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Weighted almost convergence and related infinite matrices

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Abstract

The purpose of this paper is to introduce the notion of weighted almost convergence of a sequence and prove that this sequence endowed with the sup-norm $\|\cdot\|_\infty$ is a BK-space. We also define the notions of weighted almost conservative and regular matrices and obtain necessary and sufficient conditions for these matrix classes. Moreover, we define a weighted almost A-summable sequence and prove the related interesting result.

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Keywords: almost convergence; weighted almost convergence; regular matrix

1 Introduction and preliminaries

Let ω denote the space of all complex sequences $s = (s_j)_{j=0}^{\infty}$ (or simply write $s = (s_j)$). Any vector subspace of ω is called a sequence space. By $\mathbb{N}$ we denote the set of natural numbers, and by $\mathbb{R}$ the set of real numbers. We use the standard notation ℓ_∞, c and c_0 to denote the sets of all bounded, convergent and null sequences of real numbers, respectively, where each of the sets is a Banach space with the sup-norm $\|.\|_\infty$ defined by $\|s\|_\infty = \sup_{j\in\mathbb{N}} |s_j|$. We write the space ℓ_p of all absolutely p-summable series by

$$\ell_p = \left\{ s \in \omega : \sum_{j=0}^{\infty} |s_j|^p < \infty \ (1 \le p < \infty) \right\}.$$

Clearly, ℓ_p is a Banach space with the following norm:

$$\|s\|_p = \left(\sum_{j=0}^{\infty} |s_j|^p \right)^{1/p}.$$

For $p = 1$, we obtain the set l_1 of all absolutely summable sequences. For any sequence $s = (s_j)$, let $s^{[n]} = \sum_{j=0}^{n} s_j e_j$ be its n-section, where e_j is the sequence with 1 in place j and 0 elsewhere and $e = (1, 1, 1, \dots)$.

A sequence space X is called a BK-space if it is a Banach space with continuous coordinates $p_j : X \to \mathbb{C}$, the set of complex fields, and $p_j(s) = s_j$ for all $s = (s_j) \in X$ and every $j \in \mathbb{N}$. A BK-space $X \supset \psi$, the set of all finite sequences that terminate in zeros, is said to have AK if every sequence $s = (s_j) \in X$ has a unique representation $s = \sum_{j=0}^{\infty} s_j e_j$.

Let X and Y be two sequence spaces, and let $A = (a_{n,k})$ be an infinite matrix. If, for each $s = (s_k)$ in X, the series

$$A_n s = \sum_k a_{n,k} s_k = \sum_{k=0}^{\infty} a_{n,k} s_k \tag{1}$$

converges for each $n \in \mathbb{N}$ and the sequence $As = (A_n s)$ belongs to Y, then we say that matrix A maps X into Y. By the symbol (X, Y) we denote the set of all such matrices which map X into Y. The series in (1) is called *A-transform* of s whenever the series converges for $n = 0, 1, \ldots$. We say that $s = (s_k)$ is *A-summable* to the limit λ if $A_n s$ converges to λ $(n \to \infty)$.

The sequence $s = (s_k)$ of ℓ_∞ is said to be almost convergent, denoted by f, if all of its Banach limits [1] are equal. We denote such a class by the symbol f, and one writes $f\text{-}\lim s = \lambda$ if λ is the common value of all Banach limits of the sequence $s = (s_k)$. For a bounded sequence $s = (s_k)$, Lorentz [2] proved that $f\text{-}\lim s = \lambda$ if and only if

$$\lim_{k\to\infty} \frac{s_m + s_{m+1} + \cdots + s_{m+k}}{k+1} = \lambda$$

uniformly in m. This notion was later used to (i) define and study conservative and regular matrices [3]; (ii) introduce related sequence spaces derived by the domain of matrices [4–6]; (iii) study some related matrix transformations [7–9]; (iv) define related sequence spaces derived as the domain of the generalized weighted mean and determine duals of these spaces [10, 11]. As an extension of the notion of almost convergence, Kayaduman and Şengönül [12, 13] defined Cesàro and Riesz almost convergence and established related core theorems. The almost strongly regular matrices for single sequences were introduced and characterized [14], and for double sequences they were studied by Mursaleen [15] (also refer to [16–19]). As an application of almost convergence, Mohiuddine [20] proved a Korovkin-type approximation theorem for a sequence of linear positive operators and also obtained some of its generalizations. Başar and Kirişçi [21] determined the duals of the sequence space f and other related spaces/series and investigated some useful characterizations.

