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On (p,q)-Szász-Mirakyan operators and their approximation properties

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Abstract

In the present paper, we introduce a new modification of Szász-Mirakyan operators based on (p,q)-integers and investigate their approximation properties. We obtain weighted approximation and Voronovskaya-type theorem for new operators.

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1 Introduction and preliminaries

In the last two decades, there has been intensive research on the approximation of functions by positive linear operators introduced by using q-calculus. Lupas [1] was the first who used q-calculus to define q-Bernstein polynomials, and later Phillips [2] proposed a generalization of Bernstein polynomials based on q-integers. Very recently, Mursaleen et al. applied (p,q)-calculus in approximation theory and introduced the first (p,q)-analogue of Bernstein operators [3]. They investigated the uniform convergence and convergence rate of the operators and also obtained a Voronovskaya-type theorem. Also, (p,q)-analogues of Bernstein-Stancu operators [4], Bleimann-Butzer-Hahn operators [5], and Bernstein-Schurer operarors [6] were defined and their approximation properties were investigated. Most recently, the (p,q)-analogues of some more operators were defined and their approximation properties were studied in [7–17], and [18]. In this paper, we introduce a (p,q)-analogue of Szász-Mirakyan operators. Let us recall some notation and definitions of (p,q)-calculus. Let $0 < q < p \leq 1$. For nonnegative integers k and n such that $n \geq k \geq 0$, the (p,q)-integer, (p,q)-factorial, and (p,q)-binomial are respectively defined by

$$[k]_{p,q} := \frac{p^k - q^k}{p - q},$$

$$[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots 1, & k \geq 1, \\ 1, & k = 0, \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

In the case of $p = 1$, these notations reduce to q-analogues, and we can easily see that $[n]_{p,q} = p^{n-1}[n]_{q/p}$. Further, the (p,q)-power basis is defined by

$$(x \oplus a)^n_{p,q} := (x+a)(px+qa)\bigl(p^2x+q^2a\bigr)\cdots\bigl(p^{n-1}x+q^{n-1}a\bigr)$$

and

$$(x \ominus a)^n_{p,q} := (x-a)(px-qa)\bigl(p^2x-q^2a\bigr)\cdots\bigl(p^{n-1}x-q^{n-1}a\bigr).$$

Also the (p,q)-derivative of a function f, denoted by $D_{p,q}f$, is defined by

$$(D_{p,q}f)(x) := \frac{f(px)-f(qx)}{(p-q)x}, \quad x \neq 0, \qquad (D_{p,q}f)(0) := f'(0)$$

provided that f is differentiable at 0. The formula for the (p,q)-derivative of a product is

$$D_{p,q}\bigl(u(x)v(x)\bigr) := D_{p,q}\bigl(u(x)\bigr)v(qx) + D_{p,q}\bigl(v(x)\bigr)u(qx).$$

For more details on (p,q)-calculus, we refer the readers to [19, 20] and the references therein. There are two (p,q)-analogues of the exponential function:

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!} \tag{1.1}$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!}$$

which satisfy the equality $e_{p,q}(x)E_{p,q}(-x) = 1$. For $p = 1$, $e_{p,q}(x)$ and $E_{p,q}(x)$ reduce to the q-exponential functions. Here we note that the interval of convergence of $e_{p,q}(x)$ is $|x| < 1/(p-q)$ for $|p| < 1$ and $|q| < 1$, and series (1.1) converges for all $x \in \mathbb{R}$, $|p| < 1$, and $|q| < 1$.

2 Construction of operators and auxiliary results

We first define the analogue of Szász-Mirakyan operators via (p,q)-calculus as follows.

Definition 2.1 Let $0 < q < p \leq 1$ and $n \in \mathbb{N}$. For $f : [0,\infty) \to \mathbb{R}$, we define the (p,q)-analogue of Szász-Mirakyan operators by

$$S_{n,p,q}(f;x) = \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} e_{p,q}\bigl(-[n]_{p,q}q^{-k}x\bigr) f\biggl(\frac{[k]_{p,q}}{p^{k-1}[n]_{p,q}}\biggr). \tag{2.1}$$

Operators (2.1) are linear and positive. For $p = 1$, they turn out to be the q-Szász-Mirakyan operators defined in [21].

