Lu *Journal of Inequalities and Applications* (2017) 2017:37
DOI 10.1186/s13660-016-1285-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

# A new result on the existence of periodic solutions for Liénard equations with a singularity of repulsive type

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**Abstract**

In this paper, the problem of the existence of a periodic solution is studied for the second order differential equation with a singularity of repulsive type

$$x''(t) + f(x(t))x'(t) - g(x(t)) + \varphi(t)x(t) = h(t),$$

where $g(x)$ is singular at $x = 0$, $\varphi$ and $h$ are $T$-periodic functions. By using the continuation theorem of Manásevich and Mawhin, a new result on the existence of positive periodic solution is obtained. It is interesting that the sign of the function $\varphi(t)$ is allowed to change for $t \in [0, T]$.

**MSC:** 34C25; 34B16; 34B18

**Keywords:** Liénard equation; topological degree; singularity; periodic solution

## 1 Introduction

The aim of this paper is to search for positive $T$-periodic solutions for a second order differential equation with a singularity in the following form:

$$x''(t) + f(x(t))x'(t) - g(x(t)) + \varphi(t)x(t) = h(t), \tag{1.1}$$

where $f : [0, \infty) \to R$ is an arbitrary continuous function, $g \in C((0, +\infty), (0, +\infty))$, and $g(x)$ is singular of repulsive type at $x = 0$, *i.e.*, $g(x) \to +\infty$ as $x \to 0^+$, $\varphi, h : R \to R$ are $T$-periodic functions with $h \in L^2([0, T], R)$ and $\varphi \in C([0, T], R)$, and the sign of the function $\varphi$ is allowed to change for $t \in [0, T]$.

The study of the problem of periodic solutions to scalar equations with a singularity began with work of Forbat and Huaux [1, 2], where the singular term in the equations models the restoring force caused by a compressed perfect gas (see [3–6] and the references therein). In the past years, many works used the methods, such as the approaches of critical point theory [7–12], the techniques of some fixed point theorems [13–15], and the approaches of topological degree theory, in particular, of some continuation theorems of Mawhin (see [6, 16–22]), to study the existence of positive periodic solutions for some second order ordinary differential equations with singularities. For example, in [15], by using a fixed point theorem in cones, the existence of positive periodic solutions to equation

(1.1) was investigated for the conservative case, *i.e.*, $f(x) \equiv 0$. But the function $\varphi(t)$ is required to be $\varphi(t) \geq 0$ for all $t \in [0, T]$. The method of topological degree theory, together with the technique of upper and lower solutions, was first used by Lazer and Solimini in the pioneering paper [18] for considering the problem of a periodic solution to a second order differential equations with singularities. Jebelean and Mawhin in [6] considered the problem of a $p$-Laplacian Liénard equation of the form

$$\left(\left|x'\right|^{p-2}x'\right)' + f(x)x' + g(x) = h(t) \tag{1.2}$$

and

$$\left(\left|x'\right|^{p-2}x'\right)' + f(x)x' - g(x) = h(t), \tag{1.3}$$

where $p > 1$ is a constant, $f : [0, +\infty) \to R$ is an arbitrary continuous function, $h : R \to R$ is a $T$-periodic function with $h \in L^\infty([0, T], R)$, $g : (0, +\infty) \to (0, +\infty)$ is continuous, $g(x) \to +\infty$ as $x \to 0^+$. They extended the results of Lazer and Solimini in [16] to equation (1.2) and equation (1.3). For equation (1.3), the crucial condition is that the function $g(x)$ is bounded, which means that equation (1.3) is not singular at $x = +\infty$.

By using a continuation theorem of Mawhin, Zhang in [18] studied the problem of periodic solutions of the Liénard equation with a singularity of repulsive type,

$$x'' + f(x)x' + g(t, x) = 0, \tag{1.4}$$

where $f : R \to R$ is continuous, $g : R \times (0, +\infty) \to R$ is an $L^2$-Carathéodory function with $T$-periodic in the first argument, and it is singular at $x = 0$, *i.e.*, $g(t,x)$ is unbounded as $x \to 0^+$. Different from the equation studied in [6, 16], which is only singular at $x = 0$, equation (1.4) is provided with both singularities at $x = +\infty$ and at $x = 0$. In [19], Wang further studied the existence of positive periodic solutions for a delay Liénard equation with a singularity of repulsive type

$$x'' + f(x)x' + g\big(t, x(t - \tau)\big) = 0. \tag{1.5}$$

In [18, 19], the following balance condition between the singular force at the origin and at infinity is needed.

