]^2 \int_0^{\frac{\ln[(2+\xi)(1+\cos\alpha)]}{\ln A_{\eta,\beta}(n)}} \frac{\ln u}{u^\lambda - 1} u^{\lambda_1 - 1}\, du$$
$$= O\left(\frac{1}{\ln^{\lambda_1/2} A_{\eta,\beta}(n)}\right) \in (0,1). \quad (14)$$

Lemma 4 *If* $(\varsigma,\gamma) = (\xi,\alpha)$ (*or* (η,β)), $\rho > 0$, *then we have*

$$H_\rho(\varsigma,\gamma) := \sum_{|k|=2}^{\infty} \frac{\ln^{-1-\rho} A_{\varsigma,\gamma}(k)}{A_{\varsigma,\gamma}(k)} = \frac{1}{\rho}\left(2\csc^2\gamma + o(1)\right) \quad \left(\rho \to 0^+\right). \quad (15)$$

Proof By (5) we obtain

$$\begin{aligned}
H_\rho(\varsigma,\gamma) &= \sum_{k=-2}^{-\infty} \frac{\ln^{-1-\rho}[(k-\varsigma)(\cos\gamma - 1)]}{(k-\varsigma)(\cos\gamma - 1)} + \sum_{k=2}^{\infty} \frac{\ln^{-1-\rho}[(k-\varsigma)(\cos\gamma + 1)]}{(k-\varsigma)(\cos\gamma + 1)} \\
&= \sum_{k=2}^{\infty}\left\{\frac{\ln^{-1-\rho}[(k+\varsigma)(1-\cos\gamma)]}{(k-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(k-\varsigma)(\cos\gamma+1)]}{(k-\varsigma)(\cos\gamma+1)}\right\} \\
&< \int_{\frac{3}{2}}^{\infty}\left\{\frac{\ln^{-1-\rho}[(y+\varsigma)(1-\cos\gamma)]}{(y-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(y-\varsigma)(\cos\gamma+1)]}{(y-\varsigma)(\cos\gamma+1)}\right\} dy \\
&= \frac{1}{\rho}\left\{\frac{\ln^{-\rho}[(\frac{3}{2}+\varsigma)(1-\cos\gamma)]}{1-\cos\gamma} + \frac{\ln^{-\rho}[(\frac{3}{2}-\varsigma)(1+\cos\gamma)]}{1+\cos\gamma}\right\} \\
&= \frac{1}{\rho}\left(\frac{1}{1-\cos\gamma} + \frac{1}{1+\cos\gamma} + o_1(1)\right) = \frac{1}{\rho}\left(2\csc^2\gamma + o_1(1)\right) \quad \left(\rho \to 0^+\right)
\end{aligned}$$

and

$$\begin{aligned}
H_\rho(\varsigma,\gamma) &= \sum_{k=2}^{\infty}\left\{\frac{\ln^{-1-\rho}[(k+\varsigma)(1-\cos\gamma)]}{(k-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(k-\varsigma)(\cos\gamma+1)]}{(k-\varsigma)(\cos\gamma+1)}\right\} \\
&> \int_{2}^{\infty}\left\{\frac{\ln^{-1-\rho}[(y+\varsigma)(1-\cos\gamma)]}{(y-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(y-\varsigma)(\cos\gamma+1)]}{(y-\varsigma)(\cos\gamma+1)}\right\} dy \\
&= \frac{1}{\rho}\left\{\frac{\ln^{-\rho}[(2+\varsigma)(1-\cos\gamma)]}{1-\cos\gamma} + \frac{\ln^{-\rho}[(2-\varsigma)(1+\cos\gamma)]}{1+\cos\gamma}\right\} \\
&= \frac{1}{\rho}\left(\frac{1}{1-\cos\gamma} + \frac{1}{1+\cos\gamma} + o_2(1)\right) = \frac{1}{\rho}\left(2\csc^2\gamma + o_2(1)\right) \quad \left(\rho \to 0^+\right).
\end{aligned}$$