Lemma 2.1 *Let* $0 < q < p \leq 1$ *and* $n \in \mathbb{N}$. *We have*

$$S_{n,p,q}\bigl(t^{m+1};x\bigr) = \sum_{j=0}^{m} \binom{m}{j} \frac{q^j x}{p^j [n]^{m-j}_{p,q}} S_{n,p,q}\bigl(t^j;q^{-1}x\bigr). \tag{2.2}$$

Proof Using the identity

$$[k+1]_{p,q} = p^k + q[k]_{p,q},$$

we can write

$$\begin{aligned}
S_{n,p,q}\big(t^{m+1};x\big) &= \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \left(\frac{[k]_{p,q}}{p^{k-1}[n]_{p,q}}\right)^{m+1} e_{p,q}\big(-[n]_{p,q}q^{-k}x\big) \\
&= \sum_{k=0}^{\infty} \frac{p^k p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{[k+1]_{p,q}^m x}{p^{k(m+1)}[n]_{p,q}^m} e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&= \sum_{k=0}^{\infty} \frac{p^k p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{[k+1]_{p,q}^m x}{p^{km+k}[n]_{p,q}^m} e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&= \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{(p^k + q[k]_{p,q})^m x}{p^{km}[n]_{p,q}^m} e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&= \sum_{k=0}^{\infty} \frac{x}{p^{km}[n]_{p,q}^m} \frac{p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \\
&\quad \times \sum_{j=0}^{m} \binom{m}{j} p^{k(m-j)} q^j [k]_{p,q}^j e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{q^j x}{p^j [n]_{p,q}^{m-j}} \\
&\quad \times \sum_{k=0}^{\infty} \frac{[k]_{p,q}^j}{p^{j(k-1)}[n]_{p,q}^j} \frac{p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{q^j x}{p^j [n]_{p,q}^{m-j}} S_{n,p,q}\big(t^j; q^{-1}x\big),
\end{aligned}$$

as desired. □

Lemma 2.2 *Let* $0 < q < p \leq 1$ *and* $n \in \mathbb{N}$. *We have*

(i) $S_{n,p,q}(1;x) = 1$,

(ii) $S_{n,p,q}(t;x) = x$,

(iii) $S_{n,p,q}(t^2;x) = \frac{x^2}{p} + \frac{x}{[n]_{p,q}}$,

(iv) $S_{n,p,q}(t^3;x) = \frac{x^3}{p^3} + \frac{2p+q}{p^2[n]_{p,q}}x^2 + \frac{x}{[n]_{p,q}^2}$,

(v) $S_{n,p,q}(t^4;x) = \frac{x^4}{p^6} + \frac{3p^2+2pq+q^2}{p^5[n]_{p,q}}x^3 + \frac{3p^2+3pq+q^2}{p^3[n]_{p,q}^2}x^2 + \frac{x}{[n]_{p,q}^3}$.

Proof Since the proof of each equality uses the same method, we give the proof for only last three equalities. Using (2.2), we get

(iii)

$$\begin{aligned}
S_{n,p,q}\big(t^2;x\big) &= \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{[k]_{p,q}^2}{p^{2k-2}[n]_{p,q}^2} e_{p,q}\big(-[n]_{p,q}q^{-k}x\big) \\
&= \sum_{k=0}^{\infty} \frac{p^k p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{[k+1]_{p,q}x}{p^{2k}[n]_{p,q}} e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&= \sum_{k=0}^{\infty} \frac{p^k p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{p^k x}{p^{2k}[n]_{p,q}} e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&\quad + \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{q[k]_{p,q}x}{p^k[n]_{p,q}} e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&= \frac{x}{[n]_{p,q}} + \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{2k} q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{x^2}{p} e_{p,q}\big(-[n]_{p,q}q^{-(k+2)}x\big) \\
&= \frac{x^2}{p} + \frac{x}{[n]_{p,q}}.
\end{aligned}$$

