Tian and Zhang *Journal of Inequalities and Applications* (2017) 2017:207
DOI 10.1186/s13660-017-1480-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

The regularized CQ algorithm without *a priori* knowledge of operator norm for solving the split feasibility problem

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Abstract

The split feasibility problem (SFP) is finding a point $x \in C$ such that $Ax \in Q$, where C and Q are nonempty closed convex subsets of Hilbert spaces H_1 and H_2, and $A : H_1 \to H_2$ is a bounded linear operator. Byrne's CQ algorithm is an effective algorithm to solve the SFP, but it needs to compute $\|A\|$, and sometimes $\|A\|$ is difficult to work out. López introduced a choice of stepsize λ_n, $\lambda_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$, $0 < \rho_n < 4$. However, he only obtained weak convergence theorems. In order to overcome the drawbacks, in this paper, we first provide a regularized CQ algorithm without computing $\|A\|$ to find the minimum-norm solution of the SFP and then obtain a strong convergence theorem.

MSC: 58E35; 47H09; 65J15

Keywords: split feasibility problem; regularized CQ algorithm; minimum-norm solution; strong convergence; operator norm

1 Introduction

Let H_1 and H_2 be real Hilbert spaces and let C and Q be nonempty closed convex subsets of H_1 and H_2, and let $A : H_1 \to H_2$ be a bounded linear operator. Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers.

In 1994, Censor and Elfving [1] came up with the split feasibility problem (SFP) in finite-dimensional Hilbert spaces. In infinite-dimensional Hilbert spaces, it can be formulated as

$$\textit{Find } x \in C \textit{ such that } Ax \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2, and $A : H_1 \to H_2$ is a bounded linear operator. Suppose that SFP (1.1) is solvable, and let S denote its solution set. The SFP is widely applied to signal processing, image reconstruction and biomedical engineering [2–4].

So far, some authors have studied SFP (1.1) [5–17]. Others have also found a lot of algorithms to study the split equality fixed point problem and the minimization problem [18–20]. Byrne's CQ algorithm is an effective method to solve SFP (1.1). A sequence $\{x_n\}$, generated by the formula

$$x_{n+1} = P_C\big(x_n - \lambda_n A^*(I - P_Q)Ax_n\big), \quad \forall n \geq 0, \tag{1.2}$$

where the parameters $\lambda_n \in (0, \frac{2}{\|A\|^2})$, $P_C : H \to C$, and $P_Q : H \to Q$, is a set of orthogonal projections.

As is well-known, Cencor and Elfving's algorithm needs to compute A^{-1}, and Byrne's CQ algorithm needs to compute $\|A\|$. However, they are difficult to calculate.

Consider the following convex minimization problem:

$$\min_{x\in C} f(x), \tag{1.3}$$

where

$$f(x) = \frac{1}{2}\big\|(I - P_Q)Ax\big\|^2, \tag{1.4}$$

$$\nabla f(x) = A^*(I - P_Q)Ax, \tag{1.5}$$

$f(x)$ is differentiable and the gradient ∇f is L-Lipschitz with $L > 0$.

The gradient-projection algorithm [21] is the most effective method to solve (1.3). A sequence $\{x_n\}$ is generated by the recursive formula

$$x_{n+1} = P_C(I - \lambda_n \nabla f)x_n, \quad \forall n \geq 0, \tag{1.6}$$

where the parameter $\lambda_n \in (0, \frac{2}{L})$. Then we know that Byrne's CQ algorithm is a special case of the gradient-projection algorithm.

In Byrne's CQ algorithm, λ_n depends on the operator norm $\|A\|$. However, it is difficult to compute. In 2005, Yang [22] considered λ_n as follows:

$$\lambda_n := \frac{\rho_n}{\|\nabla f(x_n)\|},$$

where $\rho_n > 0$ and satisfies

$$\sum_{n=0}^{\infty} \rho_n = \infty, \qquad \sum_{n=0}^{\infty} \rho_n^2 < \infty.$$

In 2012, López [23] introduced λ_n as follows:

$$\lambda_n := \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2},$$

where $0 < \rho_n < 4$. However, López's algorithm only has weak convergence.

In 2013, Yao [24] introduced a self-adaptive method for the SFP and obtained a strong convergence theorem. However, the algorithm is difficult to work out.