($h_1$) There exist constants $0 < D_1 < D_2$ such that if $x$ is a positive continuous $T$-periodic function satisfying

$$\int_0^T g\big(t, x(t)\big)\, dt = 0,$$

then

$$D_1 \leq x(\tau) \leq D_2, \quad \text{for some } \tau \in [0, T]. \tag{1.6}$$

From the proof of [18, 19], we see that the balance condition ($h_1$) is crucial for estimating *a priori bounds* of periodic solutions. Now, the question is how to investigate the existence of positive periodic solutions for the equations like equation (1.4) or equation (1.5) without the balance condition ($h_1$).

Motivated by this, in this paper, we study the existence of positive $T$-periodic solutions for equation (1.1) under the condition that the sign of the function $\varphi$ is allowed to change for $t \in [0,T]$. For this case, the balance condition ($h_1$) may not be satisfied. By using the continuation theorem of Manásevich and Mawhin, a new result on the existence of positive periodic solutions is obtained.

## 2 Preliminary lemmas

Throughout this paper, let $C_T = \{x \in C(R,R) : x(t+T) = x(t) \text{ for all } t \in R\}$ with the norm defined by $|x|_\infty = \max_{t\in[0,T]} |x(t)|$. For any $T$-periodic solution $y(t)$ with $y \in L^1([0,T],R)$, $y_+(t)$ and $y_-(t)$ denote $\max\{y(t),0\}$ and $-\min\{y(t),0\}$, respectively, and $\bar{y} = \frac{1}{T}\int_0^T y(s)\,ds$. Clearly, $y(t) = y_+(t) - y_-(t)$ for all $t \in R$, and $\bar{y} = \overline{y_+} - \overline{y_-}$.

The following lemma is a consequence of Theorem 3.1 in [23].

**Lemma 1** *Assume that there exist positive constants $M_0$, $M_1$, and $M_2$ with $0 < M_0 < M_1$, such that the following conditions hold.*

1. *For each $\lambda \in (0,1]$, each possible positive $T$-periodic solution $x$ to the equation*

$$u'' + \lambda f(u)u' - \lambda g(u) + \lambda\varphi(t)u = \lambda h(t)$$

*satisfies the inequalities $M_0 < x(t) < M_1$ and $|x'(t)| < M_2$ for all $t \in [0,T]$.*

2. *Each possible solution $c$ to the equation*

$$g(c) - c\bar{\varphi} + \bar{h} = 0$$

*satisfies the inequality $M_0 < c < M_1$.*

3. *We have*

$$\bigl(g(M_0) - \bar{\varphi}M_0 + \bar{h}\bigr)\bigl(g(M_1) - \bar{\varphi}M_1 + \bar{h}\bigr) < 0.$$

*Then equation* (1.1) *has at least one $T$-periodic solution $u$ such that $M_0 < u(t) < M_1$ for all $t \in [0,T]$.*

**Lemma 2** ([19]) *Let $x$ be a continuous $T$-periodic continuously differential function. Then, for any $\tau \in (0,T]$,*

$$\left(\int_0^T \bigl|x(s)\bigr|^2\,ds\right)^{1/2} \le \frac{T}{\pi}\left(\int_0^T \bigl|x'(s)\bigr|^2\,ds\right)^{1/2} + \sqrt{T}\bigl|x(\tau)\bigr|.$$

In order to study the existence of positive periodic solutions to equation (1.1), we list the following assumptions.

[$H_1$] The function $\varphi(t)$ satisfies the following conditions:

$$\int_0^T \varphi_+(s)\,ds > 0, \qquad \sigma := \frac{\int_0^T \varphi_-(s)\,ds}{\int_0^T \varphi_+(s)\,ds} \in [0,1)$$

and

$$\sigma_1 := \frac{T}{\pi}|\varphi_+|_\infty^{1/2} + \frac{T^{\frac{1}{2}}(\int_0^T \varphi_-(s)^2\,ds)^{\frac{1}{2}}}{\int_0^T \varphi_+(s)\,ds} \in (0,1);$$

[$H_2$] there are constants $A > 0$ and $M > 0$ such that $g(x) \in (0,A)$ for all $x > M$;
[$H_3$] $\int_0^1 g(s)\,ds = +\infty$.

**Remark 1** If assumptions [$H_1$]-[$H_2$] hold, then there are constants $D_1$ and $D_2$ with $0 < D_1 < D_2$ such that

$$g(x) - \bar{\varphi}x + \bar{h} > 0 \quad \text{for all } x \in (0, D_1)$$

and

$$g(x) - \bar{\varphi}x + \bar{h} < 0 \quad \text{for all } x \in (D_2, \infty).$$

Furthermore, assumption $\sigma_1 \in (0,1)$ in [$H_1$] is different from the corresponding condition $\int_0^T \varphi_+(s)\,ds < \frac{4}{T}$ in [20].