(iv)

$$\begin{aligned}
S_{n,p,q}\big(t^3;x\big) &= \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{[k]_{p,q}^3}{p^{3k-3}[n]_{p,q}^3} e_{p,q}\big(-[n]_{p,q}q^{-k}x\big) \\
&= \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{(p^{2k} + 2p^k q[k]_{p,q} + q^2[k]_{p,q}^2)}{p^{2k}[n]_{p,q}^2} \\
&\quad \times x e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&= \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{x}{[n]_{p,q}^2} e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&\quad + \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{2q[k]_{p,q}}{p^k[n]_{p,q}^2} x e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&\quad + \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^k q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{q^2[k]_{p,q}^2}{p^{2k}[n]_{p,q}^2} x e_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big) \\
&= \frac{x}{[n]_{p,q}^2} + \frac{2x^2}{p[n]_{p,q}} \\
&\quad + \sum_{k=0}^{\infty} \frac{p^k p^{\frac{k(k-1)}{2}}}{q^{2k} q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{qx^2(p^k + q[k]_{p,q})}{p^{2k+2}[n]_{p,q}} e_{p,q}\big(-[n]_{p,q}q^{-(k+2)}x\big) \\
&= \frac{x}{[n]_{p,q}^2} + \frac{2x^2}{p[n]_{p,q}} + \frac{qx^2}{p^2[n]_{p,q}} \\
&\quad + \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}}}{q^{2k} q^{\frac{k(k-1)}{2}}} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} \frac{q^2x^2[k]_{p,q}}{p^{k+2}[n]_{p,q}} e_{p,q}\big(-[n]_{p,q}q^{-(k+2)}x\big) \\
&= \frac{x^3}{p^3} + \frac{2p+q}{p^2[n]_{p,q}} x^2 + \frac{x}{[n]_{p,q}^2}.
\end{aligned}$$

(v)

$$\begin{aligned}
S_{n,p,q}\big(t^4;x\big) &= \sum_{k=0}^{\infty}\frac{p^{\frac{k(k-1)}{2}}}{q^{\frac{k(k-1)}{2}}}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!}\frac{[k]_{p,q}^4}{p^{4k-4}[n]_{p,q}^4}e_{p,q}\big(-[n]_{p,q}q^{-k}x\big)\\
&= \sum_{k=0}^{\infty}\frac{p^{\frac{k(k-1)}{2}}}{q^kq^{\frac{k(k-1)}{2}}}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!}\frac{(p^{3k}+3p^{2k}q[k]_{p,q}+3p^kq^2[k]_{p,q}^2+q^3[k]_{p,q}^3)}{p^{3k}[n]_{p,q}^3}\\
&\quad\times xe_{p,q}\big(-[n]_{p,q}q^{-(k+1)}x\big)\\
&= \frac{x}{[n]_{p,q}^3}+\frac{3x^2}{p[n]_{p,q}^2}+\frac{3qx^2}{p^2[n]_{p,q}^2}+\frac{3x^3}{p^3[n]_{p,q}}\\
&\quad+\sum_{k=0}^{\infty}\frac{p^{\frac{k(k-1)}{2}}}{q^{2k}q^{\frac{k(k-1)}{2}}}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!}\frac{q^2x^2(p^{2k}+2p^kq[k]_{p,q}+q^2[k]_{p,q}^2)}{p^{2k+3}[n]_{p,q}^2}\\
&\quad\times e_{p,q}\big(-[n]_{p,q}q^{-(k+2)}x\big)\\
&= \frac{x}{[n]_{p,q}^3}+\frac{3x^2}{p[n]_{p,q}^2}+\frac{3qx^2}{p^2[n]_{p,q}^2}+\frac{3x^3}{p^3[n]_{p,q}}+\frac{q^2x^2}{p^3[n]_{p,q}^2}+\frac{2qx^3}{p^4[n]_{p,q}}\\
&\quad+\sum_{k=0}^{\infty}\frac{p^{\frac{k(k-1)}{2}}}{q^{3k}q^{\frac{k(k-1)}{2}}}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!}\frac{q^2x^3(p^k+q[k]_{p,q})}{p^{k+5}[n]_{p,q}}e_{p,q}\big(-[n]_{p,q}q^{-(k+3)}x\big)\\
&= \frac{x^4}{p^6}+\frac{3p^2+2pq+q^2}{p^5[n]_{p,q}}x^3+\frac{3p^2+3pq+q^2}{p^3[n]_{p,q}^2}x^2+\frac{x}{[n]_{p,q}^3}. \qquad \square
\end{aligned}$$