Therefore, (15) is valid. □

3 Main results and a few particular cases

Theorem 1 *Suppose that* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$k(\lambda_1) := k_\beta^{1/p}(\lambda_1) k_\alpha^{1/q}(\lambda_1) = \frac{2\pi^2 \csc^{2/p}\beta \csc^{2/q}\alpha}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2}. \quad (16)$$

If $a_m, b_n \geq 0$ $(|m|, |n| \in \mathbf{N}\backslash\{1\})$, *satisfy*

$$0 < \sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p < \infty, \qquad 0 < \sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q < \infty,$$

then for

$$\begin{aligned}\theta(\lambda_2, m) &= \left[\frac{\lambda}{\pi} \sin\left(\frac{\pi \lambda_1}{\lambda}\right)\right]^2 \int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}} \frac{\ln u}{u^\lambda - 1} u^{\lambda_2 - 1}\, du \\ &= O\left(\frac{1}{\ln^{\lambda_2/2} A_{\xi,\alpha}(m)}\right) \in (0,1),\end{aligned}$$

we obtain the following equivalent reverse Mulholland-type inequalities:

$$\begin{aligned} I &:= \sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} \frac{\ln(\ln A_{\xi,\alpha}(m)/\ln A_{\eta,\beta}(n))}{\ln^\lambda A_{\xi,\alpha}(m) - \ln^\lambda A_{\eta,\beta}(n)} a_m b_n \\ &> \frac{2\pi^2 \csc^{2/p}\beta \csc^{2/q}\alpha}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2} \\ &\quad\times \left[\sum_{|m|=2}^{\infty} \bigl(1-\theta(\lambda_2, m)\bigr) \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}} \\ &\quad\times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{17}$$

$$\begin{aligned} J_1 &:= \left\{\sum_{|n|=2}^{\infty} \frac{\ln^{p\lambda_2-1} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \left[\sum_{|m|=2}^{\infty} \frac{\ln(\ln A_{\xi,\alpha}(m)/\ln A_{\eta,\beta}(n))}{\ln^\lambda A_{\xi,\alpha}(m) - \ln^\lambda A_{\eta,\beta}(n)} a_m\right]^p\right\}^{\frac{1}{p}} \\ &> \frac{2\pi^2 \csc^{2/p}\beta \csc^{2/q}\alpha}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2} \left[\sum_{|m|=2}^{\infty} \bigl(1-\theta(\lambda_2, m)\bigr) \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}}, \end{aligned} \tag{18}$$

$$\begin{aligned} J_2 &:= \left[\sum_{|m|=2}^{\infty} \frac{\ln^{q\lambda_1-1} A_{\xi,\alpha}(m)}{(1-\theta(\lambda_2, m))^{q-1} A_{\xi,\alpha}(m)} \left(\sum_{|n|=2}^{\infty} \frac{\ln(\ln A_{\xi,\alpha}(m)/\ln A_{\eta,\beta}(n))}{\ln^\lambda A_{\xi,\alpha}(m) - \ln^\lambda A_{\eta,\beta}(n)} b_n\right)^q\right]^{\frac{1}{q}} \\ &> \frac{2\pi^2 \csc^{2/p}\beta \csc^{2/q}\alpha}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{19}$$

Particularly, (i) *for* $\alpha = \beta = \frac{\pi}{2}, \xi, \eta \in [0, \frac{1}{2}]$, *setting*

$$\begin{aligned}\theta_1(\lambda_2, m) &:= \left[\frac{\lambda}{\pi} \sin\left(\frac{\pi \lambda_1}{\lambda}\right)\right]^2 \int_0^{\frac{\ln(2+\eta)}{\ln|m-\xi|}} \frac{\ln u}{u^\lambda - 1} u^{\lambda_2 - 1}\, du \\ &= O\left(\frac{1}{\ln^{\lambda_2/2} |m-\xi|}\right) \in (0,1),\end{aligned}$$

we have the following equivalent reverse Mulholland-type inequalities:

$$\begin{aligned}
&\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{\ln(|m-\xi|/|n-\eta|)}{\ln^{\lambda}|m-\xi|-\ln^{\lambda}|n-\eta|}a_m b_n \\
&\quad > \frac{2\pi^2}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2} \\
&\qquad\times\left[\sum_{|m|=2}^{\infty}\bigl(1-\theta_1(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}|m-\xi|}{|m-\xi|^{1-p}}a_m^p\right]^{\frac{1}{p}} \\
&\qquad\times\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}|n-\eta|}{|n-\eta|^{1-q}}b_n^q\right]^{\frac{1}{q}},
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\left[\sum_{|n|=2}^{\infty}\frac{\ln^{p\lambda_2-1}|n-\eta|}{|n-\eta|}\left(\sum_{|m|=2}^{\infty}\frac{\ln(|m-\xi|/|n-\eta|)}{\ln^{\lambda}|m-\xi|-\ln^{\lambda}|n-\eta|}a_m\right)^p\right]^{\frac{1}{p}} \\
&\quad > \frac{2\pi^2}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2}\left[\sum_{|m|=2}^{\infty}\bigl(1-\theta_1(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}|m-\xi|}{|m-\xi|^{1-p}}a_m^p\right]^{\frac{1}{p}},
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\left[\sum_{|m|=2}^{\infty}\frac{\ln^{q\lambda_1-1}|m-\xi|}{(1-\theta_1(\lambda_2,m))^{q-1}|m-\xi|}\left(\sum_{|n|=2}^{\infty}\frac{\ln(|m-\xi|/|n-\eta|)a_m}{\ln^{\lambda}|m-\xi|-\ln^{\lambda}|n-\eta|}b_n\right)^q\right]^{\frac{1}{q}} \\
&\quad > \frac{2\pi^2}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}|n-\eta|}{|n-\eta|^{1-q}}b_n^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{22}$$

(ii) *For* $\xi=\eta=0,\alpha,\beta\in[\arccos\frac{1}{3},\frac{\pi}{2}]$, *setting*

$$\begin{aligned}
\theta_2(\lambda_2,m) &:= \left[\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\right]^2\int_0^{\frac{\ln 2(1+\cos\beta)}{\ln(|m|+m\cos\alpha)}}\frac{\ln u}{u^{\lambda}-1}u^{\lambda_2-1}\,du \\
&= O\left(\frac{1}{\ln^{\lambda_2/2}A_{\xi,\alpha}(m)}\right)\in(0,1),
\end{aligned}$$

we have the following equivalent reverse Mulholland-type inequalities:

$$\begin{aligned}
&\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{\ln[\ln(|m|+m\cos\alpha)/\ln(|n|+n\cos\beta)]}{\ln^{\lambda}(|m|+m\cos\alpha)-\ln^{\lambda}(|n|+n\cos\beta)}a_m b_n \\
&\quad > \frac{2\pi^2\csc^{2/p}\beta\csc^{2/q}\alpha}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2} \\
&\qquad\times\left[\sum_{|m|=2}^{\infty}\bigl(1-\theta_2(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}(|m|+m\cos\alpha)}{(|m|+m\cos\alpha)^{1-p}}a_m^p\right]^{\frac{1}{p}} \\
&\qquad\times\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}(|n|+n\cos\beta)}{(|n|+n\cos\beta)^{1-q}}b_n^q\right]^{\frac{1}{q}},
\end{aligned} \tag{23}$$

$$\left\{\sum_{|n|=2}^{\infty}\frac{\ln^{p\lambda_2-1}(|n|+n\cos\beta)}{|n|+n\cos\beta}\left[\sum_{|m|=2}^{\infty}\frac{\ln[\ln(|m|+m\cos\alpha)/\ln(|n|+n\cos\beta)]}{\ln^{\lambda}(|m|+m\cos\alpha)-\ln^{\lambda}(|n|+n\cos\beta)}a_m\right]^p\right\}^{\frac{1}{p}}$$

$$> \frac{2\pi^2\csc^{2/p}\beta\csc^{2/q}\alpha}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2}\left[\sum_{|m|=2}^{\infty}\big(1-\theta_2(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1}(|m|+m\cos\alpha)}{(|m|+m\cos\alpha)^{1-p}}a_m^p\right]^{\frac{1}{p}}, \tag{24}$$

$$\left[\sum_{|m|=2}^{\infty}\frac{\ln^{q\lambda_1-1}(|m|+m\cos\alpha)}{(1-\theta_2(\lambda_2,m))^{q-1}(|m|+m\cos\alpha)}\right.$$