Now, we suppose that assumptions [$H_1$] and [$H_2$] hold, and we embed equation (1.1) into the following equation family with a parameter $\lambda \in (0,1]$:

$$x'' + \lambda f(x)x' - \lambda g(x) + \lambda\varphi(t)x = \lambda h(t), \quad \lambda \in (0,1]. \tag{2.1}$$

Let

$$\Omega = \left\{x \in C_T : x'' + \lambda f(x)x' - \lambda g(x) + \lambda\varphi(t)x = \lambda h(t), \lambda \in (0,1]; x(t) > 0, \forall t \in [0,T]\right\},$$

and

$$M_0 = \frac{(\int_0^T \varphi_-(s)^2\,ds)^{\frac{1}{2}}}{\int_0^T \varphi_+(s)\,ds}A_0^2 + \frac{A + |\bar{h}|}{\overline{\varphi_+}} + |\varphi_+|_\infty A_0^2 T^{\frac{1}{2}} + A_0 T^{\frac{1}{2}}\left(\int_0 \left|h_-(t)\right|^2 dt\right)^{\frac{1}{4}}, \tag{2.2}$$

where

$$A_0 = \frac{T}{\pi(1-\sigma_1)}\left(\int_0^T \left|h_-(t)\right|^2 dt\right)^{\frac{1}{4}} + \left(\frac{(A + |\bar{h}|)T^{\frac{1}{2}}}{(1-\sigma_1)\overline{\varphi_+}}\right)^{\frac{1}{2}},$$

$A > 0$ is a constant determined by assumption [$H_2$]. Clearly, $M_0$ and $A_0$ are all independent of $(\lambda, x) \in (0,1] \times \Omega$. Let $M > 0$ be determined by assumption [$H_2$], then there is a positive integer $k_0$ such that

$$k_0 M \geq M_0. \tag{2.3}$$

**Lemma 3** *Assume that assumptions* [$H_1$]-[$H_2$] *hold, then there is an integer* $k^* > k_0$ *such that, for each function* $u \in \Omega$, *there is a point* $t_0 \in [0,T]$ *satisfying*

$$u(t_0) \leq k^* M.$$

*Proof* If the conclusion does not hold, then, for each $k > k_0$, there is a function $u_k \in \Omega$ satisfying

$$u_k(t) > kM \quad \text{for all } t \in [0, T]. \tag{2.4}$$

From the definition of $\Omega$, we see

$$u_k'' + \lambda f(u_k)u_k' - \lambda g(u_k) + \lambda\varphi(t)u_k = \lambda h(t), \quad \lambda \in (0,1], \tag{2.5}$$

and by using assumption $[\mathrm{H}_2]$,

$$0 < g\bigl(u_k(t)\bigr) < A, \quad \text{for all } t \in [0, T]. \tag{2.6}$$

By integrating equation (2.5) over the interval $[0, T]$, we have

$$\int_0^T \varphi(t)u_k(t)\,dt = \int_0^T g\bigl(u_k(t)\bigr)\,dt + \int_0^T h(t)\,dt,$$

*i.e.*,

$$\int_0^T \varphi_+(t)u_k(t)\,dt = \int_0^T \varphi_-(t)u_k(t)\,dt + \int_0^T g\bigl(u_k(t)\bigr)\,dt + \int_0^T h(t)\,dt.$$

Since $\varphi_+(t) \geq 0$ and $\varphi_-(t) \geq 0$ for all $t \in [0, T]$, it follows from the integral mean value theorem that there is a point $\xi \in [0, T]$ such that

$$\begin{aligned} u_k(\xi)\int_0^T \varphi_+(t)\,dt &= \int_0^T \varphi_-(s)u_k(s)\,ds + \int_0^T g\bigl(u_k(t)\bigr)\,dt + T\bar{h} \\ &\leq \left(\int_0^T \varphi_-(s)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^T \bigl|u_k(s)\bigr|^2\,ds\right)^{\frac{1}{2}} + \int_0^T g\bigl(u_k(t)\bigr)\,dt + T\bar{h}, \end{aligned}$$

which together with (2.6) yields

$$u_k(\xi) < \frac{(\int_0^T \varphi_-(s)^2\,ds)^{\frac{1}{2}}}{\int_0^T \varphi_+(s)\,ds}\left(\int_0^T u_k(s)^2\,ds\right)^{\frac{1}{2}} + \frac{A + |\bar{h}|}{\overline{\varphi_+}}. \tag{2.7}$$