We now recall the following result.

Lemma 1.1 ([22]) *Let X and Y be BK-spaces.* (i) *Then $(X, Y) \subset B(X, Y)$, that is, every $A \in (X, Y)$ defines an operator $\mathcal{L}_A \in B(X, Y)$ by $\mathcal{L}_A(x) = Ax$ for all $x \in X$, where $B(X, Y)$ denotes the set of all bounded linear operators from X into Y.* (ii) *Then $A \in (X, \ell_\infty)$ if and only if $\|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X < \infty$. Moreover, if $A \in (X, \ell_\infty)$, then $\|\mathcal{L}_A\| = \|A\|_{(X,\ell_\infty)}$.*

2 Weighted almost convergence

Definition 2.1 Let $t = (t_k)$ be a given sequence of nonnegative numbers such that $\liminf_k t_k > 0$ and $T_m = \sum_{k=0}^{m-1} t_k \neq 0$ for all $m \geq 1$. Then the bounded sequence $s = (s_k)$ of real or complex numbers is said to be *weighted almost convergent*, shortly $f(\bar{N})$-convergent, to λ if and only if

$$\lim_{m\to\infty} \frac{1}{T_m} \sum_{k=r}^{r+m-1} t_k s_k = \lambda \quad \text{uniformly in } r.$$

We shall use the notation $f(\bar{N})$ for the space of all sequences which are $f(\bar{N})$-convergent, that is,

$$f(\bar{N}) = \left\{ s \in l_\infty : \exists \lambda \in \mathbb{C} \ni \lim_{m\to\infty} \frac{1}{T_m} \sum_{k=r}^{r+m-1} t_k s_k = \lambda \text{ uniformly in } r; \lambda = f(\bar{N})\text{-}\lim s \right\}. \quad (2)$$

We remark that if we take $t_k = 1$ for all k, then (2) is reduced to the notion of almost convergence introduced by Lorentz [2]. Clearly, a convergent sequence is $f(\bar{N})$-convergent to the same limit, but its converse is not always true.

Example 2.2 Consider a sequence $s = (s_k)$ defined by $s_k = 1$ if k is odd and 0 for even k. Also, let $t_k = 1$ for all k. Then we see that $s = (s_k)$ is $f(\bar{N})$-convergent to 1/2 but not convergent.

Definition 2.3 The matrix A (or a matrix map A) is said to be *weighted almost conservative* if $As \in f(\bar{N})$ for all $s = (s_k) \in c$. One denotes this by $A \in (c, f(\bar{N}))$. If $A \in (c, f(\bar{N}))$ with $f(\bar{N})\text{-}\lim As = \lim s$, then we say that A is *weighted almost regular matrix*; one denotes such matrices by $A \in (c, f(\bar{N}))_R$.

Theorem 2.4 *The space* $f(\bar{N})$ *of weighted almost convergence endowed with the norm* $\|\cdot\|_\infty$ *is a BK-space.*

Proof To prove our results, first we have to prove that $f(\bar{N})$ is a Banach space normed by

$$\|s\|_{f(\bar{N})} = \sup_{m,r} \left| \Psi_{m,r}(s) \right|, \quad (3)$$

where

$$\Psi_{m,r}(s) = \frac{1}{T_m} \sum_{k=r}^{r+m-1} t_k s_k.$$

It is easy to verify that (3) defines a norm on $f(\bar{N})$. We have to show that $f(\bar{N})$ is complete. For this, we need to show that every Cauchy sequence in $f(\bar{N})$ converges to some number in $f(\bar{N})$. Let (s^k) be a Cauchy sequence in $f(\bar{N})$. Then (s_j^k) is a Cauchy sequence in $\mathbb{R}$ (for each $j = 1, 2, \dots$). By using the notion of the norm of $f(\bar{N})$, it is easy to see that $(s^k) \to s$. We have only to show that $s \in f(\bar{N})$.

Let $\epsilon > 0$ be given. Since (s^k) is a Cauchy sequence in $f(\bar{N})$, there exists $M \in \mathbb{N}$ (depending on ϵ) such that

$$\left\| s^k - s^i \right\| < \epsilon/3 \quad \text{for all } k, i > M,$$

which yields

$$\sup_{m,r} \left| \Psi\left(s^k - s^i\right) \right| < \epsilon/3.$$

Therefore we have $|\Psi(s^k - s^i)| < \epsilon/3$. Taking the limit as $m \to \infty$ gives that $|\lambda^k - \lambda^i| < \epsilon/3$ for each m, r and $k, i > M$, where $\lambda^k = f(\bar{N})\text{-}\lim_m s^k$ and $\lambda^i = f(\bar{N})\text{-}\lim_m s^i$. Let $\lambda = \lim_{r\to\infty} \lambda^i$.