In general, there are two types of algorithms to solve SFPs. One is the algorithm which depends on the norm of the operator. The other is the algorithm without *a priori* knowledge of the operator norm. The first type of algorithm needs to calculate $\|A\|$, but $\|A\|$ is not easy to work out. The second type of algorithm also has a drawback. It always has weak convergence. If we want to obtain strong convergence, we have to use the composited iterative method, but then the algorithm is difficult to calculate. In order to overcome the drawbacks, we propose a new regularized CQ algorithm without *a priori* knowledge of the operator norm to solve the SFP and we obtain a strong convergence theorem.

Consider the following regularized minimization problem:

$$\min_{x\in C} f_\beta(x) := f(x) + \frac{\beta}{2}\|x\|^2, \tag{1.7}$$

where the regularization parameter $\beta > 0$. A sequence $\{x_n\}$ is generated by the formula

$$x_{n+1} = P_C\big(I - \lambda_n(\nabla f + \beta_n I)\big)x_n, \quad \forall n \geq 0, \tag{1.8}$$

where $\nabla f(x_n) = A^*(I - P_Q)Ax_n$, $0 < \beta_n < 1$, and $\lambda_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$, $0 < \rho_n < 4$. Then, under suitable conditions, the sequence $\{x_n\}$ generated by (1.8) converges strongly to a point $z \in S$, where $z = P_S(0)$ is the minimum-norm solution of SFP (1.1).

2 Preliminaries

In this part, we introduce some lemmas and some properties that are used in the rest of the paper. Throughout this paper, let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and I be the identity operator on H_1 or H_2. If $f : H \to \mathbb{R}$ is a differentiable functional, then the gradient of f is denoted by ∇f. We use the sign '$\to$' to denote strong convergence and use the sign '$\rightharpoonup$' to denote weak convergence.

Definition 2.1 (See [25]) Let D be a nonempty subset of H, and let $T : D \to H$. Then T is firmly nonexpansive if

$$\|Tx - Ty\|^2 + \big\|(I - T)x - (I - T)y\big\|^2 \leq \|x - y\|^2, \quad \forall x, y \in D.$$

Lemma 2.2 (See [26]) *Let $T : H \to H$ be an operator. Then the following are equivalent*:

(i) *T is firmly nonexpansive*,
(ii) *$I - T$ is firmly nonexpansive*,
(iii) *$2T - I$ is nonexpansive*,
(iv) *$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$, $\forall x, y \in H$*,
(v) *$0 \leq \langle Tx - Ty, (I - T)x - (I - T)y\rangle$*.

Recall $P_C : H \to C$ is an orthogonal projection, where C is a nonempty closed convex subset of H. Then to each point $x \in H$, the unique point $P_C x \in C$ satisfies the following property:

$$\|x - P_C x\| = \inf_{y\in C}\|x - y\| =: d(x, C).$$

P_C also has the following characteristics.

Lemma 2.3 (See [27]) *For a given $x \in H$*,

(i) *$z = P_C x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0$, $\forall y \in C$*,
(ii) *$z = P_C x \Longleftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$*,
(iii) *$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, $\forall x, y \in H$*.

Lemma 2.4 (See [28]) *Let f be given by* (1.4). *Then*

(i) *f is convex and differential,*

(ii) $\nabla f(x) = A^*(I - P_Q)Ax$, $\forall x \in H$,

(iii) *f is w-lsc on H,*

(iv) *∇f is $\|A\|^2$-Lipschitz*: $\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2\|x - y\|$, $\forall x, y \in H$.

Lemma 2.5 (See [29]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0,1)$ and $\{\delta_n\}_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \alpha_n|\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 (See [30]) *Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $\{\gamma_{n_i}\}_{i\in\mathbb{N}}$ of $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_{n_i} < \gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{m_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all* (*sufficiently large*) *numbers $k \in \mathbb{N}$*:

$$\gamma_{m_k} \leq \gamma_{m_k+1}, \qquad \gamma_k \leq \gamma_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, \dots, k\}$ such that the condition

$$\gamma_n \leq \gamma_{n+1}$$

holds.