$$\left.\times\left(\sum_{|n|=2}^{\infty}\frac{\ln[\ln(|m|+m\cos\alpha)/\ln(|n|+n\cos\beta)]}{\ln^{\lambda}(|m|+m\cos\alpha)-\ln^{\lambda}(|n|+n\cos\beta)}b_n\right)^q\right]^{\frac{1}{q}}$$

$$> \frac{2\pi^2\csc^{2/p}\beta\csc^{2/q}\alpha}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}(|n|+n\cos\beta)}{(|n|+n\cos\beta)^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{25}$$

Proof Applying the reverse Hölder inequality with weight (see [17]) and (8), we find

$$\left(\sum_{|m|=2}^{\infty}k(m,n)a_m\right)^p$$

$$= \left\{\sum_{|m|=2}^{\infty}k(m,n)\left[\frac{(A_{\xi,\alpha}(m))^{\frac{1}{q}}\ln^{\frac{1-\lambda_1}{q}}A_{\xi,\alpha}(m)}{\ln^{\frac{1-\lambda_2}{p}}A_{\eta,\beta}(n)}a_m\right]\left[\frac{\ln^{\frac{1-\lambda_2}{p}}A_{\eta,\beta}(n)}{(A_{\xi,\alpha}(m))^{\frac{1}{q}}\ln^{\frac{1-\lambda_1}{q}}A_{\xi,\alpha}(m)}\right]\right\}^p$$

$$\geq \sum_{|m|=2}^{\infty}k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}}\ln^{\frac{(1-\lambda_1)p}{q}}A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2}A_{\eta,\beta}(n)}a_m^p$$

$$\times\left[\sum_{|m|=2}^{\infty}k(m,n)\frac{\ln^{\frac{(1-\lambda_2)q}{p}}A_{\eta,\beta}(n)}{A_{\xi,\alpha}(m)\ln^{1-\lambda_1}A_{\xi,\alpha}(m)}\right]^{p-1}$$

$$= \frac{(\varpi(\lambda_1,n))^{p-1}A_{\eta,\beta}(n)}{\ln^{p\lambda_2-1}A_{\eta,\beta}(n)}\sum_{|m|=2}^{\infty}k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}}\ln^{\frac{(1-\lambda_1)p}{q}}A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n)\ln^{1-\lambda_2}A_{\eta,\beta}(n)}a_m^p.$$

Then since $0 < p < 1$, by (13) this yields

$$J > k_{\alpha}^{1/q}(\lambda_1)\left[\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}}\ln^{\frac{(1-\lambda_1)p}{q}}A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n)\ln^{1-\lambda_2}A_{\eta,\beta}(n)}a_m^p\right]^{\frac{1}{p}}$$

$$= k_{\alpha}^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty}k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}}\ln^{\frac{(1-\lambda_1)p}{q}}A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n)\ln^{1-\lambda_2}A_{\eta,\beta}(n)}a_m^p\right]^{\frac{1}{p}}$$

$$= k_{\alpha}^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\omega(\lambda_2,m)\frac{n^{p(1-\lambda_1)-1}A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}}. \tag{26}$$

Combining (9) and (16), we obtain (18).

Using the reverse Hölders inequality again, we obtain

$$I = \sum_{|n|=2}^{\infty}\left[\frac{(A_{\eta,\beta}(n))^{\frac{-1}{p}}}{\ln^{\frac{1}{p}-\lambda_2} A_{\eta,\beta}(n)}\sum_{|m|=2}^{\infty} k(m,n)a_m\right]\left[\frac{\ln^{\frac{1}{p}-\lambda_2} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{\frac{-1}{p}}}b_n\right]$$
$$\geq J_1\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{27}$$

Then by (18) we obtain (17).

