

RESEARCH

Open Access



Approximation degree of Durrmeyer–Bézier type operators

Purshottam N. Agrawal¹, Serkan Araci^{2*} , Martin Bohner³ and Kumari Lipi¹

*Correspondence:
mtrsrn@hotmail.com

²Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep, Turkey
Full list of author information is available at the end of the article

Abstract

Recently, a mixed hybrid operator, generalizing the well-known Phillips operators and Baskakov–Szász type operators, was introduced. In this paper, we study Bézier variant of these new operators. We investigate the degree of approximation of these operators by means of the Lipschitz class function, the modulus of continuity, and a weighted space. We study a direct approximation theorem by means of the unified Ditzian–Totik modulus of smoothness. Furthermore, the rate of convergence for functions having derivatives of bounded variation is discussed.

MSC: 26A15; 40A35; 41A25; 41A36

Keywords: Baskakov–Szász type operators; Rate of convergence; Bounded variation; Ditzian–Totik modulus of smoothness

1 Introduction

For a continuous function h on $[0, 1]$, Bernstein [1] defined a linear positive operator in order to provide a very simple and elegant proof of the Weierstrass approximation theorem, namely

$$B_n(h; x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} h\left(\frac{k}{n}\right), \quad x \in [0, 1].$$

In order to approximate continuous functions on $[0, \infty)$, Szász [2] introduced the operator

$$S_n(h; x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} e^{-nx} h\left(\frac{k}{n}\right), \quad (1.1)$$

provided the infinite series on the right-hand side converges. Later on, for $h \in C[0, \infty)$ and $0 \leq \beta < 1$, Jain [3] proposed a modification of the operators given in (1.1), namely

$$P_n^{(\beta)}(h; x) = \sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x) h\left(\frac{k}{n}\right), \quad (1.2)$$

where

$$L_{n,k}^{(\beta)}(x) = \frac{nx(nx+k\beta)^{k-1}}{k!} e^{-(nx+k\beta)}$$

with the partition of unity

$$\sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x) = 1.$$

It is observed that the Jain operator (1.2) includes the Szász operator (1.1) as a special case for $\beta = 0$. Recently, Gupta and Greubel [4] also proposed the Durrmeyer type modification of the operators given in (1.2) as

$$D_n^{(\beta)}(h; x) = \sum_{k=0}^{\infty} \frac{\langle L_{n,k}^{(\beta)}, h \rangle}{\langle L_{n,k}^{(\beta)}, 1 \rangle} L_{n,k}^{(\beta)}(x),$$

where

$$\langle h, g \rangle = \int_0^{\infty} h(t)g(t) dt.$$

They showed that these operators converge to h without any restriction on β . The moments for these operators were obtained by using Tricomi’s hypergeometric functions and Stirling numbers of first kind, and some approximation properties of these operators were proved.

In the literature, many authors have discussed the approximation behavior of different summation-integral type operators (see [5, 6]). For $0 \leq \beta < 1$ and $c \geq 0$, Acu and Gupta [7] introduced mixed Durrmeyer type operators for $x \in [0, \infty)$ as

$$P_n^{\beta,c}(h; x) = \sum_{k=1}^{\infty} \left(\int_0^{\infty} L_{n,k-1}^{[\beta]}(t) dt \right)^{-1} p_{n,k}(x, c) \int_0^{\infty} L_{n,k-1}^{[\beta]}(t)h(t) dt + p_{n,0}(x, c)h(0), \quad (1.3)$$

where

$$p_{n,k}(x, c) = \frac{(-x)^k}{k!} \phi_{n,c}^{(k)}(x)$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & \text{if } c = 0, \\ (1 + cx)^{-n/c}, & \text{if } c > 0. \end{cases}$$

They determined the degree of approximation by means of the modulus of continuity and a weighted space. The authors also studied the approximation of functions having derivatives equivalent with a function of bounded variation. It is observed that the operator defined by (1.3) has two special cases:

- (1) If $\phi_{n,c}(x) = e^{-nx}$ and $\beta = 0$, then the Phillips operators are obtained [8].
- (2) If $\phi_{n,c}(x) = (1 + cx)^{-n/c}$ and $\beta = 0$, then one gets the Baskakov–Szász operators [9].

Zeng and Piriou [10] initiated the study of Bézier variant of Bernstein operators. Zeng and Chen [11] estimated the rate of approximation for Bézier–Bernstein–Durrmeyer operators. Zeng and Tao [12] considered Bézier–Baskakov–Durrmeyer operators for $\alpha \geq 1$ and obtained the rate of convergence. For some other contributions in this direction, we

refer to [13–21]. Motivated by the above research, we introduce the Bézier variant of the operator (1.3) as

$$\begin{aligned}
 P_{n,\alpha}^{\beta,c}(h; x) &= \sum_{k=1}^{\infty} \left(\int_0^{\infty} L_{n,k-1}^{[\beta]}(t) dt \right)^{-1} R_{n,k}^{(\alpha)}(x, c) \int_0^{\infty} L_{n,k-1}^{[\beta]}(t) h(t) dt + R_{n,0}^{(\alpha)}(x, c) h(0) \\
 &= \int_0^{\infty} K_{n,\alpha}^{\beta,c}(x; t) h(t) dt,
 \end{aligned} \tag{1.4}$$

where $\alpha \geq 1$,

$$R_{n,k}^{(\alpha)}(x, c) = [I_{n,k}(x, c)]^{\alpha} - [I_{n,k+1}(x, c)]^{\alpha} \quad \text{with } I_{n,k}(x, c) = \sum_{j=k}^{\infty} p_{n,j}(x, c),$$

and

$$K_{n,\alpha}^{\beta,c}(x; t) = \sum_{k=1}^{\infty} \left(\int_0^{\infty} L_{n,k-1}^{[\beta]}(t) dt \right)^{-1} R_{n,k}^{(\alpha)}(x, c) L_{n,k-1}^{[\beta]}(t) + R_{n,0}^{(\alpha)}(x, c) \delta(t),$$

δ being the Dirac delta function. For $\alpha = 1$, we note that $P_{n,\alpha}^{\beta,c}(h; x) = P_n^{\beta,c}(h; x)$.

Recently, Acar et al. [22] considered the Bézier variant of Bernstein–Durrmeyer type operators and studied the degree of approximation of functions having derivative of bounded variation. The order of approximation of summation-integral type operators for functions with derivatives of bounded variation is estimated in [13, 23–27].

The aim of this paper is to investigate the weighted approximation properties and a direct approximation result by means of the Ditzian–Totik modulus of smoothness $\omega_{\phi^{\tau}}(h; t)$, $0 \leq \tau \leq 1$, and the rate of convergence for functions having a derivative of bounded variation for the operators given by (1.4). Throughout this paper, C denotes a constant which may be different at each occurrence.

2 Preliminaries

In the sequel, the following auxiliary results are used to prove the main results of the paper.