Letting $i \to \infty$, one obtains

$$\left|\Psi_{mr}\left(s^k - s^i\right)\right| < \epsilon/3 \quad \text{and} \quad |\lambda^k - \lambda| < \epsilon/3 \tag{4}$$

for each m, r and $k > M$. Now, for fixed k, the above inequality holds. Since $s^k \in f(\bar{N})$, for fixed k, we get

$$\lim_{m\to\infty} \Psi_{mr}\left(s^k\right) = \lambda^k \quad \text{uniformly in } r.$$

For given $\epsilon > 0$, there exists positive integers M_0 (independent of r, but dependent upon ϵ) such that

$$\left|\Psi_{mr}\left(s^k\right) - \lambda^k\right| < \epsilon/3 \tag{5}$$

for $m > M_0$ and for all r. It follows from (4) and (5) that

$$\begin{aligned}\left|\Psi_{mr}(s) - \lambda\right| &= \left|\Psi_{mr}(s) - \Psi_{mr}\left(s^k\right) + \Psi_{mr}\left(s^k\right) - \lambda^k + \lambda^k - L\right| \\ &\leq \left|\Psi_{mr}(s) - \Psi_{mr}\left(s^k\right)\right| + \left|\Psi_{mr}\left(s^k\right) - \lambda^k\right| + \left|\lambda^k - L\right| < \epsilon.\end{aligned}$$

This proves that $f(\bar{N})$ is a Banach space normed by (3).

Since $c \subset f(\bar{N}) \subset l_\infty$, there exist positive real numbers α and β with $\alpha < \beta$ such that $\alpha\|s\|_\infty \leq \|s\|_{f(\bar{N})} \leq \beta\|s\|_\infty$. That is to say, two norms $\|\cdot\|_\infty$ and $\|\cdot\|_{f(\bar{N})}$ are equivalent. It is well known that the spaces c and l_∞ endowed with the norm $\|\cdot\|_\infty$ are BK-spaces, and hence the space $f(\bar{N})$ endowed with the norm $\|\cdot\|_\infty$ is also a BK-space. □

We prove the following characterization of weighted almost conservative matrices.

Theorem 2.5 *The matrix $A = (a_{n,k})$ is weighted almost conservative, that is, $A \in (c, f(\bar{N}))$ if and only if*

$$\sup\left\{\sum_{k=0}^{\infty} \frac{1}{T_m}\left|\sum_{n=r}^{r+m-1} t_n a_{n,k}\right| : m \in \mathbb{Z}^+\right\} < \infty; \tag{6}$$

$$\lim_{m\to\infty} \frac{1}{T_m}\sum_{n=r}^{r+m-1} t_n a_{n,k} = \lambda_k \quad \textit{exists } (k = 0, 1, 2, \dots) \textit{ uniformly in } r; \tag{7}$$

$$\lim_{m\to\infty} \frac{1}{T_m}\sum_{n=r}^{r+m-1}\sum_{k=0}^{\infty} t_n a_{n,k} = \lambda \quad \textit{exists uniformly in } r. \tag{8}$$

Proof Necessity. Let $A \in (c, f(\bar{N}))$. Since the sequences e and e_k both are convergent, so A-transforms of the sequences e_k and e belong to $f(\bar{N})$ and exist uniformly in r. It follows that (7) and (8) are valid. Let r be any nonnegative integer. One writes

$$\Phi_{mr}(s) = \frac{1}{T_m}\sum_{n=r}^{r+m-1} t_n \alpha_n(s),$$

where

$$\alpha_n(s) = \sum_{k=0}^{\infty} a_{n,k} s_k.$$

It follows that $\alpha_n \in c'$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and this yields $\Phi_{mr} \in c'$ $(m \geq 1)$. Since $A \in (c, f(\bar{N}))$,

$$\lim_{m\to\infty} \Phi_{mr}(s) = \Phi(s) \quad \text{exists uniformly in } r.$$