On the other-hand, assuming that (17) is valid, letting

$$b_n := \frac{\ln^{p\lambda_2-1} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}\left(\sum_{|m|=2}^{\infty} k(m,n)a_m\right)^{p-1}, \quad |n| \in \mathbf{N}\backslash\{1\},$$

we find

$$J_1 = \left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{p}}.$$

By (26) it follows that $J_1 > 0$. If $J_1 = \infty$, then (19) is trivially valid; if $J_1 < \infty$, then by (17) we have

$$\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q$$
$$= J_1^p = I$$
$$> k(\lambda_1)\left[\sum_{|m|=2}^{\infty}\left(1-\theta(\lambda_2,m)\right)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{q}},$$
$$J_1 = \left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{p}} > k(\lambda_1)\left[\sum_{|m|=2}^{\infty}\left(1-\theta(\lambda_2,m)\right)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}}.$$

Thus (18) is valid, which is equivalent to (17).

We further prove that (19) is equivalent to (17). Using the reverse Hölders inequality, we have

$$I = \sum_{|m|=2}^{\infty}\left[\left(1-\theta(\lambda_2,m)\right)^{\frac{1}{p}}\frac{\ln^{\frac{1}{q}-\lambda_1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{\frac{-1}{q}}}a_m\right]$$
$$\times\left[\frac{n^{\frac{-1}{q}+\lambda_1}A_{\xi,\alpha}(m)}{(1-\theta(\lambda_2,m))^{\frac{1}{p}}(A_{\xi,\alpha}(m))^{\frac{1}{q}}}\sum_{|n|=2}^{\infty} k(m,n)b_n\right]$$
$$\geq \left[\sum_{|m|=2}^{\infty}\left(1-\theta(\lambda_2,m)\right)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}} J_2, \tag{28}$$

and then (19) is valid by (17).

On the other-hand, assuming that (17) is valid, we set

$$a_m := \frac{\ln^{q\lambda_1-1} A_{\xi,\alpha}(m)}{(1-\theta(\lambda_2,m))^{q-1}A_{\xi,\alpha}(m)}\left(\sum_{|n|=2}^{\infty}\frac{\ln(\ln A_{\xi,\alpha}(m)/\ln A_{\eta,\beta}(n))}{\ln^{\lambda} A_{\xi,\alpha}(m)-\ln^{\lambda} A_{\eta,\beta}(n)}b_n\right)^{q-1}, \quad m \in \mathbf{N}\backslash\{1\},$$

and find

$$J_2 = \left[\sum_{|m|=2}^{\infty}\bigl(1-\theta(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{q}}.$$

If $J_2 = 0$, then (19) is impossible, so that $J_2 > 0$. If $J_2 = \infty$, then (19) is trivially valid; if $J_2 < \infty$, then by (17) we have

$$\begin{aligned}
&\sum_{|m|=2}^{\infty}\bigl(1-\theta(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p \\
&\quad = J_2^q = I \\
&\quad > k(\lambda_1)\left[\sum_{|m|=2}^{\infty}\bigl(1-\theta(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
&\left[\sum_{|m|=2}^{\infty}\bigl(1-\theta(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{q}} \\
&\quad = J_2 \\
&\quad > k(\lambda_1)\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Thus (19) is valid, which is equivalent to (17).

Hence, inequalities (17), (18), and (19) are equivalent. □

Theorem 2 *Under the assumptions in Theorem* 1,

$$k(\lambda_1) = \frac{2\pi^2 \csc^{2/p}\beta \csc^{2/q}\alpha}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2}$$

is the best possible constant factor in (17), (18), *and* (19).

Proof For $0 < \varepsilon < \min\{p\lambda_1, p(1-\lambda_2)\}$, we set $\tilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} (\in (0,1))$, $\tilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p} (\in (0,1))$, and

$$\tilde{a}_m := \frac{\ln^{\lambda_1-\frac{\varepsilon}{p}-1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} = \frac{\ln^{\tilde{\lambda}_1-1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \quad \bigl(|m| \in \mathbf{N}\backslash\{1\}\bigr),$$

$$\tilde{b}_n := \frac{\ln^{\lambda_2-\frac{\varepsilon}{q}-1} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} = \frac{\ln^{\tilde{\lambda}_2-\varepsilon-1} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \quad \bigl(|n| \in \mathbf{N}\backslash\{1\}\bigr).$$