It follows from $|u_k|_\infty \leq u_k(\xi) + T^{\frac{1}{2}}(\int_0^T |u_k'(s)|^2\,ds)^{\frac{1}{2}}$ that

$$|u_k|_\infty \leq \frac{(\int_0^T \varphi_-(s)^2\,ds)^{\frac{1}{2}}}{\int_0^T \varphi_+(s)\,ds}\left(\int_0^T u_k(s)^2\,ds\right)^{\frac{1}{2}} + \frac{A + \bar{h}}{\overline{\varphi_+}} + T^{\frac{1}{2}}\left(\int_0^T \bigl|u_k'(s)\bigr|^2\,ds\right)^{1/2}. \tag{2.8}$$

On the other hand, by multiplying equation (2.5) with $u_k(t)$, and integrating it over the interval $[0, T]$, we obtain

$$\int_0^T \bigl|u_k'(t)\bigr|^2\,dt = -\lambda\int_0^T g\bigl(u_k(t)\bigr)u_k(t)\,dt + \lambda\int_0^T \varphi(t)u_k^2(t)\,dt - \lambda\int_0^T h(t)u_k(t)\,dt,$$

which together with the fact of $g(x) > 0$ for all $x > 0$ gives

$$\begin{aligned}\int_0^T \left|u_k'(t)\right|^2 dt &< \lambda \int_0^T \varphi_+(t)u_k^2(t)\,dt + \lambda \int_0^T h_-(t)u_k(t)\,dt \\ &\le |\varphi_+|_\infty \int_0^T \left|u_k(t)\right|^2 dt + \left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \left|h_-(t)\right|^2 dt\right)^{\frac{1}{2}},\end{aligned}$$

*i.e.*,

$$\begin{aligned}&\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{1/2} \\ &\quad < |\varphi_+|_\infty^{1/2} \left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{\frac{1}{2}} + \left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{\frac{1}{4}} \left(\int_0^T \left|h_-(t)\right|^2 dt\right)^{\frac{1}{4}}.\end{aligned} \tag{2.9}$$

By using Lemma 2, we have

$$\left(\int_0^T \left|u_k(s)\right|^2 ds\right)^{1/2} \le \frac{T}{\pi}\left(\int_0^T \left|u_k'(s)\right|^2 ds\right)^{1/2} + \sqrt{T}\left|u_k(\xi)\right|.$$

Substituting (2.7) and (2.9) into the above formula,

$$\begin{aligned}&\left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{1/2} \\ &\quad < \frac{T}{\pi}\left[|\varphi_+|_\infty^{1/2}\left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{\frac{1}{2}} + \left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{\frac{1}{4}}\left(\int_0^T \left|h_-(t)\right|^2 dt\right)^{\frac{1}{4}}\right] \\ &\qquad + \frac{T^{\frac{1}{2}}(\int_0^T \varphi_-(s)^2\,ds)^{\frac{1}{2}}}{\int_0^T \varphi_+(s)\,ds}\left(\int_0^T u_k(s)^2\,ds\right)^{\frac{1}{2}} + \frac{(A+\bar{h})T^{\frac{1}{2}}}{\overline{\varphi_+}} \\ &\quad = \sigma_1\left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{\frac{1}{2}} + \frac{T}{\pi}\left(\int_0^T \left|h_-(t)\right|^2 dt\right)^{\frac{1}{4}}\left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{\frac{1}{4}} + \frac{(A+|\bar{h}|)T^{\frac{1}{2}}}{\overline{\varphi_+}},\end{aligned}$$

where

$$\sigma_1 = \frac{T}{\pi}|\varphi_+|_\infty^{1/2} + \frac{T^{\frac{1}{2}}(\int_0^T \varphi_-(s)^2\,ds)^{\frac{1}{2}}}{\int_0^T \varphi_+(s)\,ds} \in (0,1),$$

which is determined by assumption $[H_1]$. This gives

$$\begin{aligned}&\left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{1/2} \\ &\quad \le \frac{T}{\pi(1-\sigma_1)}\left(\int_0^T \left|h_-(t)\right|^2 dt\right)^{\frac{1}{4}}\left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{\frac{1}{4}} + \frac{(A+|\bar{h}|)T^{\frac{1}{2}}}{(1-\sigma_1)\overline{\varphi_+}},\end{aligned} \tag{2.10}$$

*i.e.*,

$$\left(\int_0^T \left|u_k(t)\right|^2 dt\right)^{\frac{1}{4}} \le A_0, \tag{2.11}$$

where

$$A_0 = \frac{T}{\pi(1-\sigma_1)}\left(\int_0^T \left|h_-(t)\right|^2 dt\right)^{\frac{1}{4}} + \left(\frac{(A+|\bar{h}|)T^{\frac{1}{2}}}{(1-\sigma_1)\overline{\varphi_+}}\right)^{\frac{1}{2}}.$$