It is clear that $(\Phi_{mr}(s))$ is bounded for $s = (s_k) \in c$ and fixed r. Hence, by the uniform boundedness principle, $(\|\Phi_{mr}\|)$ is bounded. For each $p \in \mathbb{Z}^+$ (the positive integers), the sequence $x = (x_k)$ is defined by

$$x_k = \begin{cases} \operatorname{sgn} \sum_{n=r}^{r+m-1} t_n a_{n,k} & \text{if } 0 \leq k \leq p, \\ 0 & \text{if } k > p. \end{cases}$$

Then a sequence $x \in c$, $\|x\| = 1$ and

$$\left|\Phi_{mr}(x)\right| = \frac{1}{T_m} \sum_{k=0}^{p} \left| \sum_{n=r}^{r+m-1} t_n a_{n,k} \right|. \tag{9}$$

Therefore, we obtain

$$\left|\Phi_{mr}(x)\right| \leq \|\Phi_{mr}\| \|x\| = \|\Phi_{mr}\|. \tag{10}$$

Equations (9) and (10) give that

$$\frac{1}{T_m} \sum_{k=0}^{p} \left| \sum_{n=r}^{r+m-1} t_n a_{n,k} \right| \leq \|\Phi_{mr}\| < \infty,$$

it follows that (6) is valid.

Sufficiency. Let conditions (6)-(8) hold. Let r be any nonnegative integer, and let $s_k \in c$. Then

$$\begin{aligned} \Phi_{mr}(s) &= \frac{1}{T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} s_k \\ &= \frac{1}{T_m} \sum_{k=0}^{\infty} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k, \end{aligned}$$

which gives

$$\left|\Phi_{mr}(s)\right| \leq \frac{1}{T_m} \sum_{k=0}^{\infty} \left| \sum_{n=r}^{r+m-1} t_n a_{n,k} \right| \|s\|.$$

It follows from hypothesis (6) that $|\Phi_{mr}(s)| \leq B_r \|s\|$, where B_r is a constant independent of r. Thus we have $\Phi_{mr} \in c'$ for each $m \geq 1$, which gives that a sequence $(\|\Phi_{mr}\|)$ is bounded for each nonnegative integer r. Hypotheses (7) and (8) imply that the limit of $\Phi_{mr}(e_k)$ and $\Phi_{mr}(e)$ must exist for all nonnegative integers k and r. Since $\{e, e_0, e_1, \ldots\}$ is a fundamental set in c, it follows from [23, p. 252] that $\lim_m \Phi_{mr}(s) = \Phi_r(s)$ exists and $\Phi_r \in c'$. Therefore Φ_r has the following form (see [23, p. 205]):

$$\Phi_r(s) = \xi\left(\Phi_r(e) - \sum_{k=0}^{\infty}\Phi_r(e_k)\right) + \sum_{k=0}^{\infty} s_k\Phi_r(e_k),$$

where $\xi = \lim s_k$. From (7) and (8), we see that $\Phi_r(e_k) = \lambda_k$ for a nonnegative integer k and $\Phi_r(e) = \lambda$. Therefore, for each $s \in c$ and a nonnegative integer r, we have

$$\lim_{m\to\infty} \Phi_{mr}(s) = \Phi(s)$$

with the following expression:

$$\Phi(s) = \xi\left(\lambda - \sum_{k=0}^{\infty}\lambda_k\right) + \sum_{k=0}^{\infty} s_k\lambda_k. \tag{11}$$

Since $\Phi_{mr} \in c'$, so it has the representation

$$\Phi_{mr}(s) = \xi\left(\Phi_{mr}(e) - \sum_{k=0}^{\infty}\Phi_{mr}(e_k)\right) + \sum_{k=0}^{\infty} s_k\Phi_{mr}(e_k). \tag{12}$$

We observe from (11) and (12) that the convergence of $\Phi_{mr}(s)$ to $\Phi(s)$ is uniform since $\lim_{m\to\infty}\Phi_{mr}(e_k) = \lambda_k$ and $\lim_{m\to\infty}\Phi_{mr}(e) = \lambda$ uniformly in r. Hence, A is a weighted almost conservative matrix. □

In the following theorem, we obtain the characterization of weighted almost regular matrices.