Lemma 1 (see [7]) *For the m th order moment $P_n^{\beta,c}(t^m; x)$, $m = 0, 1, 2$, we obtain*

- (1) $P_n^{\beta,c}(1; x) = 1$;
- (2) $P_n^{\beta,c}(t; x) = (1 - \beta)x + \frac{\beta(2-\beta)}{n(1-\beta)}(1 - \phi_{n,c}(x))$;
- (3) $P_n^{\beta,c}(t^2; x) = (1 - \beta)^2 \left[x^2 + \frac{x(1+cx)}{n} \right] + \frac{(1+4\beta-2\beta^2)}{n} x + \frac{\beta^2(3-\beta)}{n^2(1-\beta)}(1 - \phi_{n,c}(x))$.

Consequently, for the r th order central moment $\mu_{n,r}^{\beta,c}(x) = P_n^{\beta,c}((t-x)^r; x)$, $r = 0, 1, 2$, one has

- (1) $\mu_{n,0}^{\beta,c}(x) = 1$;
- (2) $\mu_{n,1}^{\beta,c}(x) = -\beta x + \frac{\beta(2-\beta)}{n(1-\beta)}(1 - \phi_{n,c}(x))$;
- (3) $\mu_{n,2}^{\beta,c}(x) = \left[\beta^2 + \frac{c(1-\beta)^2}{n} \right] x^2 + \frac{2-4\beta-\beta^2+\beta^3+2\beta(2-\beta)\phi_{n,c}(x)}{n(1-\beta)} x + \frac{\beta^2(3-\beta)(1-\phi_{n,c}(x))}{n^2(1-\beta)}$.

Lemma 2 (see [7]) *If $\beta = \beta(n) \rightarrow 0$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} n\beta(n) = l \in \mathbb{R}$, then*

- (1) $\lim_{n \rightarrow \infty} n\mu_{n,1}^{\beta,c}(x) = -lx$;
- (2) $\lim_{n \rightarrow \infty} n\mu_{n,2}^{\beta,c}(x) = x(cx + 2)$;
- (3) $\lim_{n \rightarrow \infty} n^2\mu_{n,4}^{\beta,c}(x) = 3x^2(cx + 2)^2$.

Remark 1 It is observed that

$$P_{n,\alpha}^{\beta,c}(1; x) = \sum_{k=0}^{\infty} R_{n,k}^{(\alpha)}(x, c) = [I_{n,0}(x, c)]^\alpha = \left[\sum_{j=0}^{\infty} p_{n,k}(x, c) \right]^\alpha = 1, \tag{2.1}$$

since $\sum_{j=0}^{\infty} p_{n,k}(x, c) = 1$.

Let $C_B[0, \infty)$ denote the space of all continuous and bounded functions on $[0, \infty)$, where the norm is defined by

$$\|h\| = \sup_{[0, \infty)} |h(x)|.$$

Lemma 3 For every $h \in C_B[0, \infty)$, we have

$$\|P_{n,\alpha}^{\beta,c}(h; \cdot)\| \leq \|h\|.$$

Lemma 3 can easily be proved using (2.1).

Remark 2 We observe that

$$\begin{aligned} 0 < R_{n,k}^{(\alpha)}(x, c) &= [I_{n,k}(x, c)]^\alpha - [I_{n,k+1}(x, c)]^\alpha \\ &\leq \alpha (I_{n,k}(x, c) - I_{n,k+1}(x, c)) \\ &= \alpha p_{n,k}(x, c), \end{aligned}$$

in view of the inequality

$$|a^\alpha - b^\alpha| \leq \alpha |a - b| \quad \text{for } 0 \leq a, b \leq 1, \alpha \geq 1.$$

Hence, from (1.4), we get

$$|P_{n,\alpha}^{\beta,c}(h; x)| \leq \alpha P_n^{\beta,c}(|h|; x).$$

3 Main results

For $x \in (0, \infty)$, $t \in [0, \infty)$, and $0 < r \leq 1$, as we can see in Özarslan and Duman [28], the Lipschitz type space is defined as

$$\text{Lip}_M^*(r) := \left\{ h \in C[0, \infty) : |h(t) - h(x)| \leq M \frac{|t - x|^r}{(t + x)^{r/2}} \right\}.$$

In the following theorem, we obtain the rate of convergence of the operators $P_{n,\alpha}^{\beta,c}$ for functions in $\text{Lip}_M^*(r)$.

Theorem 1 Let $h \in \text{Lip}_M^*(r)$ and $r \in (0, 1]$. Then, for all $x \in (0, \infty)$, we have

$$|P_{n,\alpha}^{\beta,c}(h; x) - h(x)| \leq \alpha M \left(\frac{\mu_{n,2}^{\beta,c}(x)}{x} \right)^{r/2}.$$

Proof Using Remark 2, we get

$$\begin{aligned}
 |P_{n,\alpha}^{\beta,c}(h(t);x) - h(x)| &\leq P_{n,\alpha}^{\beta,c}(|h(t) - h(x)|;x) \\
 &\leq \alpha P_n^{\beta,c}(|h(t) - h(x)|;x) \\
 &\leq \alpha M P_n^{\beta,c}\left(\frac{|t-x|^r}{(t+x)^{r/2}};x\right) \\
 &\leq \frac{\alpha M}{x^{r/2}} P_n^{\beta,c}(|t-x|^r;x).
 \end{aligned}
 \tag{3.1}$$

Taking $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$ and applying Hölder’s inequality, we obtain

$$P_n^{\beta,c}(|t-x|^r;x) \leq \{P_n^{\beta,c}((t-x)^2;x)\}^{r/2} \{P_n^{\beta,c}(1^{\frac{2}{2-r}};x)\}^{\frac{2-r}{2}} = (\mu_{n,2}^{\beta,c}(x))^{r/2}.
 \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$|P_{n,\alpha}^{\beta,c}(h(t);x) - h(x)| \leq \alpha M \left(\frac{\mu_{n,2}^{\beta,c}(x)}{x}\right)^{r/2}.$$

This completes the proof. □

In the following, we present some weighted approximation results. First, we recall some basic notations. Let $B_2[0, \infty) = \{h : [0, \infty) \rightarrow \mathbb{R} : |h(x)| \leq M_h(1+x^2) \text{ for all } x \in [0, \infty)\}$. Further, let $C_2[0, \infty)$ be the subspace of $B_2[0, \infty)$ consisting of continuous functions defined on $[0, \infty)$. The norm in $C_2[0, \infty)$ is given by

$$\|h\|_2 = \sup_{x \in [0, \infty)} \frac{|h(x)|}{1+x^2}.$$

Also, let

$$C_2^0[0, \infty) := \left\{ h \in C_2[0, \infty) : \lim_{x \rightarrow \infty} \frac{|h(x)|}{1+x^2} \text{ is finite} \right\}.$$

The next theorem provides us the degree of approximation of $P_{n,\alpha}^{\beta,c}$ in terms of the classical modulus of continuity for the functions in the weighted space $C_2[0, \infty)$.