It follows from (2.9) that

$$\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{1/2} < |\varphi_+|_\infty A_0^2 + A_0\left(\int_0^T \left|h_-(t)\right|^2 dt\right)^{\frac{1}{4}}. \tag{2.12}$$

Substituting (2.11)-(2.12) into (2.8), we have

$$|u_k|_\infty < \frac{(\int_0^T \varphi_-(s)^2\,ds)^{\frac{1}{2}}}{\int_0^T \varphi_+(s)\,ds}A_0^2 + \frac{A+|\bar{h}|}{\overline{\varphi_+}} + |\varphi_+|_\infty A_0^2 T^{\frac{1}{2}} + A_0 T^{\frac{1}{2}}\left(\int_0^T \left|h_-(t)\right|^2 dt\right)^{\frac{1}{4}},$$

which together with (2.2) yields

$$u_k(t) < M_0 \quad \text{for all } t \in [0,T]. \tag{2.13}$$

By the definition of $k_0$, we see from (2.3) that (2.13) contradicts (2.4). This contradiction implies that the conclusion of Lemma 3 is true. □

## 3 Main results

**Theorem 1** *Assume that* $[H_1]$-$[H_3]$ *hold. Then equation* (1.1) *has at leat one positive* $T$*-periodic solution.*

*Proof* Firstly, we will show that there exist $M_1, M_2$ with $M_1 > k^*M$ and $M_2 > 0$ such that each positive $T$-periodic solution $u(t)$ of equation (2.1) satisfies the inequalities

$$u(t) < M_1, \qquad \left|u'(t)\right| < M_2, \quad \text{for all } t \in [0,T]. \tag{3.1}$$

In fact, if $u$ is an arbitrary positive $T$-periodic solution of equation (2.1), then

$$u'' + \lambda f(u)u' - \lambda g(u) + \lambda\varphi(t)u = \lambda h(t), \quad \lambda \in (0,1]. \tag{3.2}$$

This implies $u \in \Omega$. So by using Lemma 3 that there is a point $t_0 \in [0,T]$ such that

$$u(t_0) \le k^*M, \tag{3.3}$$

and then

$$|u|_\infty \le k^*M + T^{1/2}\left(\int_0^T \left|u'(s)\right|^2 ds\right)^{1/2}. \tag{3.4}$$

Integrating (3.2) over the interval $[0,T]$, we have

$$-\int_0^T g\big(u(t)\big)\,dt + \int_0^T \varphi(t)u(t)\,dt = \int_0^T h(t)\,dt. \tag{3.5}$$

Since $g(x) \to +\infty$ as $x \to 0^+$, we see from (3.5) that there is a point $t_1 \in [0, T]$ such that

$$u(t_1) \geq \gamma, \tag{3.6}$$

where $\gamma < k^*M$ is a positive constant, which is independent of $\lambda \in (0,1]$. Similar to the proof of (2.9), we have

$$\begin{aligned} &\left(\int_0^T |u'(t)|^2\,dt\right)^{1/2} \\ &\quad < |\varphi_+|_\infty^{1/2}\left(\int_0^T |u(t)|^2\,dt\right)^{\frac{1}{2}} + \left(\int_0^T |u(t)|^2\,dt\right)^{\frac{1}{4}}\left(\int_0^T |h_-(t)|^2\,dt\right)^{\frac{1}{4}}. \end{aligned} \tag{3.7}$$

By using Lemma 2, we have

$$\left(\int_0^T |u(s)|^2\,ds\right)^{1/2} \leq \frac{T}{\pi}\left(\int_0^T |u'(s)|^2\,ds\right)^{1/2} + \sqrt{T}|u(t_0)|, \tag{3.8}$$

where $t_0$ is determined in (3.3). Substituting (3.7) into (3.8), we have

$$\begin{aligned} &\left(\int_0^T |u(t)|^2\,dt\right)^{1/2} \\ &\quad < \frac{T}{\pi}\left[|\varphi_+|_\infty^{1/2}\left(\int_0^T |u(t)|^2\,dt\right)^{\frac{1}{2}} + \left(\int_0^T |u(t)|^2\,dt\right)^{\frac{1}{4}}\left(\int_0^T |h_-(t)|^2\,dt\right)^{\frac{1}{4}}\right] \\ &\qquad + T^{\frac{1}{2}}k^*M \\ &\quad = \frac{T}{\pi}|\varphi_+|_\infty^{1/2}\left(\int_0^T |u(t)|^2\,dt\right)^{\frac{1}{2}} + \frac{T}{\pi}\left(\int_0^T |h_-(t)|^2\,dt\right)^{\frac{1}{4}}\left(\int_0^T |u(t)|^2\,dt\right)^{\frac{1}{4}} + T^{\frac{1}{2}}k^*M, \end{aligned}$$