Corollary 2.1 *Using Lemma* 2.2, *we immediately have the following explicit formulas for the central moments*:

$$S_{n,p,q}\big((t-x)^2;x\big) = \frac{x}{[n]_{p,q}}+\left(\frac{1}{p}-1\right)x^2, \tag{2.3}$$

$$S_{n,p,q}\big((t-x)^3;x\big) = \frac{x}{[n]_{p,q}^2}+\frac{2p+q-3p^2}{p^2[n]_{p,q}}x^2+\frac{1-3p^2+2p^3}{p^3}x^3, \tag{2.4}$$

$$\begin{aligned}
S_{n,p,q}\big((t-x)^4;x\big) &= \frac{x}{[n]_{p,q}^3}+\frac{3p^2+3pq+q^2-4p^3}{p^3[n]_{p,q}^2}x^2\\
&\quad+\frac{3p^2+2pq+q^2-8p^4-4p^3q+6p^5}{p^5[n]_{p,q}}x^3\\
&\quad+\frac{1-4p^3+6p^5-3p^6}{p^6}x^4.
\end{aligned}\tag{2.5}$$

Remark 2.1 For $q\in(0,1)$ and $p\in(q,1]$ we easily see that $\lim_{n\to\infty}[n]_{p,q}=\frac{1}{p-q}$. Hence, operators (2.1) are not approximation process with above form. To study convergence properties of the sequence of (p,q)-Szász operators, we assume that $q=(q_n)$ and $p=(p_n)$ are such that $0<q_n<p_n\le 1$ and $q_n\to 1$, $p_n\to 1$, $q_n^n\to a$, $p_n^n\to b$ as $n\to\infty$. We also assume

that

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(\frac{1}{p_n}-1\right)=\alpha,$$
$$\lim_{n\to\infty}[n]_{p_n,q_n}\frac{1-3p_n^2+2p_n^3}{p_n^3}=\gamma,$$
$$\lim_{n\to\infty}[n]_{p_n,q_n}\frac{1-4p_n^3+6p_n^5-3p_n^6}{p_n^6}=\beta.$$

It is natural to ask whether such sequences (q_n) and (p_n) exist. For example, let $c,d\in\mathbb{R}^+$ be such that $c>d$. If we choose $q_n=\frac{n}{n+c}$ and $p_n=\frac{n}{n+d}$, then $q_n\to 1$, $p_n\to 1$, $q_n^n\to e^{-c}$, $p_n^n\to e^{-d}$, and $\lim_{n\to\infty}[n]_{p,q}=\infty$ as $n\to\infty$. Also, we have $\alpha=\frac{a(e^{-d}-e^{-c})}{d-c}$, $\gamma=e^{-d}-e^{-c}$, $\beta=0$.

Corollary 2.2 *According to Remark* 2.1, *we immediately have*

$$\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,p_n,q_n}\big((t-x)^2;x\big)=x+\alpha x^2, \tag{2.6}$$
$$\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,p_n,q_n}\big((t-x)^3;x\big)=\gamma x^3, \tag{2.7}$$
$$\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,p_n,q_n}\big((t-x)^4;x\big)=\beta x^4. \tag{2.8}$$

3 Direct results

In this section, we present a local approximation theorem for the operators $S_{n,p,q}$. By $C_B[0,\infty)$ we denote the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\|f\|=\sup_{0\le x<\infty}\big|f(x)\big|.$$