Theorem 2 For $h \in C_2[0, \infty)$, we have

$$|P_{n,\alpha}^{\beta,c}(h;x) - h(x)| \leq 4\alpha M_h(1+x^2)\mu_{n,2}^{\beta,c}(x) + (1+\sqrt{\alpha})\omega_{b+1}\left(h; \sqrt{\mu_{n,2}^{\beta,c}(x)}\right),$$

where $\omega_{b+1}(h;\delta)$ is the modulus of continuity of h on $[0, b+1]$.

Proof From [29], for $x \in [0, b]$ and $t \geq 0$, we obtain

$$|h(t) - h(x)| \leq 4M_h(t-x)^2(1+x^2) + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{b+1}(h;\delta).$$

Applying Remark 2 and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} |P_{n,\alpha}^{\beta,c}(h;x) - h(x)| &\leq 4M_h(1+x^2)P_{n,\alpha}^{\beta,c}((t-x)^2;x) + \omega_{b+1}(h;\delta) \left(1 + \frac{1}{\delta}(\alpha\mu_{n,2}^{\beta,c}(x))^{1/2}\right) \\ &\leq 4\alpha M_h(1+x^2)\mu_{n,2}^{\beta,c}(x) + \omega_{b+1}(h;\delta) \left(1 + \frac{\sqrt{\alpha}}{\delta}(\mu_{n,2}^{\beta,c}(x))^{1/2}\right). \end{aligned}$$

Choosing $\delta = \sqrt{\mu_{n,2}^{\beta,c}(x)}$, we get the desired result. □

To determine the rate of convergence for functions in $C_2^0[0, \infty)$, Yüksel and Ispir [6] introduced the weighted modulus of continuity as

$$\Omega(h;\delta) = \sup_{x \in [0,\infty), 0 < \eta < \delta} \frac{|h(x+\eta) - h(x)|}{1+(x+\eta)^2}.$$

In the following lemma, we state the properties of the weighted modulus of continuity $\Omega(h;\delta)$.

Lemma 4 (see [6]) *Let $h \in C_2^0[0, \infty)$. Then the following results hold.*

- (1) $\Omega(h;\delta)$ is monotonically increasing in δ .
- (2) $\lim_{\delta \rightarrow 0^+} \Omega(h;\delta) = 0$.
- (3) For each $m \in \mathbb{N}$, $\Omega(h;m\delta) \leq m\Omega(h;\delta)$.
- (4) For each $\lambda \in [0, \infty)$, $\Omega(h;\lambda\delta) \leq (1+\lambda)\Omega(h;\delta)$.

Theorem 3 *Let $h \in C_2^0[0, \infty)$, $\beta = \beta(n) \rightarrow 0$ as $n \rightarrow \infty$ with $\lim_{n \rightarrow \infty} n\beta = l \in \mathbb{R}$, and $b > 0$. Then*

$$\lim_{n \rightarrow \infty} \sup_{x \in [0,\infty)} \frac{|P_{n,\alpha}^{\beta,c}(h;x) - h(x)|}{(1+x^2)^{1+b}} = 0.$$

Proof Let $x_0 \in [0, \infty)$ be arbitrary but fixed. Then

$$\begin{aligned} &\sup_{x \in [0,\infty)} \frac{|P_{n,\alpha}^{\beta,c}(h;x) - h(x)|}{(1+x^2)^{1+b}} \\ &\leq \sup_{x \leq x_0} \frac{|P_{n,\alpha}^{\beta,c}(h;x) - h(x)|}{(1+x^2)^{1+b}} + \sup_{x > x_0} \frac{|P_{n,\alpha}^{\beta,c}(h;x) - h(x)|}{(1+x^2)^{1+b}} \\ &\leq \|P_{n,\alpha}^{\beta,c}(h;x) - h(x)\|_{C[0,x_0]} + \|h\|_2 \sup_{x > x_0} \frac{P_{n,\alpha}^{\beta,c}(1+t^2;x)}{(1+x^2)^{1+b}} + \sup_{x > x_0} \frac{|h(x)|}{(1+x^2)^{1+b}}. \end{aligned} \tag{3.3}$$

Since $|h(x)| \leq \|h\|_2(1+x^2)$, we have

$$\sup_{x > x_0} \frac{|h(x)|}{(1+x^2)^{1+b}} \leq \frac{\|h\|_2}{(1+x_0^2)^b}.$$

Let $\varepsilon > 0$ be arbitrary. We choose x_0 to be so large that

$$\frac{\|h\|_2}{(1+x_0^2)^b} < \frac{\varepsilon}{6}. \tag{3.4}$$

For $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$|P_{n,\alpha}^{\beta,c}(1+t^2;x) - (1+x^2)| < \frac{\varepsilon}{3\|h\|_2} \quad \text{for all } n \geq n_1.$$

Hence, using (3.4), we get

$$\begin{aligned} \|h\|_2 \sup_{x \geq x_0} \frac{P_{n,\alpha}^{\beta,c}(1+t^2;x)}{(1+x^2)^{1+b}} &\leq \|h\|_2 \sup_{x \geq x_0} \frac{1}{(1+x^2)^{1+b}} \left((1+x^2) + \frac{\varepsilon}{3\|h\|_2} \right) \\ &\leq \|h\|_2 \sup_{x \geq x_0} \left(\frac{1}{(1+x^2)^b} + \frac{\varepsilon}{3\|h\|_2(1+x^2)^{1+b}} \right) \\ &\leq \|h\|_2 \sup_{x \geq x_0} \left(\frac{1}{(1+x^2)^b} + \frac{\varepsilon}{3\|h\|_2} \right) \\ &\leq \frac{\|h\|_2}{(1+x_0^2)^b} + \frac{\varepsilon}{3} \\ &\leq \frac{\varepsilon}{2}. \end{aligned} \tag{3.5}$$

Applying Theorem 2, we can find $n_2 \in \mathbb{N}$ such that

$$\|P_{n,\alpha}^{\beta,c}(h;x) - h(x)\|_{C[0,x_0]} < \frac{\varepsilon}{3}, \tag{3.6}$$

for all n greater than equal to n_2 . Combining (3.3)–(3.6), we obtain

$$\sup_{x \in [0,\infty)} \frac{|P_{n,\alpha}^{\beta,c}(h;x) - h(x)|}{(1+x^2)^{1+b}} < \varepsilon.$$

This proves the required result. □

In the following theorem, we establish the rate of convergence of the operators $P_{n,\alpha}^{\beta,c}$ in terms of the weighted modulus of continuity Ω .

Theorem 4 *Let $h \in C_2^0[0, \infty)$. If $\beta = \beta(n) \rightarrow 0$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} n\beta(n) = l \in \mathbb{R}$, then, for sufficiently large n , we have*

$$\sup_{x \in [0,\infty)} \frac{|P_{n,\alpha}^{\beta,c}(h;x) - h(x)|}{(1+x^2)^{\frac{5}{2}}} \leq C\Omega\left(h; \frac{1}{\sqrt{n}}\right),$$

where C is a positive constant independent of h and n .