which results in

$$\begin{aligned} &\left(1 - \frac{T}{\pi}|\varphi_+|_\infty^{1/2}\right)\left(\int_0^T |u(t)|^2\,dt\right)^{1/2} \\ &\quad < \frac{T}{\pi}\left(\int_0^T |h_-(t)|^2\,dt\right)^{\frac{1}{4}}\left(\int_0^T |u(t)|^2\,dt\right)^{\frac{1}{4}} + T^{\frac{1}{2}}k^*M. \end{aligned} \tag{3.9}$$

Since $\frac{T}{\pi}|\varphi_+|_\infty^{1/2} < \sigma_1 \in (0,1)$, it follows from (3.9) that there is a constant $\rho > 0$, which is independent of $\lambda \in (0,1]$, such that

$$\left(\int_0^T |u(t)|^2\,dt\right)^{1/2} < \rho,$$

and then by (3.7), we have

$$\left(\int_0^T |u'(t)|^2\,dt\right)^{1/2} < |\varphi_+|_\infty^{1/2}\rho + \left(\int_0^T |h_-(t)|^2\,dt\right)^{\frac{1}{4}}\rho^{1/2}.$$

It follows from (3.4) that

$$|u|_\infty < k^*M + T^{1/2}|\varphi_+|_\infty^{1/2}\rho + (T\rho)^{1/2}\left(\int_0^T \left|h_-(t)\right|^2 dt\right)^{\frac{1}{4}} := M_1,$$

*i.e.*,

$$u(t) < M_1, \quad \text{for all } t \in [0, T]. \tag{3.10}$$

Now, if $u$ attains its maximum over $[0, T]$ at $t_2 \in [0, T]$, then $u'(t_2) = 0$ and we deduce from (3.2) that

$$u'(t) = \lambda \int_{t_2}^{t} \left[-f(u)u' + g(u) - \varphi(t)u + h(t)\right] dt$$

for all $t \in [t_2, t_2 + T]$. Thus, if $F' = f$, then

$$\begin{aligned}
\left|u'(t)\right| &\le \lambda\left|F\big(u(t)\big) - F\big(u(t_2)\big)\right| + \lambda \int_{t_2}^{t_2+T} g\big(u(t)\big)\, dt \\
&\quad + \lambda \int_{t_2}^{t_2+T} \left|\varphi(s)\right|u(s)\, ds + \lambda \int_{t_2}^{t_2+T} \left|h(s)\right| ds \\
&\le 2\lambda \max_{0\le u\le M_1} \left|F(u)\right| + \lambda \int_0^T g\big(u(s)\big)\, ds + \lambda T\overline{|\varphi|}|u|_\infty + \lambda T\overline{|h|}.
\end{aligned} \tag{3.11}$$

From (3.2), we see that

$$\begin{aligned}
\int_0^T g\big(u(s)\big)\, ds &= \int_0^T \varphi(t)u(t)\, dt - T\bar{h} \\
&\le T\overline{\varphi_+}|u|_\infty + T\overline{h_-}.
\end{aligned}$$

It follows from (3.10) and (3.11) that

$$\begin{aligned}
\left|u'(t)\right| &\le 2\lambda\left(\max_{0\le u\le M_1} \left|F(u)\right| + T\overline{|\varphi|}|u|_\infty + T\overline{|h|}\right) \\
&< 2\lambda\left(\max_{0\le u\le M_1} \left|F(u)\right| + M_1 T\overline{|\varphi|} + T\overline{|h|}\right) \\
&:= \lambda M_2, \quad t \in [0, T],
\end{aligned} \tag{3.12}$$

and then

$$\left|u'(t)\right| < M_2, \quad \text{for all } t \in [0, T]. \tag{3.13}$$

Equations (3.10) and (3.13) imply that (3.1) holds.