Further, let us consider the following K-functional:

$$K_2(f,\delta)=\inf_{g\in W^2}\big\{\|f-g\|+\delta\big\|g''\big\|\big\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By Theorem 2.4 of [22] there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta}), \tag{3.1}$$

where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\le\sqrt{\delta}}\sup_{x\in[0,\infty)}\big|f(x+2h)-2f(x+h)+f(x)\big|$$

is the second-order modulus of smoothness of $f\in C_B[0,\infty)$. The usual modulus of continuity of $f\in C_B[0,\infty)$ is defined by

$$\omega(f,\delta)=\sup_{0<h\le\delta}\sup_{x\in[0,\infty)}\big|f(x+h)-f(x)\big|.$$

Theorem 3.1 *Let $p,q \in (0,1)$ be such that $q < p$. Then we have*

$$\left|S_{n,p,q}(f;x) - f(x)\right| \leq C\omega_2\bigl(f;\delta_n(x)\bigr)$$

for every $x \in [0,\infty)$ and $f \in C_B[0,\infty)$, where

$$\delta_n^2(x) = \frac{x}{[n]_{p,q}} + \left(\frac{1}{p} - 1\right)x^2.$$

Proof Let $g \in \mathcal{W}^2$. Then from the Taylor expansion we get

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,\mathrm{d}u, \quad t \in [0,\mathcal{A}], \mathcal{A} > 0.$$

Now by Corollary 2.1 we have

$$\begin{aligned} S_{n,p,q}(g;x) &= g(x) + S_{n,p,q}\left(\int_x^t (t-u)g''(u)\,\mathrm{d}u;x\right), \\ \left|S_{n,p,q}(g;x) - g(x)\right| &\leq S_{n,p,q}\left(\left|\int_x^t \left|(t-u)\right|\left|g''(u)\right|\mathrm{d}u;x\right|\right) \\ &\leq S_{n,p,q}\bigl((t-x)^2;x\bigr)\left\|g''\right\|. \end{aligned}$$

Hence we get

$$\left|S_{n,p,q}(g;x) - g(x)\right| \leq \left\|g''\right\|\left(\frac{x}{[n]_{p,q}} + \left(\frac{1}{p} - 1\right)x^2\right).$$

On the other hand, we have

$$\left|S_{n,p,q}(f;x) - f(x)\right| \leq \left|S_{n,p,q}\bigl((f-g);x\bigr) - (f-g)(x)\right| + \left|S_{n,p,q}(g;x) - g(x)\right|.$$

Since

$$\left|S_{n,p,q}(f;x)\right| \leq \|f\|,$$

we have

$$\left|S_{n,p,q}(f;x) - f(x)\right| \leq \|f - g\| + \left\|g''\right\|\left(\frac{x}{[n]_{p,q}} + \left(\frac{1}{p} - 1\right)x^2\right).$$

Now taking the infimum on the right-hand side over all $g \in \mathcal{W}^2$, we get

$$\left|S_{n,p,q}(f;x) - f(x)\right| \leq \mathcal{C}K_2\bigl(f,\delta_n^2(x)\bigr).$$

By the property of a K-functional we get

$$\left|S_{n,p,q}(f;x) - f(x)\right| \leq \mathcal{C}\omega_2\bigl(f,\delta_n(x)\bigr).$$

This completes the proof. □

4 Weighted approximation by $S_{n,p,q}$

Now we give approximation properties of the operators $S_{n,p,q}$ on the interval $[0,\infty)$. Since

$$\begin{aligned} S_{n,p,q}\bigl(1+t^2;x\bigr) &= 1+\left(\frac{1}{p}-1\right)x^2+\frac{x}{[n]_{p,q}} \\ &\leq 1+x^2+x, \end{aligned}$$

$x\leq 1$ for $x\in[0,1]$, and $x\leq x^2$ for $x\in(1,\infty)$, we have

$$S_{n,p,q}\bigl(1+t^2;x\bigr)\leq 2\bigl(1+x^2\bigr),$$

which says that $S_{n,p,q}$ are linear positive operators acting from $C_2[0,\infty)$ to $B_2[0,\infty)$. For more details, see [23, 24], and [25].