By (15) and (13) we find

$$\begin{aligned}
\tilde{I}_2 &:= \left[\sum_{|m|=2}^{\infty}\big(1-\theta(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1}A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}\tilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}\tilde{b}_n^q\right]^{\frac{1}{q}} \\
&= \left[\sum_{|m|=2}^{\infty}\frac{\ln^{-1-\varepsilon}A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)}-\sum_{|m|=2}^{\infty}\frac{O(\ln^{-1-(\frac{\lambda_2}{2}+\varepsilon)}A_{\xi,\alpha}(m))}{A_{\xi,\alpha}(m)}\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{-1-\varepsilon}A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}\right]^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\big(2\csc^2\alpha+o(1)-\varepsilon O(1)\big)^{\frac{1}{p}}\big(2\csc^2\beta+\tilde{o}(1)\big)^{\frac{1}{q}} \quad \big(\varepsilon\to 0^+\big), \\
\tilde{I} &= \sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}k(m,n)\tilde{a}_m\tilde{b}_n = \sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty}k(m,n)\frac{\ln^{\tilde{\lambda}_1-1}A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)}\frac{\ln^{\tilde{\lambda}_2-\varepsilon-1}A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \\
&= \sum_{|n|=2}^{\infty}\varpi(\tilde{\lambda}_1,n)\frac{\ln^{-1-\varepsilon}A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} < k_\alpha(\tilde{\lambda}_1)\sum_{|n|=2}^{\infty}\frac{\ln^{-1-\varepsilon}A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \\
&= \frac{1}{\varepsilon}k_\alpha(\tilde{\lambda}_1)\big(2\csc^2\beta+o(1)\big).
\end{aligned}$$

If there exists a positive number $k\geq k(\lambda_1)$ such that (17) is still valid when replacing $k(\lambda_1)$ by k, then, in particular, we have

$$\varepsilon\tilde{I}=\varepsilon\sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty}k(m,n)\tilde{a}_m\tilde{b}_n>\varepsilon k\tilde{I}_2.$$

We obtain from the previous results that

$$\begin{aligned}
&k_\beta\left(\lambda_1+\frac{\varepsilon}{q}\right)\big(2\csc^2\alpha+o(1)\big) \\
&\quad > k\big(2\csc^2\alpha+o(1)-\varepsilon O(1)\big)^{\frac{1}{p}}\big(2\csc^2\beta+\tilde{o}(1)\big)^{\frac{1}{q}},
\end{aligned}$$

and then

$$\frac{4\pi^2}{\lambda^2\sin^2(\frac{\pi\lambda_1}{\lambda})}\csc^2\beta\csc^2\alpha\geq 2k\csc^{\frac{2}{p}}\alpha\csc^{\frac{2}{q}}\beta \quad \big(\varepsilon\to 0^+\big),$$

namely, $k(\lambda_1)=\frac{2\pi^2}{\lambda^2\sin^2(\frac{\pi\lambda_1}{\lambda})}\csc^{\frac{2}{p}}\beta\csc^{\frac{2}{q}}\alpha\geq k$. Hence, $k=k(\lambda_1)$ is the best possible constant factor of (17).

The constant factor $k(\lambda_1)$ in (18) and (19) is still the best possible. Otherwise, we would reach a contradiction by (27) and (28) that the constant factor in (17) is not the best possible. □

Remark 2 (i) For $\xi=\eta=0$ in (20), setting

$$\tilde{\theta}_1(\lambda_2,m):=\left[\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\right]^2\int_0^{\frac{\ln 2}{\ln|m|}}\frac{\ln u}{u^\lambda-1}u^{\lambda_2-1}\,du=O\left(\frac{1}{\ln^{\lambda_2/2}|m|}\right)\in(0,1),$$

we have the following new inequality:

$$\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{\ln(\ln|m|/\ln|n|)}{\ln^{\lambda}|m|-\ln^{\lambda}|n|}a_m b_n$$
$$>\frac{2\pi^2}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2}$$
$$\times\left[\sum_{|m|=2}^{\infty}\bigl(1-\tilde{\theta}_1(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}|m|}{|m|^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}|n|}{|n|^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{29}$$