Below, we will show that there exists a constant $\gamma_0 \in (0, \gamma)$, such that each positive $T$-periodic solution of equation (2.1) satisfies

$$u(t) > \gamma_0 \quad \text{for all } t \in [0, T]. \tag{3.14}$$

Suppose that $u(t)$ is an arbitrary positive $T$-periodic solution of equation (2.1), then

$$u'' + \lambda f(u)u' - \lambda g(u) + \lambda\varphi(t)u = \lambda h(t), \quad \lambda \in (0,1]. \tag{3.15}$$

Let $t_1$ be determined in (3.6). Multiplying (3.15) by $u'(t)$ and integrating it over the interval $[t_1, t]$ (or $[t, t_1]$), we get

$$\frac{|u'(t)|^2}{2} - \frac{|u'(t_1)|^2}{2} + \lambda\int_{t_1}^{t} f(u)\bigl(u'\bigr)^2\,dt = \lambda\int_{t_1}^{t} g(u)u'\,dt - \lambda\int_{t_1}^{t}\varphi(t)uu'\,dt + \lambda\int_{t_1}^{t} h(t)u'\,dt,$$

which yields the estimate

$$\begin{aligned}\lambda\left|\int_{u(t)}^{u(t_1)} g(s)\,ds\right| \le{}& \frac{|u'(t)|^2}{2} + \frac{|u'(t_1)|^2}{2} + \lambda\int_0^T \bigl|f(u)\bigr|\bigl(u'\bigr)^2\,dt \\ &+ \lambda\int_0^T \bigl|\varphi(t)uu'\bigr|\,dt + \lambda\int_0^T \bigl|h(t)u'\bigr|\,dt.\end{aligned}$$

From (3.10) and (3.12), we get

$$\lambda\left|\int_{u(t)}^{u(t_1)} g(s)\,ds\right| \le \lambda M_2^2 + \lambda\max_{0\le u\le M_1}\bigl|f(u)\bigr|TM_2^2 + \lambda M_1M_2T\overline{|\varphi|} + \lambda M_2T\overline{|h|},$$

which gives

$$\left|\int_{u(t)}^{u(t_1)} g(s)\,ds\right| \le M_3, \quad \text{for all } t \in [t_1, t_1 + T], \tag{3.16}$$

with

$$M_3 = M_2^2 + \max_{0\le u\le M_1}\bigl|f(u)\bigr|TM_2^2 + M_1M_2T\overline{|\varphi|} + M_2T\overline{|h|}.$$

From $[\mathrm{H}_3]$ there exists $\gamma_0 \in (0, \gamma)$ such that

$$\int_{\eta}^{\gamma} g(u)\,du > M_3, \quad \text{for all } \eta \in (0, \gamma_0]. \tag{3.17}$$

Therefore, if there is a $t^* \in [t_1, t_1 + T]$ such that $u(t^*) \le \gamma_0$, then from (3.17) we get

$$\int_{u(t^*)}^{\gamma} g(s)\,ds > M_3,$$

which contradicts (3.16). This contradiction gives that $u(t) > \gamma_0$ for all $t \in [0, T]$. So (3.14) holds. Let $m_0 = \min\{D_1, \gamma_0\}$ and $m_1 \in (M_1 + D_2, +\infty)$ be two constants, then from (3.1) and (3.14), we see that each possible positive $T$-periodic solution $u$ to equation (2.1) satisfies

$$m_0 < u(t) < m_1, \quad \bigl|u'(t)\bigr| < M_2.$$

This implies that condition 1 and condition 2 of Lemma 1 are satisfied. Also, we can deduce from Remark 1 that

$$g(c) - \bar{\varphi}c + \bar{h} > 0, \quad \text{for } c \in (0, m_0]$$

and

$$g(c) - \bar{\varphi}c + \bar{h} < 0, \quad \text{for } c \in [m_1, +\infty),$$

which results in

$$\big(g(m_0) - \bar{\varphi}m_0 + \bar{h}\big)\big(g(m_1) - \bar{\varphi}m_1 + \bar{h}\big) < 0.$$

So condition 3 of Lemma 1 holds. By using Lemma 1, we see that equation (1.1) has at least one positive $T$-periodic solution. The proof is complete. □

Let us consider the equation

$$x'' + f(x)x' - \frac{1}{x^\gamma} + \varphi(t)x = h(t), \tag{3.18}$$

where $f : [0, +\infty) \to R$ is an arbitrary continuous function, $\varphi, h : R \to R$ are $T$-periodic functions with $h \in L^1([0, T], R)$ and $\varphi \in C([0, T], R)$, and the sign of the function $\varphi$ is allowed to change for $t \in [0, T]$, $\gamma \geq 1$ is a constant. Corresponding to equation (1.1), $g(x) = \frac{1}{x^\gamma}$. For this case, $g(x) \to +\infty$ as $x \to 0^+$, and assumptions [$H_2$]-[$H_3$] are satisfied. Thus, by using Theorem 1, we have the following results.