Proof For $x \in (0, \infty)$ and $\delta > 0$, using the definition of weighted modulus of continuity and Lemma 4, we have

$$\begin{aligned} |h(t) - h(x)| &\leq (1 + (x + |x - t|)^2)\Omega(h; |t - x|) \\ &\leq 2(1 + x^2)(1 + (t - x)^2) \left(1 + \frac{|t - x|}{\delta} \right) \Omega(h; \delta). \end{aligned}$$

Applying $P_{n,\alpha}^{\beta,c}(\cdot; x)$ to both sides of the above inequality, we can write

$$\begin{aligned}
 & |P_{n,\alpha}^{\beta,c}(h; x) - h(x)| \\
 & \leq 2(1 + x^2)\Omega(h; \delta) \left(1 + P_{n,\alpha}^{\beta,c}((t-x)^2; x) + P_{n,\alpha}^{\beta,c} \left((1 + (t-x)^2) \frac{|t-x|}{\delta}; x \right) \right). \tag{3.7}
 \end{aligned}$$

From Lemma 2, for sufficiently large n , it follows that

$$n\mu_{n,2}^{\beta,c}(x) \leq Cx(cx + 2) \quad \text{and} \quad n^2\mu_{n,4}^{\beta,c}(x) \leq Cx^2(cx + 2)^2, \tag{3.8}$$

where C is a positive constant. Now, applying the Cauchy–Schwarz inequality in the last term of (3.7), we obtain

$$\begin{aligned}
 & P_{n,\alpha}^{\beta,c} \left((1 + (t-x)^2) \frac{|t-x|}{\delta}; x \right) \\
 & \leq \frac{1}{\delta} (\alpha\mu_{n,2}^{\beta,c}(x))^{1/2} + \frac{1}{\delta} (\alpha\mu_{n,4}^{\beta,c}(x))^{1/2} (\alpha\mu_{n,2}^{\beta,c}(x))^{1/2}. \tag{3.9}
 \end{aligned}$$

Combining estimates (3.7)–(3.9) and taking

$$C = 2(1 + \sqrt{\alpha C} + 2\alpha C) \quad \text{and} \quad \delta = \frac{1}{\sqrt{n}},$$

we reach the required result. □

Now our aim is to discuss the rate of convergence in terms of the unified Ditzian–Totik modulus of smoothness $\omega_{\phi^\tau}(h, t)$, $0 \leq \tau \leq 1$. First, we define the Ditzian–Totik modulus of smoothness and the Peetre K -functional. Let $\phi(x) = \sqrt{x(2 + cx)}$ and $h \in C_B[0, \infty)$. The modulus $\omega_{\phi^\tau}(h, t)$, $0 \leq \tau \leq 1$, is defined as

$$\omega_{\phi^\tau}(h, t) = \sup_{0 \leq j \leq t} \sup_{x \pm \frac{j\phi^\tau(x)}{2} \in [0, \infty)} \left| h \left(x + \frac{j\phi^\tau(x)}{2} \right) - h \left(x - \frac{j\phi^\tau(x)}{2} \right) \right|,$$

and the appropriate K -functional is given by

$$K_{\phi^\tau}(h, t) = \inf_{g \in W_\tau} \{ \|h - g\| + t \|\phi^\tau g'\| \},$$

where W_τ is the subspace of the space of locally absolutely continuous functions g on $[0, \infty)$, with $\|\phi^\tau g'\| < \infty$. By [30, Theorem 2.1.1], there exists a constant $N > 0$ such that

$$N^{-1}\omega_{\phi^\tau}(h, t) \leq K_{\phi^\tau}(h, t) \leq N\omega_{\phi^\tau}(h, t). \tag{3.10}$$

Theorem 5 *If $h \in C_B[0, \infty)$, then*

$$|P_{n,\alpha}^{\beta,c}(h; x) - h(x)| \leq C\omega_{\phi^\tau} \left(h, \frac{\phi^{1-\tau}(x)}{\sqrt{n}} \right)$$

for sufficiently large n , where C is independent of h and n .

Proof By the definition of $K_{\phi^\tau}(h, t)$, there exists a function $g \in W_\tau$ such that

$$\|h - g\| + \frac{\phi^{1-\tau}(x)}{\sqrt{n}} \|\phi^\tau g'\| \leq 2K_{\phi^\tau} \left(h, \frac{\phi^{1-\tau}(x)}{\sqrt{n}} \right). \tag{3.11}$$

We can write

$$\begin{aligned} |P_{n,\alpha}^{\beta,c}(h; x) - h(x)| &\leq |P_{n,\alpha}^{\beta,c}(h - g; x)| + |P_{n,\alpha}^{\beta,c}(g; x) - g(x)| + |g(x) - h(x)| \\ &\leq 2\|h - g\| + |P_{n,\alpha}^{\beta,c}(g; x) - g(x)|. \end{aligned} \tag{3.12}$$

Since $g \in W_\tau$, we obtain

$$g(t) = g(x) + G(t), \quad \text{where } G(t) := \int_x^t g'(u) \, du,$$

and so

$$|P_{n,\alpha}^{\beta,c}(g; x) - g(x)| \leq P_{n,\alpha}^{\beta,c}(|G|; x). \tag{3.13}$$

By applying Hölder’s inequality, we get

$$|G(t)| \leq \|\phi^\tau g'\| \left| \int_x^t \frac{du}{\phi^\tau(u)} \right| \leq \|\phi^\tau g'\| |t - x|^{1-\tau} \left| \int_x^t \frac{du}{\phi(u)} \right|^\tau. \tag{3.14}$$

Now

$$\begin{aligned} \left| \int_x^t \frac{du}{\phi(u)} \right| &\leq \left| \int_x^t \frac{du}{\sqrt{u}} \right| \left(\frac{1}{\sqrt{2+cx}} + \frac{1}{\sqrt{2+ct}} \right) \\ &= 2|\sqrt{t} - \sqrt{x}| \left(\frac{1}{\sqrt{2+cx}} + \frac{1}{\sqrt{2+ct}} \right) \\ &= \frac{2|t-x|}{\sqrt{t} + \sqrt{x}} \left(\frac{1}{\sqrt{2+cx}} + \frac{1}{\sqrt{2+ct}} \right) \\ &\leq \frac{2|t-x|}{\sqrt{x}} \left(\frac{1}{\sqrt{2+cx}} + \frac{1}{\sqrt{2+ct}} \right), \end{aligned}$$

the inequality $|a + b|^\tau \leq |a|^\tau + |b|^\tau$, $0 \leq \tau \leq 1$, and (3.14) imply

$$\begin{aligned} |G(t)| &\leq \frac{2^\tau \|\phi^\tau g'\| |t-x|}{x^{\tau/2}} \left(\frac{1}{\sqrt{2+cx}} + \frac{1}{\sqrt{2+ct}} \right)^\tau \\ &\leq \frac{2^\tau \|\phi^\tau g'\| |t-x|}{x^{\tau/2}} \left(\frac{1}{(2+cx)^{\tau/2}} + \frac{1}{(2+ct)^{\tau/2}} \right). \end{aligned} \tag{3.15}$$