3 Main results

In this paper, we always assume that $f : H \to \mathbb{R}$ is a real-valued convex function, where $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$, the gradient $\nabla f(x) = A^*(I - P_Q)Ax$, C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

Algorithm 3.1 Choose an initial guess $x_0 \in H$ arbitrarily. Assume that the nth iterate $x_n \in C$ has been constructed and $\nabla f(x_n) \neq 0$. Then we calculate the $(n + 1)$th iterate x_{n+1} via the formula

$$x_{n+1} = P_C\big(x_n - \lambda_n\big(A^*(I - P_Q)Ax_n + \beta_n x_n\big)\big), \quad \forall n \geq 0, \tag{3.1}$$

where λ_n is chosen as follows:

$$\lambda_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$$

with $0 < \rho_n < 4$. If $\nabla f(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of SFP (1.1) and the iterative process stops. Otherwise, we set $n := n + 1$ and go to (3.1) to evaluate the next iterate x_{n+2}.

Theorem 3.1 *Suppose that $S \neq \emptyset$ and the parameters $\{\beta_n\}$ and $\{\rho_n\}$ satisfy the following conditions*:

(i) $\{\beta_n\} \subset (0,1)$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,

(ii) $\varepsilon \leq \rho_n \leq 4 - \varepsilon$ *for some* $\varepsilon > 0$ *small enough.*

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 *converges strongly to $z \in S$, where* $z = P_S(0)$.

Proof Let $x^* \in S$. Since minimization is an exactly fixed point of its projection mapping, we have $x^* = P_C x^*$ and $Ax^* = P_Q Ax^*$.

By (3.1) and the nonexpansivity of P_C, we derive

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P_C(x_n - \lambda_n(A^*(I-P_Q)Ax_n + \beta_n x_n)) - P_C x^*\|^2 \\
&\leq \|(1-\lambda_n\beta_n)x_n - \lambda_n A^*(I-P_Q)Ax_n - x^*\|^2 \\
&= \left\| \lambda_n\beta_n(-x^*) + (1-\lambda_n\beta_n)\left(x_n - \frac{\lambda_n}{1-\lambda_n\beta_n}A^*(I-P_Q)Ax_n - x^*\right)\right\|^2 \\
&= \lambda_n\beta_n\|x^*\|^2 + (1-\lambda_n\beta_n)\left\|x_n - \frac{\lambda_n}{1-\lambda_n\beta_n}A^*(I-P_Q)Ax_n - x^*\right\|^2 \\
&\quad - \lambda_n\beta_n(1-\lambda_n\beta_n)\left\|x_n - \frac{\lambda_n}{1-\lambda_n\beta_n}A^*(I-P_Q)Ax_n\right\|^2 \\
&\leq \lambda_n\beta_n\|x^*\|^2 + (1-\lambda_n\beta_n)\left\|x_n - \frac{\lambda_n}{1-\lambda_n\beta_n}A^*(I-P_Q)Ax_n - x^*\right\|^2.
\end{aligned} \tag{3.2}$$

Since P_Q is firmly nonexpansive, from Lemma 2.2, we deduce that $I - P_Q$ is also firmly nonexpansive. Hence, we have

$$\begin{aligned}
\langle A^*(I-P_Q)Ax_n, x_n - x^*\rangle &= \langle (I-P_Q)Ax_n, Ax_n - Ax^*\rangle \\
&= \langle (I-P_Q)Ax_n - (I-P_Q)Ax^*, Ax_n - Ax^*\rangle \\
&\geq \|(I-P_Q)Ax_n\|^2 \\
&= 2f(x_n).
\end{aligned} \tag{3.3}$$

Note that $\nabla f(x_n) = A^*(I-P_Q)Ax_n$. From (3.3), we obtain

$$\begin{aligned}
&\left\|x_n - \frac{\lambda_n}{1-\lambda_n\beta_n}A^*(I-P_Q)Ax_n - x^*\right\|^2 \\
&\quad = \|x_n - x^*\|^2 + \frac{\lambda_n^2}{(1-\lambda_n\beta_n)^2}\|A^*(I-P_Q)Ax_n\|^2 - \frac{2\lambda_n}{1-\lambda_n\beta_n}\langle A^*(I-P_Q)Ax_n, x_n - x^*\rangle \\
&\quad = \|x_n - x^*\|^2 + \frac{\lambda_n^2}{(1-\lambda_n\beta_n)^2}\|\nabla f(x_n)\|^2 - \frac{2\lambda_n}{1-\lambda_n\beta_n}\langle \nabla f(x_n), x_n - x^*\rangle \\
&\quad \leq \|x_n - x^*\|^2 + \frac{\lambda_n^2}{(1-\lambda_n\beta_n)^2}\|\nabla f(x_n)\|^2 - \frac{4\lambda_n}{1-\lambda_n\beta_n}f(x_n) \\
&\quad = \|x_n - x^*\|^2 + \frac{1}{(1-\lambda_n\beta_n)^2}\cdot\frac{\rho_n^2 f(x_n)^2}{\|\nabla f(x_n)\|^4}\cdot\|\nabla f(x_n)\|^2 \\
&\qquad - \frac{4\rho_n f(x_n)}{(1-\lambda_n\beta_n)\|\nabla f(x_n)\|^2}\cdot f(x_n)
\end{aligned}$$