It follows that (20) is an extension of (29). In particular, for $\lambda=1,\lambda_1=\lambda_2=\frac{1}{2}$, setting

$$\tilde{\theta}_1(m):=\frac{1}{\pi^2}\int_0^{\frac{\ln 2}{\ln|m|}}\frac{\ln u}{u-1}u^{\frac{-1}{2}}\,du=O\biggl(\frac{1}{\ln^{1/4}|m|}\biggr)\in(0,1),$$

we have the following simple reverse Mulholland-type inequality in the whole plane:

$$\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{\ln(\ln|m|/\ln|n|)}{\ln(|m|/|n|)}a_m b_n$$
$$>2\pi^2\left[\sum_{|m|=2}^{\infty}\bigl(1-\tilde{\theta}_1(m)\bigr)\frac{\ln^{\frac{p}{2}-1}|m|}{|m|^{1-p}}a_m^p\right]^{\frac{1}{p}}\left(\sum_{|n|=2}^{\infty}\frac{\ln^{\frac{q}{2}-1}|n|}{|n|^{1-q}}b_n^q\right)^{\frac{1}{q}}. \tag{30}$$

(ii) If $a_{-m}=a_m, b_{-n}=b_n$ $(m,n\in\mathbf{N}\backslash\{1\})$, for $m\in\mathbf{N}\backslash\{1\}$, setting

$$\widehat{\theta}_1(\lambda_2,m):=\biggl[\frac{\lambda}{\pi}\sin\biggl(\frac{\pi\lambda_1}{\lambda}\biggr)\biggr]^2\int_0^{\frac{\ln(2+\eta)}{\ln(m-\xi)}}\frac{\ln u}{u^{\lambda}-1}u^{\lambda_2-1}\,du=O\biggl(\frac{1}{\ln^{\lambda_2/2}(m-\xi)}\biggr)\in(0,1),$$
$$\breve{\theta}_1(\lambda_2,m):=\biggl[\frac{\lambda}{\pi}\sin\biggl(\frac{\pi\lambda_1}{\lambda}\biggr)\biggr]^2\int_0^{\frac{\ln(2+\eta)}{\ln(m+\xi)}}\frac{\ln u}{u^{\lambda}-1}u^{\lambda_2-1}\,du=O\biggl(\frac{1}{\ln^{\lambda_2/2}(m+\xi)}\biggr)\in(0,1),$$

(20) reduces to

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\biggl\{\frac{\ln[\ln(m-\xi)/\ln(n-\eta)]}{\ln^{\lambda}(m-\xi)-\ln^{\lambda}(n-\eta)}+\frac{\ln[\ln(m-\xi)/\ln(n+\eta)]}{\ln^{\lambda}(m-\xi)-\ln^{\lambda}(n+\eta)}$$
$$+\frac{\ln[\ln(m+\xi)/\ln(n-\eta)]}{\ln^{\lambda}(m+\xi)-\ln^{\lambda}(n-\eta)}+\frac{\ln[\ln(m+\xi)/\ln(n+\eta)]}{\ln^{\lambda}(m+\xi)-\ln^{\lambda}(n+\eta)}\biggr\}a_m b_n$$
$$>\frac{2\pi^2}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2}\left\{\sum_{m=2}^{\infty}\biggl[\bigl(1-\widehat{\theta}_1(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}}\right.$$
$$\left.+\bigl(1-\breve{\theta}_1(\lambda_2,m)\bigr)\frac{\ln^{p(1-\lambda_1)-1}(m+\xi)}{(m+\xi)^{1-p}}\biggr]a_m^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=2}^{\infty}\biggl[\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}}+\frac{\ln^{q(1-\lambda_2)-1}(n+\eta)}{(n+\eta)^{1-q}}\biggr]b_n^q\right\}^{\frac{1}{q}}. \tag{31}$$