Thus, from (3.13), (3.15), and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} |P_{n,\alpha}^{\beta,c}(g; x) - g(x)| &\leq \frac{2^\tau \|\phi^\tau g'\|}{x^{\tau/2}} P_{n,\alpha}^{\beta,c} \left(|t-x| \left(\frac{1}{(2+cx)^{\tau/2}} + \frac{1}{(2+ct)^{\tau/2}} \right); x \right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{2^\tau \|\phi^\tau g'\|}{x^{\tau/2}} \left(\frac{1}{(2+cx)^{\tau/2}} \sqrt{\alpha \mu_{n,2}^{\beta,c}} + \sqrt{\alpha \mu_{n,2}^{\beta,c}} \sqrt{P_{n,\alpha}^{\beta,c}((2+ct)^{-\tau}; x)} \right) \\ &\leq 2^\tau \|\phi^\tau g'\| \sqrt{\alpha \mu_{n,2}^{\beta,c}} \left\{ \phi^{-\tau}(x) + x^{-\tau/2} \sqrt{P_{n,\alpha}^{\beta,c}((2+ct)^{-\tau}; x)} \right\}. \end{aligned} \tag{3.16}$$

Note that for each $x \in (0, \infty)$,

$$P_{n,\alpha}^{\beta,c}((2+ct)^{-\tau}; x) \rightarrow (2+cx)^{-\tau} \quad \text{as } n \rightarrow \infty,$$

and thus, for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$P_{n,\alpha}^{\beta,c}((2+ct)^{-\tau}; x) \leq (2+cx)^{-\tau} + \varepsilon \quad \text{for all } n \geq n_0.$$

Choosing $\varepsilon = (2+cx)^{-\tau}$, we obtain

$$P_{n,\alpha}^{\beta,c}((2+ct)^{-\tau}; x) \leq 2(2+cx)^{-\tau} \quad \text{for all } n \geq n_0.$$

Therefore, using (3.8) and (3.16), we get

$$\begin{aligned} |P_{n,\alpha}^{\beta,c}(g; x) - g(x)| &\leq 2^\tau \|\phi^\tau g'\| \sqrt{\frac{\alpha C \phi^2(x)}{n}} \left\{ \phi^{-\tau}(x) + \sqrt{2} x^{-\tau/2} (2+cx)^{-\tau/2} \right\} \\ &\leq 2^\tau (1 + \sqrt{2}) \|\phi^\tau g'\| \phi^{1-\tau}(x) \sqrt{\frac{\alpha C}{n}} \end{aligned} \tag{3.17}$$

for sufficiently large n . Thus, from (3.12), (3.17), and (3.11) (in that order), we find

$$\begin{aligned} |P_{n,\alpha}^{\beta,c}(h; x) - h(x)| &\leq 2 \|h - g\| + 2^\tau (1 + \sqrt{2}) \|\phi^\tau g'\| \phi^{1-\tau}(x) \sqrt{\frac{\alpha C}{n}} \\ &\leq C' \left\{ \|h - g\| + \frac{\phi^{1-\tau}(x)}{\sqrt{n}} \|\phi^\tau g'\| \right\} \\ &\leq CK_{\phi^\tau} \left(h, \frac{\phi^{1-\tau}(x)}{\sqrt{n}} \right), \end{aligned} \tag{3.18}$$

where $C' = \max\{2, 2^\tau(1 + \sqrt{2})\sqrt{\alpha C}\}$ and $C = 2C'$. By using relation (3.10), we reach the required result. □

Lastly, we obtain the convergence rate for functions having derivatives equivalent with a function of bounded variation. Let $\text{DBV}[0, \infty)$ be the class of functions $h \in \text{B}_2[0, \infty)$ having a derivative of bounded variation on every finite subinterval of $[0, \infty)$. The function $h \in \text{DBV}[0, \infty)$ has the representation

$$h(x) = \int_0^x j(t) dt + h(0),$$

where j is a function of bounded variation on each finite subinterval of $[0, \infty)$. For this purpose, we use the following auxiliary result.

Lemma 5 For fixed $u \in (0, \infty)$ and sufficiently large n , we have

$$\xi_{n,\alpha}^{\beta,c}(u, v) := \int_0^v K_{n,\alpha}^{\beta,c}(u; t) dt \leq \alpha \frac{Cu(2 + cu)}{n} (u - v)^{-2}, \quad 0 \leq v < u, \tag{3.19}$$

and

$$1 - \xi_{n,\alpha}^{\beta,c}(u, w) = \int_w^\infty K_{n,\alpha}^{\beta,c}(u; t) dt \leq \alpha \frac{Cu(2 + cu)}{n} (w - u)^{-2}, \quad u < w < \infty, \tag{3.20}$$

where C is a positive constant.

Proof Applying Remark 2 and using (3.8), we have

$$\begin{aligned} \xi_{n,\alpha}^{\beta,c}(u, v) &= \int_0^v K_{n,\alpha}^{\beta,c}(u; t) dt \\ &\leq \int_0^v \left(\frac{u - t}{u - v}\right)^2 K_{n,\alpha}^{\beta,c}(u; t) dt \\ &= (u - v)^{-2} \int_0^v (u - t)^2 K_{n,\alpha}^{\beta,c}(u; t) dt \\ &\leq \frac{P_{n,\alpha}^{\beta,c}((t - u)^2; u)}{(u - v)^2} \leq \frac{\alpha \mu_{n,2}^{\beta,c}(u)}{(u - v)^2} \\ &\leq \alpha \frac{Cu(2 + cu)}{n} (u - v)^{-2}, \end{aligned}$$

showing (3.19). Similarly, applying Remark 2 and using (3.8), we get

$$\begin{aligned} 1 - \xi_{n,\alpha}^{\beta,c}(u, w) &= \int_w^\infty K_{n,\alpha}^{\beta,c}(u; t) dt \\ &\leq \int_w^\infty \left(\frac{t - u}{w - u}\right)^2 K_{n,\alpha}^{\beta,c}(u; t) dt \\ &= (w - u)^{-2} \int_w^\infty (u - t)^2 K_{n,\alpha}^{\beta,c}(u; t) dt \\ &\leq \frac{P_{n,\alpha}^{\beta,c}((u - t)^2; u)}{(w - u)^2} \leq \frac{\alpha \mu_{n,2}^{\beta,c}(u)}{(w - u)^2} \\ &\leq \alpha \frac{Cu(2 + cu)}{n} (w - u)^{-2}, \end{aligned}$$

showing (3.20). □

Theorem 6 Let $h \in \text{DBV}[0, \infty)$. Then, for every $x \in (0, \infty)$ and sufficiently large n , we have

$$\begin{aligned} |P_{n,\alpha}^{\beta,c}(h; x) - h(x)| &\leq \frac{1}{\alpha + 1} |h'(x+) + \alpha h'(x-)| \sqrt{\frac{\alpha Cx(2 + cx)}{n}} \\ &\quad + \frac{\alpha}{\alpha + 1} |h'(x+) - h'(x-)| \sqrt{\frac{\alpha Cx(2 + cx)}{n}} \\ &\quad + \frac{\alpha C(2 + cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x - \frac{x}{k}}^{x + \frac{x}{k}} h'_x \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{x}{\sqrt{n}} \left(\int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} h'_x \right) + \frac{\alpha C(2+cx)}{nx} |h(2x) - h(x) - xh'(x+)| \\
 & + |h'(x+)| \sqrt{\frac{\alpha Cx(2+cx)}{n}} \\
 & + \left(4M + \frac{M + |h(x)|}{x^2} \right) \frac{\alpha Cx(2+cx)}{n},
 \end{aligned}$$

where $\int_a^b h(x)$ represents the total variation of h on $[a, b]$, M is a constant, and h'_x is defined by

$$h'_x(u) = \begin{cases} h'(u) - h'(x-), & \text{if } 0 \leq u < x, \\ 0, & \text{if } u = x, \\ h'(u) - h'(x+), & \text{if } x < u < \infty. \end{cases} \tag{3.21}$$