$$= \|x_n - x^*\|^2 + \frac{\rho_n^2 f(x_n)^2}{(1-\lambda_n\beta_n)^2\|\nabla f(x_n)\|^2} - \frac{4\rho_n f(x_n)^2}{(1-\lambda_n\beta_n)\|\nabla f(x_n)\|^2}$$
$$= \|x_n - x^*\|^2 - \rho_n\left(4 - \frac{\rho_n}{1-\lambda_n\beta_n}\right)\cdot\frac{f(x_n)^2}{(1-\lambda_n\beta_n)\|\nabla f(x_n)\|^2}. \tag{3.4}$$

By condition (ii), without loss of generality, we assume that $(4 - \frac{\rho_n}{1-\lambda_n\beta_n}) > 0$ for all $n \geq 0$. Thus from (3.2) and (3.4), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \lambda_n\beta_n\|x^*\|^2 + (1-\lambda_n\beta_n)\bigg(\|x_n - x^*\|^2 \\ &\quad - \rho_n\left(4 - \frac{\rho_n}{1-\lambda_n\beta_n}\right)\cdot\frac{f(x_n)^2}{(1-\lambda_n\beta_n)\|\nabla f(x_n)\|^2}\bigg) \\ &= \lambda_n\beta_n\|x^*\|^2 + (1-\lambda_n\beta_n)\|x_n - x^*\|^2 \\ &\quad - \rho_n\left(4 - \frac{\rho_n}{1-\lambda_n\beta_n}\right)\cdot\frac{f(x_n)^2}{\|\nabla f(x_n)\|^2} \\ &\leq \lambda_n\beta_n\|x^*\|^2 + (1-\lambda_n\beta_n)\|x_n - x^*\|^2 \\ &\leq \max\{\|x^*\|^2, \|x_n - x^*\|^2\}.\end{aligned} \tag{3.5}$$

Hence, $\{x_n\}$ is bounded.

Let $z = P_S 0$. From (3.5), we deduce

$$\begin{aligned}0 &\leq \rho_n\left(4 - \frac{\rho_n}{1-\lambda_n\beta_n}\right)\frac{f(x_n)^2}{\|\nabla f(x_n)\|^2} \\ &\leq \lambda_n\beta_n\|z\|^2 + (1-\lambda_n\beta_n)\|x_n - z\|^2 - \|x_{n+1} - z\|^2.\end{aligned} \tag{3.6}$$

We consider the following two cases.

Case 1. One has $\|x_{n+1} - z\| \leq \|x_n - z\|$ for every $n \geq n_0$ large enough.

In this case, $\lim_{n\to\infty}\|x_n - z\|$ exists as finite and hence

$$\lim_{n\to\infty}\left(\|x_{n+1} - z\| - \|x_n - z\|\right) = 0. \tag{3.7}$$

This, together with (3.6), implies that

$$\rho_n\left(4 - \frac{\rho_n}{1-\lambda_n\beta_n}\right)\frac{f(x_n)^2}{\|\nabla f(x_n)\|^2} \to 0.$$

Since $\liminf_{n\to\infty}\rho_n(4 - \frac{\rho_n}{1-\lambda_n\beta_n}) \geq \varepsilon_0$ (where $\varepsilon_0 > 0$ is a constant), we get

$$\frac{f(x_n)^2}{\|\nabla f(x_n)\|^2} \to 0.$$

Noting that $\|\nabla f(x_n)\|^2$ is bounded, we deduce immediately that

$$\lim_{n\to\infty} f(x_n) = 0. \tag{3.8}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle -z, x_n - z\rangle \le 0. \tag{3.9}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ satisfying $x_{n_i} \rightharpoonup \hat{z}$ and

$$\limsup_{n\to\infty}\langle -z, x_n - z\rangle = \lim_{i\to\infty}\langle -z, x_{n_i} - z\rangle.$$