Theorem 4.1 *Let the sequence of linear positive operators* (L_n) *acting from* $C_2[0,\infty)$ *to* $B_2[0,\infty)$ *satisfy the condition*

$$\lim_{n\to\infty}\|L_ne_i-e_i\|_2=0,\quad i=0,1,2.$$

Then, for any function $f\in C_2^*[0,\infty)$,

$$\lim_{n\to\infty}\|L_nf-f\|_2=0.$$

Theorem 4.2 *Let* $q=q_n\in(0,1)$ *and* $p=p_n\in(q,1)$ *be such that* $q_n\to 1$ *and* $p_n\to 1$ *as* $n\to\infty$. *Then, for each function* $f\in C_2^*[0,\infty)$, *we get*

$$\lim_{n\to\infty}\|S_{n,p_n,q_n}f-f\|_2=0.$$

Proof According to Theorem 4.1, it is sufficient to verify the condition

$$\lim_{n\to\infty}\|S_{n,p_n,q_n}e_i-e_i\|_2=0,\quad i=0,1,2. \tag{4.1}$$

By Lemma 2.1(i), (ii) it is clear that

$$\begin{aligned} &\lim_{n\to\infty}\bigl\|S_{n,p_n,q_n}(1;x)-1\bigr\|_2=0, \\ &\lim_{n\to\infty}\bigl\|S_{n,p_n,q_n}(t;x)-x\bigr\|_2=0, \end{aligned}$$

and by Lemma 2.1(iii) we have

$$\begin{aligned} \lim_{n\to\infty}\bigl\|S_{n,p_n,q_n}\bigl(t^2;x\bigr)-x^2\bigr\|_2 &= \sup_{x\geq 0}\frac{(\frac{1}{p_n}-1)x^2+\frac{x}{[n]_{p_n,q_n}}}{1+x^2} \\ &\leq\left(\frac{1}{p_n}-1\right)+\frac{1}{[n]_{p_n,q_n}}. \end{aligned}$$

The last inequality means that (4.1) holds for $i=2$. By Theorem 4.1 the proof is complete. □

The weighted modulus of continuity is given by

$$\Omega(f;\delta) = \sup_{0\le h<\delta, x\in[0,\infty)} \frac{|f(x+h)-f(x)|}{(1+h^2)+(1+x^2)} \tag{4.2}$$

for $f \in C_2[0,\infty)$. We know that, for every $f \in C_2^*[0,\infty)$, $\Omega(\cdot;\delta)$ has the properties

$$\lim_{\delta\to 0} \Omega(f;\delta) = 0$$

and

$$\Omega(f;\lambda\delta) \le 2(1+\lambda)\bigl(1+\delta^2\bigr)\Omega(f;\delta), \quad \lambda > 0. \tag{4.3}$$

For $f \in C_2[0,\infty)$, from (4.2) and (4.3) we can write

$$\begin{aligned} \bigl|f(t)-f(x)\bigr| &\le \bigl(1+(t-x)^2\bigr)\bigl(1+x^2\bigr)\Omega\bigl(f;|t-x|\bigr) \\ &\le 2\left(1+\frac{|t-x|}{\delta}\right)\bigl(1+\delta^2\bigr)\Omega(f;\delta)\bigl(1+(t-x)^2\bigr)\bigl(1+x^2\bigr). \end{aligned} \tag{4.4}$$

All concepts mentioned can be found in [26].

Theorem 4.3 *Let $0 < q = q_n < p = p_n \le 1$ be such that $q_n \to 1$ and $p_n \to 1$ as $n \to \infty$. Then, for each function $f \in C_2^*[0,\infty)$, there exists a positive constant A such that*

$$\sup_{x\in[0,\infty)} \frac{|S_{n,p,q}(f;x)-f(x)|}{(1+x^2)^{\frac{5}{2}}} \le A\Omega\left(f;\frac{1}{\sqrt{\beta_{p,q}(n)}}\right),$$

where $\beta_{p,q}(n) = \max\{\frac{1}{p}-1, \frac{1}{[n]_{p,q}}\}$, and A is a positive constant.