Proof For any $h \in \text{DBV}[0, \infty)$, from (3.21), we may write

$$\begin{aligned}
 h'(u) & = h'_x(u) + \frac{1}{\alpha + 1} (h'(x+) + \alpha h'(x-)) \\
 & + \frac{1}{2} (h'(x+) - h'(x-)) \left(\text{sgn}(u - x) + \frac{\alpha - 1}{\alpha + 1} \right) \\
 & + \delta_x(u) \left[h'(u) - \frac{1}{2} (h'(x+) + h'(x-)) \right],
 \end{aligned} \tag{3.22}$$

where

$$\delta_x(t) = \begin{cases} 1, & \text{if } t = x, \\ 0, & \text{if } t \neq x. \end{cases}$$

Since $P_{n,\alpha}^{\beta,c}(1; x) = 1$, using (1.4), for every $x \in (0, \infty)$, we get

$$\begin{aligned}
 P_{n,\alpha}^{\beta,c}(h; x) - h(x) & = \int_0^\infty K_{n,\alpha}^{\beta,c}(x; t) (h(t) - h(x)) dt \\
 & = \int_0^\infty K_{n,\alpha}^{\beta,c}(x; t) \left(\int_x^t h'(u) du \right) dt.
 \end{aligned} \tag{3.23}$$

From (3.22) and (3.23), we get

$$\begin{aligned}
 & P_{n,\alpha}^{\beta,c}(h; x) - h(x) \\
 & = \int_0^\infty K_{n,\alpha}^{\beta,c}(x; t) \int_x^t \left[h'_x(u) + \frac{1}{\alpha + 1} (h'(x+) + \alpha h'(x-)) + \frac{1}{2} (h'(x+) - h'(x-)) \right. \\
 & \quad \left. \times \left(\text{sgn}(u - x) + \frac{\alpha - 1}{\alpha + 1} \right) + \delta_x(u) \left[h'(u) - \frac{1}{2} (h'(x+) + h'(x-)) \right] du \right] dt \\
 & = C_1 + C_2 + C_3 + C_{n,\alpha}^{\beta,c}(h'_x, x) + D_{n,\alpha}^{\beta,c}(h'_x, x),
 \end{aligned}$$

where

$$\begin{aligned}
 C_1 &= \int_0^\infty \left(\int_x^t \frac{1}{\alpha + 1} (h'(x+) + \alpha h'(x-)) \, du \right) K_{n,\alpha}^{\beta,c}(x; t) \, dt, \\
 C_2 &= \int_0^\infty K_{n,\alpha}^{\beta,c}(x; t) \left(\int_x^t \frac{1}{2} (h'(x+) - h'(x-)) \left(\operatorname{sgn}(u - x) + \frac{\alpha - 1}{\alpha + 1} \right) \, du \right) \, dt, \\
 C_3 &= \int_0^\infty \left(\int_x^t \left(h'(u) - \frac{1}{2} (h'(x+) + h'(x-)) \right) \delta_x(u) \, du \right) K_{n,\alpha}^{\beta,c}(x; t) \, dt, \\
 C_{n,\alpha}^{\beta,c}(h'_x, x) &= \int_0^x \left(\int_x^t h'_x(u) \, du \right) K_{n,\alpha}^{\beta,c}(x; t) \, dt,
 \end{aligned}$$

and

$$D_{n,\alpha}^{\beta,c}(h'_x, x) = \int_x^\infty \left(\int_x^t h'_x(u) \, du \right) K_{n,\alpha}^{\beta,c}(x; t) \, dt.$$

Obviously,

$$C_3 = \int_0^\infty \left(\int_x^t \left(h'(u) - \frac{1}{2} (h'(x+) + h'(x-)) \right) \delta_x(u) \, du \right) K_{n,\alpha}^{\beta,c}(x; t) \, dt = 0. \tag{3.24}$$

Next, using (1.4), we get

$$\begin{aligned}
 C_1 &= \int_0^\infty \left(\int_x^t \frac{1}{\alpha + 1} (h'(x+) + \alpha h'(x-)) \, du \right) K_{n,\alpha}^{\beta,c}(x; t) \, dt \\
 &= \frac{1}{\alpha + 1} (h'(x+) + \alpha h'(x-)) \int_0^\infty (t - x) K_{n,\alpha}^{\beta,c}(x; t) \, dt \\
 &= \frac{1}{\alpha + 1} (h'(x+) + \alpha h'(x-)) P_{n,\alpha}^{\beta,c}((t - x); x)
 \end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
 C_2 &= \int_0^\infty K_{n,\alpha}^{\beta,c}(x; t) \left(\int_x^t \frac{1}{2} (h'(x+) - h'(x-)) \left(\operatorname{sgn}(u - x) + \frac{\alpha - 1}{\alpha + 1} \right) \, du \right) \, dt \\
 &= \frac{1}{2} (h'(x+) - h'(x-)) \left[- \int_0^x \left(\int_t^x \left(\operatorname{sgn}(u - x) + \frac{\alpha - 1}{\alpha + 1} \right) \, du \right) K_{n,\alpha}^{\beta,c}(x; t) \, dt \right. \\
 &\quad \left. + \int_x^\infty \left(\int_x^t \left(\operatorname{sgn}(u - x) + \frac{\alpha - 1}{\alpha + 1} \right) \, du \right) K_{n,\alpha}^{\beta,c}(x; t) \, dt \right] \\
 &\leq \frac{\alpha}{\alpha + 1} |h'(x+) - h'(x-)| \int_0^\infty |t - x| K_{n,\alpha}^{\beta,c}(x; t) \, dt \\
 &= \frac{\alpha}{\alpha + 1} |h'(x+) - h'(x-)| P_{n,\alpha}^{\beta,c}(|t - x|; x).
 \end{aligned} \tag{3.26}$$