In particular, for $\xi = \eta = 0, \lambda = 1, \lambda_1 = \lambda_2 = \frac{1}{2}$, setting

$$\widehat{\theta}_1(m) := \frac{1}{\pi^2}\int_0^{\frac{\ln 2}{\ln m}} \frac{\ln u}{u-1} u^{-\frac{1}{2}}\, du = O\left(\frac{1}{\ln^{1/4} m}\right) \in (0,1),$$

we have the following simple reverse Mulholland-type inequality:

$$\begin{aligned} &\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{\ln(\ln m/\ln n)}{\ln(m/n)} a_m b_n \\ &\quad > \pi^2 \left[\sum_{m=2}^{\infty}\bigl(1-\widehat{\theta}_1(m)\bigr)\frac{\ln^{\frac{p}{2}-1} m}{m^{1-p}} a_m^p\right]^{\frac{1}{p}} \left(\sum_{n=2}^{\infty} \frac{\ln^{\frac{q}{2}-1} n}{n^{1-q}} b_n^q\right)^{\frac{1}{q}}. \end{aligned} \tag{32}$$

4 Conclusions

In this paper, we obtain a new reverse Mulholland's inequality in the whole plane with a best possible constant factor in Theorems 1–2. Equivalent forms and a few particular cases are considered. The method of real analysis is very important and is the key to prove the reverse equivalent inequalities with the best possible constant factor. The lemmas and theorems can provide an extensive account of this type inequalities.

Acknowledgements
This work is supported by the National Natural Science Foundation (Nos. 61370186, 61640222, and 11401113) and Science and Technology Planning Project Item of Guangzhou City (No. 201707010229).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. JL participated in the design of the study and performed the numerical analysis. Both authors read and approved the final manuscript.

Publisher's Note

Received: 10 January 2018 Accepted: 2 April 2018 Published online: 10 April 2018

References

1. Hardy, G.H., Littlewood, J.E., Polya, G.: Inequalities. Cambridge University Press, Cambridge (1934)
2. Mitrinović, D.S., Pečarić, J.E., Fink, A.M.: Inequalities Involving Functions and Their Integrals and Derivatives. Kluwer Academic Publishers, Boston (1991)
3. Yang, B.: A new Hilbert's type integral inequality. Soochow J. Math. **33**(4), 849–859 (2007)
4. Hong, Y.: All-sided generalization about Hardy–Hilbert integral inequalities. Acta Math. Sin. **44**(4), 619–626 (2001)
5. Krnić, M., Pečarić, J.E.: General Hilbert's and Hardy's inequalities. Math. Inequal. Appl. **8**(1), 29–51 (2005)
6. Perić, I., Vuković, P.: Multiple Hilbert's type inequalities with a homogeneous kernel. Banach J. Math. Anal. **5**(2), 33–43 (2011)
7. He, B.: A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor. J. Math. Anal. Appl. **431**, 902–990 (2015)
8. Adiyasuren, V., Tserendorj, B., Krnić, M.: Multiple Hilbert-type inequalities involving some differential operators. Banach J. Math. Anal. **10**(2), 320–337 (2016)
9. Li, Y., He, B.: On inequalities of Hilbert's type. Bull. Aust. Math. Soc. **76**(1), 1–13 (2007)
10. Krnić, M., Vuković, P.: On a multidimensional version of the Hilbert type inequality. Anal. Math. **38**(4), 291–303 (2012)
11. Huang, Q., Yang, B.: A more accurate half-discrete Hilbert inequality with a nonhomogeneous kernel. J. Funct. Spaces Appl. **2013**, Article ID 628250 (2013)
12. He, B., Wang, Q.: A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor. J. Math. Anal. Appl. **431**(2), 889–902 (2015)
13. Yang, B., Chen, Q.: A new extension of Hardy–Hilbert's inequality in the whole plane. J. Funct. Spaces **2016**, Article ID 9197476 (2016)
14. Xin, D., Yang, B., Chen, Q.: A discrete Hilbert-type inequality in the whole plane. J. Inequal. Appl. **2016**, 133 (2016)

15. Zhong, Y., Yang, B., Chen, Q.: A more accurate Mulholland-type inequality in the whole plane. J. Inequal. Appl., **2017** 315 (2017)
16. Yang, B.: A more accurate multidimensional Hardy–Hilbert's inequality. J. Appl. Anal. Comput. **8**(2), 559–573 (2018)
17. Kuang, J.: Applied Inequalities. Shangdong Science Technic Press, Jinan (2010) (in Chinese)