Proof Since $S_{n,p,q}(1;x) = 1$, using the monotonicity of $S_{n,p,q}$, we can write

$$\bigl|S_{n,p,q}(f;x)-f(x)\bigr| \le S_{n,p,q}\bigl(\bigl|f(t)-f(x)\bigr|;x\bigr).$$

On the other hand, from (4.4) we have that

$$\begin{aligned} \bigl|S_{n,p,q}(f;x)-f(x)\bigr| &\le 2\bigl(1+\delta^2\bigr)\Omega(f;\delta)\bigl(1+x^2\bigr)\left[S_{n,p,q}\left(\left(1+\frac{|t-x|}{\delta}\right)\bigl(1+(t-x)^2\bigr);x\right)\right] \\ &\le 2\bigl(1+\delta^2\bigr)\Omega(f;\delta)\bigl(1+x^2\bigr)\biggl\{S_{n,p,q}(1;x)+S_{n,p,q}\bigl((t-x)^2;x\bigr) \\ &\quad + \frac{1}{\delta}S_{n,p,q}\bigl(|t-x|;x\bigr)+\frac{1}{\delta}S_{n,p,q}\bigl(|t-x|(t-x)^2;x\bigr)\biggr\}. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we can write

$$\begin{aligned} \bigl|S_{n,p,q}(f;x)-f(x)\bigr| &\le 2\bigl(1+\delta^2\bigr)\Omega(f;\delta)\bigl(1+x^2\bigr)\biggl\{S_{n,p,q}(1;x)+S_{n,p,q}\bigl((t-x)^2;x\bigr) \\ &\quad + \frac{1}{\delta}\sqrt{S_{n,p,q}\bigl((t-x)^2;x\bigr)}+\frac{1}{\delta}\sqrt{S_{n,p,q}\bigl((t-x)^4;x\bigr)}\sqrt{S_{n,p,q}\bigl((t-x)^2;x\bigr)}\biggr\}. \end{aligned}$$

On the other hand, using (2.3), we have

$$\begin{aligned} S_{n,p,q}\big((t-x)^2;x\big) &\leq \frac{x}{[n]_{p,q}} + \left(\frac{1}{p}-1\right)x^2 \\ &\leq C_1 O\big(\beta_{p,q}(n)\big)\big(1+x^2\big), \end{aligned}$$

where $C_1 > 0$ and $\beta_{p,q}(n) = \max\{\frac{1}{p}-1, \frac{1}{[n]_{p,q}}\}$. Since $\lim_{n\to\infty}\frac{1}{p_n} = 1$ and $\lim_{n\to\infty}\frac{1}{[n]_{p,q}} = 0$, there exists a positive constant A_2 such that

$$S_{n,p,q}\big((t-x)^2;x\big) \leq A_2\big(1+x^2\big).$$

Also, using (2.5), we get

$$S_{n,p,q}\big((t-x)^4;x\big)^{\frac{1}{2}} \leq A_3\big(1+x^2\big)$$

and

$$S_{n,p,q}\left(\frac{(t-x)^2}{\delta^2};x\right)^{\frac{1}{2}} \leq \frac{A_4}{\delta} O\big(\beta_{p,q}(n)\big)^{\frac{1}{2}}\big(1+x^2\big)^{\frac{1}{2}}$$

for $A_3 > 0$ and $A_4 > 0$. So we have

$$\begin{aligned} \big|S_{n,p,q}(f;x) - f(x)\big| \leq{}& 2\left(1+\frac{1}{\beta_{p,q}(n)}\right)\Omega\left(f;\frac{1}{\sqrt{\beta_{p,q}(n)}}\right)\big(1+x^2\big)\bigg\{1 + A_2\big(1+x^2\big) \\ &+ \frac{A_4}{\delta} O\big(\beta_{p,q}(n)\big)^{\frac{1}{2}}\big(1+x^2\big)^{\frac{1}{2}} \\ &+ A_3\big(1+x^2\big)\frac{A_4}{\delta} O\big(\beta_{p,q}(n)\big)^{\frac{1}{2}}\big(1+x^2\big)^{\frac{1}{2}}\bigg\}. \end{aligned}$$