By the lower semicontinuity of f, we get

$$0 \le f(\hat{z}) \le \liminf_{i\to\infty} f(x_{n_i}) = \lim_{n\to\infty} f(x_n) = 0.$$

So

$$f(\hat{z}) = \frac{1}{2}\big\|(I - P_Q)A\hat{z}\big\|^2 = 0.$$

That is, $\hat{z}$ is a minimizer of f, and $\hat{z} \in S$. Therefore

$$\begin{aligned}\limsup_{n\to\infty}\langle -z, x_n - z\rangle &= \lim_{i\to\infty}\langle -z, x_{n_i} - z\rangle \\ &= \langle -z, \hat{z} - z\rangle \\ &\le 0. \end{aligned}\tag{3.10}$$

Then we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \big\|P_C\big(x_n - \lambda_n A^*(I - P_Q)Ax_n - \lambda_n\beta_n x_n\big) - P_C z\big\|^2 \\ &\le \big\|(1 - \lambda_n\beta_n)x_n - \lambda_n A^*(I - P_Q)Ax_n - z\big\|^2 \\ &= \left\|\lambda_n\beta_n(-z) + (1 - \lambda_n\beta_n)\left(x_n - \frac{\lambda_n}{1 - \lambda_n\beta_n}A^*(I - P_Q)Ax_n - z\right)\right\|^2 \\ &= (\lambda_n\beta_n)^2\|z\|^2 \\ &\quad + (1 - \lambda_n\beta_n)^2\left\|x_n - \frac{\lambda_n}{1 - \lambda_n\beta_n}A^*(I - P_Q)Ax_n - z\right\|^2 \\ &\quad + 2(1 - \lambda_n\beta_n)\lambda_n\beta_n\left\langle x_n - \frac{\lambda_n}{1 - \lambda_n\beta_n}A^*(I - P_Q)Ax_n - z, -z\right\rangle \\ &\le (1 - \lambda_n\beta_n)^2\|x_n - z\|^2 + (\lambda_n\beta_n)^2\|z\|^2 \\ &\quad + 2(1 - \lambda_n\beta_n)\lambda_n\beta_n\langle x_n - z, -z\rangle + 2\lambda_n^2\beta_n\big\langle\nabla f(x_n), z\big\rangle \\ &\le (1 - \lambda_n\beta_n)\|x_n - z\|^2 \\ &\quad + \lambda_n\beta_n\big(\lambda_n\beta_n\|z\|^2 + 2(1 - \lambda_n\beta_n)\langle x_n - z, -z\rangle + 2\lambda_n\big\|\nabla f(x_n)\big\| \cdot \|z\|\big). \end{aligned}$$

Note that $\|\nabla f(x_n)\|^2$ is bounded, and that $\lambda_n\|\nabla f(x_n)\| = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2} \cdot \|\nabla f(x_n)\|$. Thus $\lambda_n\|\nabla f(x_n)\| \to 0$ by (3.8). From Lemma 2.5, we deduce that

$$x_n \to z.$$

Case 2. There exists a subsequence $\{\|x_{n_j} - z\|\}$ of $\{\|x_n - z\|\}$ such that

$$\|x_{n_j} - z\| < \|x_{n_j+1} - z\| \quad \text{for all } j \geq 1.$$

By Lemma 2.6, there exists a strictly nondecreasing sequence $\{m_k\}$ of positive integers such that $\lim_{k\to\infty} m_k = +\infty$ and the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$\|x_{m_k} - z\| \leq \|x_{m_k+1} - z\|, \qquad \|x_k - z\| \leq \|x_{m_k+1} - z\|. \tag{3.11}$$