Theorem 2.6 *The matrix* $A \in (c, f(\bar{N}))_R$ *if and only if*

$$\sup\left\{\sum_{k=0}^{\infty}\frac{1}{T_m}\left|\sum_{n=r}^{r+m-1} t_n a_{n,k}\right| : m \in \mathbb{Z}^+\right\} < \infty; \tag{13}$$

$$\lim_{m\to\infty}\frac{1}{T_m}\sum_{n=r}^{r+m-1} t_n a_{n,k} = 0 \quad \textit{uniformly in } r\ (k \in \mathbb{N}_0); \tag{14}$$

$$\lim_{m\to\infty}\frac{1}{T_m}\sum_{n=r}^{r+m-1}\sum_{k=0}^{\infty} t_n a_{n,k} = 1 \quad \textit{uniformly in } r. \tag{15}$$

Proof Necessity. Let $A \in (c, f(\bar{N}))_R$. We see that condition (13) holds by using the fact that A is also weighted almost conservative. Take $e_k, e \in c$. Then A-transforms of the sequences e_k and e are weighted almost convergent to 0 and 1, respectively, since $e_k \to 0$ and $e \to 1$. Hence $e_k \in c$ gives condition (14) and $e \in c$ proves the validity of (15).

Sufficiency. Let conditions (13)-(15) hold. It is easy to see that A is weighted almost conservative. So, for each $(s_k) \in c$, $\lim_{m\to\infty}\Phi_{mr}(s) = \Phi(s)$ uniformly in r. Thus we obtain from (11) and our hypotheses (13)-(15) that $\Phi(s) = \xi = \lim s_k$. This yields A is weighted almost regular. □

We now obtain necessary and sufficient conditions for the matrix A which transform the absolutely convergent series into the space of weighted almost convergence.

Theorem 2.7 *The matrix $A \in (l_1, f(\bar{N}))$ if and only if*

$$\sup_{k,m,r} \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} \right| < \infty, \tag{16}$$

$$\lim_{m\to\infty} \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} = \lambda_k \quad \text{exists for each } k \in \mathbb{N}_0 \text{ uniformly in } r. \tag{17}$$

Proof Necessity. Let $A \in (l_1, f(\bar{N}))$. Condition (17) follows since $e_k \in l_1$. Let Φ_{mr} be a continuous linear functional on l_1 defined by

$$\Phi_{mr}(s) = \frac{1}{T_m} \sum_{k=0}^{\infty} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k.$$

Then we have

$$\left|\Phi_{mr}(s)\right| \leq \sup_k \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} \right| \|s\|_1,$$

which yields

$$\|\Phi_{mr}\| \leq \sup_k \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} \right|. \tag{18}$$

For any fixed $k \in \mathbb{N}_0$, we define a sequence $s = (s_j)$ by

$$s_j = \begin{cases} \operatorname{sgn} \sum_{n=r}^{r+m-1} t_n a_{n,k} & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Then we have $\|s\|_1 = 1$ and

$$\left|\Phi_{mr}(s)\right| = \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} s_k \right| = \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} \right| \|s\|_1,$$

so

$$\|\Phi_{mr}\| \geq \sup_k \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} \right|. \tag{19}$$

We obtain from (18) and (19) that

$$\|\Phi_{mr}\| = \sup_k \left| \frac{1}{T_m} \sum_{n=r}^{r+m-1} t_n a_{n,k} \right|.$$

Since $A \in (l_1, f(\bar{N}))$, for any $s \in l_1$, we have

$$\sup_{m,r}\left|\Phi_{mr}(s)\right| = \sup_{m,r}\left|\frac{1}{T_m}\sum_{k=0}^{\infty}\sum_{n=r}^{r+m-1} t_n a_{n,k} s_k\right| < \infty. \tag{20}$$

By using the uniform boundedness theorem, Equation (20) becomes

$$\sup_{m,r}\|\Phi_{mr}\| = \sup_{k,m,r}\left|\frac{1}{T_m}\sum_{n=r}^{r+m-1} t_n a_{n,k}\right| < \infty.$$

This proves the validity of (16).

Sufficiency. Let conditions (16) and (17) hold, and let $s = (s_k) \in l_1$. In virtue of these conditions, we see that

$$\lim_{m\to\infty}\frac{1}{T_m}\sum_{k=0}^{\infty}\sum_{n=r}^{r+m-1} t_n a_{n,k} s_k = \sum_{k=0}^{\infty}\lambda_k s_k \quad \text{uniformly in } r, \tag{21}$$

it also converges absolutely. Furthermore, $\frac{1}{T_m}\sum_{k=0}^{\infty}\sum_{n=r}^{r+m-1} t_n a_{n,k} s_k$ converges absolutely for each m and r.