Choosing $\delta = \beta_{p,q}(n)^{\frac{1}{2}}$, we obtain

$$\begin{aligned} \big|S_{n,p,q}(f;x) - f(x)\big| \leq{}& 2\big(1+\beta_{p,q}(n)\big)\Omega\left(f;\frac{1}{\sqrt{\beta_{p,q}(n)}}\right)\big(1+x^2\big)\big\{1 + A_2\big(1+x^2\big) \\ &+ CA_4\big(1+x^2\big)^{\frac{1}{2}} + C_1A_3A_4\big(1+x^2\big)^{\frac{3}{2}}\big\}. \end{aligned}$$

For $0 < q < p \leq 1$, we have $\beta_{p,q}(n) \leq 1$. Hence we can write

$$\sup_{x\in[0,\infty)} \frac{|S_{n,p,q}(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq A\Omega\left(f;\frac{1}{\sqrt{\beta_{p,q}(n)}}\right),$$

where $A = 4(1 + A_2 + CA_4 + C_1A_3A_4)$, and the result follows. □

5 Voronovskaya-type theorem for $S_{n,p,q}$

Here we give a Voronovskaya-type theorem for $S_{n,p,q}$.

Theorem 5.1 *Let $0 < q_n < p_n \leq 1$ be such that $q_n \to 1$, $p_n \to 1$, $q_n^n \to a$, and $p_n^n \to b$ as $n \to \infty$. Then, for each function $f \in C_2^*[0,\infty)$ such that $f', f'' \in C_2^*[0,\infty)$, we have*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\big(S_{n,p_n,q_n}(f;x) - f(x)\big) = \big(x+\alpha x^2\big)f''(x)$$

uniformly on any $[0,A]$, $A > 0$.

Proof Let $f, f', f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$. By the Taylor formula we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + h(t,x)(t-x)^2, \tag{5.1}$$

where $h(t,x)$ is the remainder of the Peano form. Then $h(\cdot,x) \in C_2^*[0,\infty)$ and $\lim_{t\to x} h(t,x) = 0$ for n large enough. Applying operators (2.1) to both sides of (5.1), we get

$$\begin{aligned} {}_{p_n,q_n}\bigl(S_{n,p_n,q_n}(f;x) - f(x)\bigr) = {} & [n]_{p_n,q_n} f'(x) S_{n,p_n,q_n}\bigl((t-x);x\bigr) \\ & + [n]_{p_n,q_n} f''(x) S_{n,p_n,q_n}\bigl((t-x)^2;x\bigr) \\ & + S_{n,p_n,q_n}\bigl(h(t,x)(t-x)^2;x\bigr). \end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$S_{n,p_n,q_n}\bigl(h(t,x)(t-x)^2;x\bigr) \le \sqrt{S_{n,p_n,q_n}\bigl(h^2(t,x);x\bigr)}\sqrt{S_{n,p_n,q_n}\bigl((t-x)^4;x\bigr)}. \tag{5.2}$$

Observe that $h^2(x,x) = 0$ and $h^2(\cdot,x) \in C_2^*[0,\infty)$. Then it follows from Theorem 4.3 that

$$\lim_{n\to\infty} S_{n,p_n,q_n}\bigl(h^2(t,x);x\bigr) = h^2(x,x) = 0 \tag{5.3}$$

uniformly with respect to $x \in [0,A]$. Hence, from (5.2), (5.3), and (2.8) we obtain

$$\lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\bigl(h(t,x)(t-x)^2;x\bigr) = 0 \tag{5.4}$$

and

$$S_{n,p,q}\bigl((t-x);x\bigr) = 0.$$

Then using (2.6) and (5.4), we have

$$\begin{aligned} \lim_{n\to\infty} [n]_{p_n,q_n}\bigl(S_{n,p_n,q_n}(f;x) - f(x)\bigr) &= f'(x) \lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\bigl((t-x);x\bigr) \\ &\quad + f''(x) \lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\bigl((t-x)^2;x\bigr) \\ &\quad + \lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\bigl(h(t,x)(t-x)^2;x\bigr) \\ &= \bigl(x + \alpha x^2\bigr) f''(x), \end{aligned}$$

as desired. □

6 Conclusion

In this paper, we have constructed a new modification of Szász-Mirakyan operators based on (p,q)-integers and investigated their approximation properties. We have obtained a weighted approximation and Voronovskaya-type theorem for our new operators.

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