We have

$$\begin{aligned}
\|x_{n+1} - z\| &= \bigl\|P_C\bigl(x_n - \lambda_n A^*(I - P_Q)Ax_n - \lambda_n\beta_n x_n\bigr) - P_C z\bigr\| \\
&\leq \bigl\|(1 - \lambda_n\beta_n)x_n - \lambda_n A^*(I - P_Q)Ax_n - z\bigr\| \\
&= \left\|\lambda_n\beta_n(-z) + (1 - \lambda_n\beta_n)\left(x_n - \frac{\lambda_n}{1 - \lambda_n\beta_n}A^*(I - P_Q)Ax_n - z\right)\right\| \\
&\leq \lambda_n\beta_n\bigl\|(-z)\bigr\| + (1 - \lambda_n\beta_n)\left\|x_n - \frac{\lambda_n}{1 - \lambda_n\beta_n}A^*(I - P_Q)Ax_n - z\right\| \\
&\leq \lambda_n\beta_n\|z\| + (1 - \lambda_n\beta_n)\|x_n - z\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
0 &\leq \lim_{k\to\infty}\bigl(\|x_{m_k+1} - z\| - \|x_{m_k} - z\|\bigr) \\
&\leq \limsup_{n\to\infty}\bigl(\|x_{n+1} - z\| - \|x_n - z\|\bigr) \\
&\leq \limsup_{n\to\infty}\bigl(\lambda_n\beta_n\|z\| + (1 - \lambda_n\beta_n)\|x_n - z\| - \|x_n - z\|\bigr) \\
&= \limsup_{n\to\infty}\lambda_n\beta_n\bigl(\|z\| - \|x_n - z\|\bigr) \\
&= 0.
\end{aligned}$$

Hence,

$$\lim_{k\to\infty}\bigl(\|x_{m_k+1} - z\| - \|x_{m_k} - z\|\bigr) = 0. \tag{3.12}$$

By a similar argument to that of Case 1, we prove that

$$\begin{aligned}
&\limsup_{k\to\infty}\langle -z, x_{m_k} - z\rangle \leq 0, \\
&\|x_{m_k+1} - z\|^2 \leq (1 - \lambda_{m_k}\beta_{m_k})\|x_{m_k} - z\|^2 + \lambda_{m_k}\beta_{m_k}\sigma_{m_k},
\end{aligned} \tag{3.13}$$

where

$$\sigma_{m_k} = \lambda_{m_k}\beta_{m_k}\|z\|^2 + 2(1 - \lambda_{m_k}\beta_{m_k})\langle x_{m_k} - z, -z\rangle + 2\lambda_{m_k}\bigl\|\nabla f(x_{m_k})\bigr\| \cdot \|z\|.$$

In particular, from (3.13), we get

$$\lambda_{m_k}\beta_{m_k}\|x_{m_k} - z\|^2 \leq \|x_{m_k} - z\|^2 - \|x_{m_k+1} - z\|^2 + \lambda_{m_k}\beta_{m_k}\sigma_{m_k}. \tag{3.14}$$

Since $\|x_{m_k} - z\| \le \|x_{m_k+1} - z\|$, we deduce that

$$\|x_{m_k} - z\|^2 - \|x_{m_k+1} - z\|^2 \le 0.$$

Then, from (3.14), we have

$$\lambda_{m_k}\beta_{m_k}\|x_{m_k} - z\|^2 \le \lambda_{m_k}\beta_{m_k}\sigma_{m_k}.$$

Then

$$\limsup_{k\to\infty}\|x_{m_k} - z\|^2 \le \limsup_{k\to\infty}\sigma_{m_k} \le 0. \tag{3.15}$$

Then, from (3.12), we deduce that

$$\limsup_{k\to\infty}\|x_{m_k+1} - z\| = 0. \tag{3.16}$$

Thus, from (3.11) and (3.16), we conclude that

$$\limsup_{k\to\infty}\|x_k - z\| \le \limsup_{k\to\infty}\|x_{m_k+1} - z\| = 0.$$

Therefore, $x_n \to z$. This completes the proof. □

4 Conclusion

Recently, the SFP has been studied extensively by many authors. However, some algorithms need to compute $\|A\|$, and this is not an easy thing to work out. Others do not need to compute $\|A\|$, but the algorithms always have weak convergence. If we want to obtain strong convergence theorems, the algorithms are complex and difficult to calculate. We try to get over the drawbacks. In this article, we use the regularized CQ algorithm without computing $\|A\|$ to find the minimum-norm solution of the SFP, where $\lambda_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}$, $0 < \rho_n < 4$. Then, under suitable conditions, the explicit strong convergence theorem is obtained.

Acknowledgements
The authors thank the referees for their helping comments, which notably improved the presentation of this paper. This work was supported by the Fundamental Research Funds for the Central Universities [grant number 3122017072]. MT was supported by the Foundation of Tianjin Key Laboratory for Advanced Signal Processing. H-FZ was supported in part by the Technology Innovation Funds of Civil Aviation University of China for Graduate (Y17-39).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors read and approved the final manuscript.

Publisher's Note

Received: 16 May 2017 Accepted: 25 August 2017 Published online: 05 September 2017

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