Combining (3.23)–(3.26), applying Remark 2 and the Cauchy–Schwarz inequality, and using (3.8), we obtain

$$\begin{aligned}
 &|P_{n,\alpha}^{\beta,c}(h; x) - h(x)| \\
 &\leq \frac{1}{\alpha + 1} |h'(x+) + \alpha h'(x-)| (\alpha P_n^{\beta,c}((t - x)^2; x))^{1/2} + \frac{\alpha}{\alpha + 1} |h'(x+) - h'(x-)|
 \end{aligned}$$

$$\begin{aligned}
 & \times (\alpha P_n^{\beta,c}((t-x)^2; x))^{1/2} + |C_{n,\alpha}^{\beta,c}(h'_x, x)| + |D_{n,\alpha}^{\beta,c}(h'_x, x)| \\
 \leq & \frac{1}{\alpha+1} |h'(x+) + \alpha h'(x-)| \sqrt{\frac{\alpha Cx(2+cx)}{n}} + \frac{\alpha}{\alpha+1} |h'(x+) - h'(x-)| \\
 & \times \sqrt{\frac{\alpha Cx(2+cx)}{n}} + |C_{n,\alpha}^{\beta,c}(h'_x, x)| + |D_{n,\alpha}^{\beta,c}(h'_x, x)|. \tag{3.27}
 \end{aligned}$$

Now we estimate $C_{n,\alpha}^{\beta,c}(h'_x, x)$ and $D_{n,\alpha}^{\beta,c}(h'_x, x)$. Since

$$\int_a^b \frac{d}{dt} \xi_{n,\alpha}^{\beta,c}(x, t) dt \leq 1 \quad \text{for all } [a, b] \subseteq [0, \infty),$$

substituting $y = x - x/\sqrt{n}$ and applying Lemma 5, we get

$$\begin{aligned}
 |C_{n,\alpha}^{\beta,c}(h'_x, x)| &= \left| \int_0^x \left(\int_x^t h'_x(u) du \right) d_t \xi_{n,\alpha}^{\beta,c}(x, t) \right| = \left| \int_0^x \xi_{n,\alpha}^{\beta,c}(x, t) h'_x(t) dt \right| \\
 &\leq \int_0^y |h'_x(t)| |\xi_{n,\alpha}^{\beta,c}(x, t)| dt + \int_y^x |h'_x(t)| |\xi_{n,\alpha}^{\beta,c}(x, t)| dt \\
 &\leq \frac{\alpha Cx(2+cx)}{n} \int_0^y \left(\bigvee_t^x h'_x \right) (x-t)^{-2} dt + \int_y^x \left(\bigvee_t^x h'_x \right) dt \\
 &\leq \frac{\alpha Cx(2+cx)}{n} \int_0^y \left(\bigvee_t^x h'_x \right) (x-t)^{-2} dt + \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x h'_x \right) \\
 &= \frac{\alpha Cx(2+cx)}{n} \int_0^{x-\frac{x}{\sqrt{n}}} \left(\bigvee_t^x h'_x \right) (x-t)^{-2} dt + \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x h'_x \right).
 \end{aligned}$$

Substituting $u = x/(x-t)$, we obtain

$$\begin{aligned}
 \frac{\alpha Cx(2+cx)}{n} \int_0^{x-\frac{x}{\sqrt{n}}} (x-t)^{-2} \left(\bigvee_t^x h'_x \right) dt &= \frac{\alpha Cx(2+cx)}{n} x^{-1} \int_1^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{u}}^x h'_x \right) du \\
 &\leq \frac{\alpha C(2+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^x h'_x \right).
 \end{aligned}$$

Thus,

$$|C_{n,\alpha}^{\beta,c}(h'_x, x)| \leq \frac{\alpha C(2+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^x h'_x \right) + \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x h'_x \right). \tag{3.28}$$

Again, using the Cauchy–Schwarz inequality, integration by parts, and applying Lemma 5 to estimate $D_{n,\alpha}^{\beta,c}(h'_x, x)$, we get

$$\begin{aligned}
 |D_{n,\alpha}^{\beta,c}(h'_x, x)| &\leq \left| \int_{2x}^\infty \left(\int_x^t h'_x(u) du \right) K_{n,\alpha}^{\beta,c}(x; t) dt \right| \\
 &\quad + \left| \int_x^{2x} \left(\int_x^t h'_x(u) du \right) d_t (1 - \xi_{n,\alpha}^{\beta,c}(x, t)) \right|
 \end{aligned}$$

$$\begin{aligned} &\leq \left| \int_{2x}^{\infty} h(t)K_{n,\alpha}^{\beta,c}(x;t) dt \right| + |h(x)| \left| \int_{2x}^{\infty} K_{n,\alpha}^{\beta,c}(x;t) dt \right| \\ &\quad + |h'(x+)| \left(\int_{2x}^{\infty} (t-x)^2 K_{n,\alpha}^{\beta,c}(x;t) dt \right)^{1/2} \\ &\quad + \frac{\alpha C(2+cx)}{nx} \left| \int_x^{2x} (h'(u) - h'(x+)) du \right| + \left| \int_x^{x+\frac{x}{\sqrt{n}}} h'_x(t) dt \right| \\ &\quad + \frac{\alpha Cx(2+cx)}{n} \left| \int_{x+\frac{x}{\sqrt{n}}}^{2x} (t-x)^{-2} h'_x(t) dt \right|. \end{aligned}$$

$D_{n,\alpha}^{\beta,c}(h'_x, x)$ is estimated in a manner similar to $C_{n,\alpha}^{\beta,c}(h'_x, x)$. Putting $t = x + x/u$ and using (3.8), we get

$$\begin{aligned} |D_{n,\alpha}^{\beta,c}(h'_x, x)| &\leq M \int_{2x}^{\infty} (1+t^2)K_{n,\alpha}^{\beta,c}(x;t) dt \\ &\quad + |h(x)| \int_{2x}^{\infty} K_{n,\alpha}^{\beta,c}(x;t) dt + |h'(x+)| \sqrt{\frac{\alpha Cx(2+cx)}{n}} \\ &\quad + \frac{\alpha C(2+cx)}{nx} |h(2x) - h(x) - xh'(x+)| + \frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} h'_x \right) \\ &\quad + \frac{\alpha C(2+cx)}{n} \left| \int_{x+\frac{x}{\sqrt{n}}}^{2x} (t-x)^{-2} h'_x(t) dt \right| \\ &\leq M \int_{2x}^{\infty} (1+t^2)K_{n,\alpha}^{\beta,c}(x;t) dt + |h(x)| \int_{2x}^{\infty} K_{n,\alpha}^{\beta,c}(x;t) dt \\ &\quad + |h'(x+)| \sqrt{\frac{\alpha Cx(2+cx)}{n}} \\ &\quad + \frac{\alpha C(2+cx)}{nx} |h(2x) - h(x) - xh'(x+)| + \frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} h'_x \right) \\ &\quad + \frac{\alpha C(2+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} h'_x \right). \tag{3.29} \end{aligned}$$