**Corollary 1** *Assume that the function $\varphi(t)$ satisfies the following conditions*:

$$\int_0^T \varphi_+(s)\,ds > 0, \qquad \sigma := \frac{\int_0^T \varphi_-(s)\,ds}{\int_0^T \varphi_+(s)\,ds} \in [0, 1)$$

*and*

$$\sigma_1 := \frac{T}{\pi}|\varphi_+|_\infty^{1/2} + \frac{T^{\frac{1}{2}}(\int_0^T \varphi_-(s)^2\,ds)^{\frac{1}{2}}}{\int_0^T \varphi_+(s)\,ds} \in (0, 1).$$

*Then*, *equation* (3.18) *possesses at least one positive $T$-periodic solution*.

**Remark 2** Corresponding to equation (1.4) and equation (1.5), the function $g(t, x)$ associated to equation (3.18) can be regarded as

$$g(t, u) = -\frac{1}{u^\gamma} + \varphi(t)u - h(t), \quad (t, u) \in [0, T] \times (0, +\infty). \tag{3.19}$$

For the case of $\varphi(t) \geq 0$ for all $t \in [0, T]$, we see that if $x$ is a positive $T$-periodic continuous function satisfying $\int_0^T g(t, x(t))\,dt = 0$, then

$$\int_0^T \frac{1}{x^\gamma(t)}\,dt = \int_0^T \varphi(t)x(t)\,dt - \int_0^T h(t)\,dt. \tag{3.20}$$

By applying the integral mean value theorem to the term $\int_0^T \varphi(t)x(t)\,dt$ in equation (3.20), one can easily verify that $g(t, u)$ determined in (3.19) satisfies the balance condition ($h_1$). However, if the sign of the function $\varphi(t)$ is changeable for $t \in [0, T]$, then it is unclear from

(3.20) whether the balance condition ($h_1$) is satisfied. For this case, the main results of [18, 19] cannot be applied to equation (3.18).

**Corollary 2** *Assume that the function* $\varphi(t)$ *satisfies* $\varphi(t) \geq 0$ *for all* $t \in [0, T]$ *with* $\int_0^T \varphi(s)\,ds > 0$, *and*

$$|\varphi|_\infty < \left(\frac{\pi}{T}\right)^2.$$

*Then, equation* (3.18) *possesses at least one positive* $T$*-periodic solution.*

**Example 1** Consider the following equation:

$$x''(t) + f\bigl(x(t)\bigr)x'(t) - \frac{1}{x^2(t)} + a(1 + 2\sin 2t)x(t) = \cos 2t, \tag{3.21}$$

where $f$ is an arbitrary continuous function, $a \in (0, +\infty)$ is a constant. Corresponding to equation (3.18), we have $\gamma = 2$, $\varphi(t) = a(1 + 2\sin 2t)$ and $h(t) = \cos 2t$, $T = \pi$. By simply calculating, we can verify that

$$\int_0^T \varphi_+(t)\,dt = \left(\frac{2\pi}{3} + \frac{3}{2}\right)a, \qquad \int_0^T \varphi_-(t)\,dt = \left(\frac{3}{2} - \frac{\pi}{3}\right)a,$$
$$\int_0^T \bigl(\varphi_-(t)\bigr)^2\,dt = \frac{3\pi a}{2},$$

and then

$$\sigma := \frac{\int_0^T \varphi_-(s)\,ds}{\int_0^T \varphi_+(s)\,ds} = \frac{9 - 2\pi}{4\pi + 9} \in (0, 1)$$

and

$$\sigma_1 := \frac{T}{\pi}|\varphi_+|_\infty^{1/2} + \frac{T^{\frac{1}{2}}(\int_0^T \varphi_-(s)^2\,ds)^{\frac{1}{2}}}{\int_0^T \varphi_+(s)\,ds} = \sqrt{3a} + \frac{3\pi\sqrt{6}}{4\pi + 9}.$$

Thus, if $0 < a < \frac{1}{3}(\frac{4\pi+9-3\pi\sqrt{6}}{4\pi+9})^2$, then $\sigma_1 \in (0, 1)$. By using Corollary 1, we see that equation (3.21) has at least one positive $\pi$-periodic solution.

**Remark 3** Since the sign of $\varphi(t) = 1 + 2\sin t$ is changed for $t \in [0, T]$, whether the right inequality of (1.6) in the balance condition ($h_1$) is satisfied remains unclear. So the conclusion of the example cannot be obtained by using the main results in [18, 19].

**Competing interests**
The author declares to have no competing interests.

**Acknowledgements**
The work is sponsored by the National Natural Science Foundation of China (No. 11271197).

Received: 19 August 2016 Accepted: 22 December 2016 Published online: 07 February 2017

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