Let $\epsilon > 0$ be given. Then there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k>k_0}|s_k| < \epsilon. \tag{22}$$

By condition (17), we can find some $m_0 \in \mathbb{N}$ such that

$$\left|\sum_{k\le k_0}\left[\frac{1}{T_m}\sum_{n=r}^{r+m-1} t_n a_{n,k} - \lambda_k\right]s_k\right| < \epsilon \tag{23}$$

for all $m > m_0$ uniformly in r. Now

$$\begin{aligned}\left|\sum_{k=0}^{\infty}\left[\frac{1}{T_m}\sum_{n=r}^{r+m-1} t_n a_{n,k} - \lambda_k\right]s_k\right| &\le \left|\sum_{k\le k_0}\left[\frac{1}{T_m}\sum_{n=r}^{r+m-1} t_n a_{n,k} - \lambda_k\right]s_k\right| \\ &\quad + \sum_{k>k_0}\left|\frac{1}{T_m}\sum_{n=r}^{r+m-1} t_n a_{n,k} - \lambda_k\right||s_k|\end{aligned} \tag{24}$$

for all $m > m_0$ uniformly in r. By using Equations (22) and (23) and our hypotheses in the above inequality, we see that (21) holds, and hence the sufficiency part. □

Theorem 2.8 *If the matrix A in $(l_1, f(\bar{N}))$, then $\|\mathcal{L}_A\| = \|A\|$.*

Proof Let $A \in (l_1, f(\bar{N}))$. Then we have

$$\|\mathcal{L}_A(s)\| = \sup_{m,r}\left|\frac{1}{T_m}\sum_{k=0}^{\infty}\sum_{n=r}^{r+m-1} t_n a_{n,k} s_k\right| \le \sup_{m,r}\sum_{k=0}^{\infty}\left|\frac{1}{T_m}\sum_{n=r}^{r+m-1} t_n a_{n,k}\right||s_k|,$$

which gives $\|\mathcal{L}_A(s)\| \leq \|A\|\|s\|_1$. This implies that $\|\mathcal{L}_A\| \leq \|A\|$. Also, $\mathcal{L}_A \in B(l_1, f(\bar{N}))$ gives

$$\|\mathcal{L}_A(s)\| = \|As\| \leq \|\mathcal{L}_A\|\|s\|_1.$$

Taking $s = (e_k)$ and using the fact that $\|e_k\|_1 = 1$ $\forall k$, one obtains $\|A\| \leq \|\mathcal{L}_A\|$. Hence we conclude that $\|\mathcal{L}_A\| = \|A\|$. □

Definition 2.9 Let $t = (t_k)_{k\in\mathbb{N}}$ be a given sequence of nonnegative numbers such that $\liminf_k t_k > 0$ and $T_m = \sum_{k=0}^{m-1} t_k \neq 0$ for all $m \geq 1$. A sequence $s = (s_k)$ is said to be *weighted almost A-summable* to $\lambda \in \mathbb{C}$ if the A-transform of sequence $s = (s_k)$ is weighted almost convergent to λ; equivalently, we can write

$$\lim_m \sigma_{mr}(s) = \lambda \quad \text{uniformly in } r,$$

where

$$\sigma_{mr}(s) = \frac{1}{T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} s_k.$$

In the applications of summability theory to function theory, it is important to know the region in which $S = (S_k(z))$, the sequence of partial sums of the geometric series is A-summable to $\frac{1}{1-z}$ for a given matrix A. In the following theorem, we find the region in which S is weighted almost A-summable to $\frac{1}{1-z}$.

Theorem 2.10 *Let $A = (a_{n,k})$ be a matrix such that* (15) *holds. The sequence $(S_k(z))$ is weighted almost A-summable to $\frac{1}{1-z}$ if and only if $z \in R$, where*

$$R = \left\{ z = \left(z^k\right) : \lim_m \sigma_{mr}(z) = 0 \text{ uniformly in } r \right\}.$$

Proof One writes

$$\begin{aligned} \sigma_{mr} &= \frac{1}{T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} S_k(z) \\ &= \frac{1}{T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} \frac{1-z^{k+1}}{1-z} \\ &= \frac{1}{(1-z)T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} - \frac{z}{(1-z)T_m} \sum_{n=r}^{r+m-1} \sum_{k=0}^{\infty} t_n a_{n,k} z^k. \end{aligned}$$

Taking the limit as $m \to \infty$ in the above equality and using condition (15), one obtains

$$\lim_{m\to\infty} \sigma_{mr} = \frac{1}{1-z} \quad \text{uniformly in } r$$

if and only if $z \in R$. This completes the proof. □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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