For $t \geq 2x$, we have $t \leq 2(t-x)$ and $x \leq t-x$. Now, using (3.8), we obtain

$$\begin{aligned} &M \int_{2x}^{\infty} t^2 K_{n,\alpha}^{\beta,c}(x;t) dt + (M + |h(x)|) \int_{2x}^{\infty} K_{n,\alpha}^{\beta,c}(x;t) dt \\ &\leq 4M \int_{2x}^{\infty} (t-x)^2 K_{n,\alpha}^{\beta,c}(x;t) dt + (M + |h(x)|) \int_{2x}^{\infty} \frac{(t-x)^2}{x^2} K_{n,\alpha}^{\beta,c}(x;t) dt \\ &\leq 4M \int_0^{\infty} (t-x)^2 K_{n,\alpha}^{\beta,c}(x;t) dt + \frac{(M + |h(x)|)}{x^2} \int_0^{\infty} (t-x)^2 K_{n,\alpha}^{\beta,c}(x;t) dt \\ &= \left(4M + \frac{M + |h(x)|}{x^2} \right) \frac{\alpha Cx(2+cx)}{n}. \end{aligned}$$

Combining this with (3.27)–(3.29) yields the desired result. □

4 Conclusion

The Bézier variant of a sequence of mixed hybrid operators has been introduced and the rate of convergence by means of the Lipschitz class and the modulus of continuity has been established. The weighted approximation properties and a direct approximation theorem have been obtained. The approximation of functions with derivatives of bounded variation has been studied.

Acknowledgements

The last author expresses her sincere thanks to Ms. Sheetal Deshwal and Ms. Tarul Garg for their continued support and helpful discussions during the preparation of the paper.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have equally contributed to this work. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, India. ²Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep, Turkey. ³Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, USA.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 5 December 2017 Accepted: 17 January 2018 Published online: 22 February 2018

References

- Bernstein, S.N.: Démonstration du théorème de Weierstrass fondée sur la calcul des probabilités. *Comm. Soc. Math. Kharkov* **13**(2), 1–2 (1912)
- Szász, O.: Generalization of S. Bernstein's polynomials to the infinite interval. *J. Res. Natl. Bur. Stand.* **45**(3), 239–245 (1950)
- Jain, G.C.: Approximation of functions by a new class of linear operators. *J. Aust. Math. Soc.* **13**, 271–276 (1972)
- Gupta, V., Greubel, G.C.: Moment estimations of new Szász–Mirakyan–Durrmeyer operators. *Appl. Math. Comput.* **271**, 540–547 (2015)
- Gupta, V., Agarwal, R.P.: *Convergence Estimates in Approximation Theory*. Springer, Cham (2014)
- Yüksel, I., Ispir, N.: Weighted approximation by a certain family of summation integral-type operators. *Comput. Math. Appl.* **52**(10–11), 1463–1470 (2007) 2006
- Acu, A.M., Gupta, V.: Direct results for certain summation-integral type Baskakov–Szász operators. *Results Math.* **72**(3), 1161–1180 (2017)
- Phillips, R.S.: An inversion formula for Laplace transforms and semi-groups of linear operators. *Ann. Math. (2)* **59**, 325–356 (1954)
- Agrawal, P.N., Mohammad, A.J.: Linear combination of a new sequence of linear positive operators. *Rev. Unión Mat. Argent.* **44**(1), 33–41 (2003)
- Zeng, X.-M., Piriou, A.: On the rate of convergence of two Bernstein–Bézier type operators for bounded variation functions. *J. Approx. Theory* **95**(3), 369–387 (1998)
- Zeng, X.-M., Chen, W.: On the rate of convergence of the generalized Durrmeyer type operators for functions of bounded variation. *J. Approx. Theory* **102**(1), 1–12 (2000)
- Zeng, X.-M., Tao, W.: Rate of convergence of the integral type Lupas–Bézier operators. *Kyungpook Math. J.* **43**(4), 593–604 (2003)
- Agrawal, P.N., Ispir, N., Kajla, A.: Approximation properties of Bezier-summation-integral type operators based on Polya–Bernstein functions. *Appl. Math. Comput.* **259**, 533–539 (2015)
- Agrawal, P.N., Mohammad, A.J.: On simultaneous approximation by a linear combination of a new sequence of linear positive operators. *Turk. J. Math.* **28**(4), 353–365 (2004)
- Aral, A.: A generalization of Szász–Mirakyan operators based on q -integers. *Math. Comput. Model.* **47**(9–10), 1052–1062 (2008)
- Chang, G.-Z.: Generalized Bernstein–Bézier polynomials. *J. Comput. Math.* **1**(4), 322–327 (1983)
- Goyal, M., Agrawal, P.N.: Bézier variant of the generalized Baskakov Kantorovich operators. *Boll. Unione Mat. Ital.* **8**(4), 229–238 (2016)
- Guo, S.S., Liu, G.F., Song, Z.J.: Approximation by Bernstein–Durrmeyer–Bézier operators in L_p spaces. *Acta Math. Sci. Ser. A Chin. Ed.* **30**(6), 1424–1434 (2010)
- Gupta, V., Mohapatra, R.N.: Approximation by Durrmeyer–Bézier operators. *Nonlinear Anal., Real World Appl.* **9**(4), 1491–1498 (2008)
- Wang, P., Zhou, Y.: A new estimate on the rate of convergence of Durrmeyer–Bézier operators. *J. Inequal. Appl.* **2009**, Article ID 702680 (2009)
- Zeng, X.-M.: On the rate of convergence of two Bernstein–Bézier type operators for bounded variation functions. II. *J. Approx. Theory* **104**(2), 330–344 (2000)

22. Acar, T., Agrawal, P.N., Neer, T.: Bezier variant of the Bernstein–Durrmeyer type operators. *Results Math.* **72**(3), 1341–1358 (2017)
23. Bojanić, R., Chêng, F.H.: Rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation. *J. Math. Anal. Appl.* **141**(1), 136–151 (1989)
24. Gupta, V., Abel, U., Ivan, M.: Rate of convergence of beta operators of second kind for functions with derivatives of bounded variation. *Int. J. Math. Math. Sci.* **23**, 3827–3833 (2005)
25. Gupta, V., Vasishtha, V., Gupta, M.K.: Rate of convergence of summation-integral type operators with derivatives of bounded variation. *J. Inequal. Pure Appl. Math.* **2003**, Article ID 34 (2003)
26. Kajla, A., Acu, A.M., Agrawal, P.N.: Baskakov–Szász-type operators based on inverse Pólya–Eggenberger distribution. *Ann. Funct. Anal.* **8**(1), 106–123 (2017)
27. Kajla, A., Goyal, M.: Blending type approximation by Bernstein–Durrmeyer type operators. *Mat. Vesn.* **70**(1), 40–54 (2018)
28. Özarıslan, M.A., Duman, O.: Local approximation behavior of modified SMK operators. *Miskolc Math. Notes* **11**(1), 87–99 (2010)
29. İbikli, E., Gadjeva, E.A.: The order of approximation of some unbounded functions by the sequences of positive linear operators. *Turk. J. Math.* **19**(3), 331–337 (1995)
30. Ditzian, Z., Totik, V.: *Moduli of Smoothness*. Springer Series in Computational Mathematics, vol. 9. Springer, New York (1987)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)
