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Analysis and construction of a family of refinable functions based on generalized Bernstein polynomials

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Abstract

In this paper, we construct a new family of refinable functions from generalized Bernstein polynomials, which include pseudo-splines of Type II. A comprehensive analysis of the refinable functions is carried out. We then prove the convergence of cascade algorithms associated with the new masks and construct Riesz wavelets whose dilation and translation form a Riesz basis for $L_2(\mathbb{R})$. Stability of the subdivision schemes, regularity and approximation orders are obtained. We also illustrate the symmetry of the corresponding refinable functions.

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Keywords: refinable functions; generalized Bernstein polynomials; cascade algorithms; Riesz wavelet bases

1 Introduction

During the last decades, the study of refinable functions has attracted considerable attention. This will be followed by a description of the development of refinable functions. Initially, B-splines appeared as a special family of refinable functions, which satisfied the scaling equation in [1]. After that, pseudo-splines of Type I were introduced by Daubechies *et al.* [2] and Selesnick [3], along with the appearance of pseudo-splines of Type II in [4]. Especially, the properties of them, such as stability, regularity, approximation orders etc., were fully analyzed in [2, 4, 5]. Later, there appeared other refinable functions that had been discovered, for instance, dual pseudo-splines [6, 7], pseudo-box splines [8], Battle-Lemarie refinable functions [9–11], Butterworth refinable functions [12, 13], pseudo-Butterworth refinable functions [14], generalized pseudo-Butterworth refinable functions [15], and so on. Notice that almost all contributions above focus on the extensions based upon pseudo-splines. However, we discover that pseudo-Butterworth refinable functions, as an extension of pseudo-splines, lack compact support, although they have exponential decay to compensate the lack. It is natural to consider a new extension of pseudo-splines, which has compact support and exponential decay, in particular, masks of new refinable functions derived from generalized Bernstein polynomials [16] by substitution and summation.

We start with generalized Bernstein polynomials, defined as

$$S_k^{(n)}(t) = \binom{n}{k} \frac{t(t+\alpha)\cdots(t+[k-1]\alpha)(1-t)(1-t+\alpha)\cdots(1-t+[n-k-1]\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+[n-1]\alpha)}, \tag{1}$$

where $\alpha \geq 0$. New masks with order (m, l, α) for given nonnegative integers m, l, satisfying $l < m - 5$, and $0 \leq \alpha < \frac{1}{3(m+l)-7}$, are defined by substituting $t = \sin^2(\frac{\omega}{2})$, $n = m + l$ in (1) and the summation of $l + 1$ terms of them as follows:

$$\tau_0^{m,l,\alpha}(\omega) := \sum_{j=0}^{l} \binom{m+l}{j} \left(\prod_{i=0}^{j-1} \left(\sin^2\left(\frac{\omega}{2}\right) + i\alpha \right) \prod_{i=0}^{m+l-j-1} \left(\cos^2\left(\frac{\omega}{2}\right) + i\alpha \right) \right) \Big/ \prod_{i=1}^{m+l-1} (1 + i\alpha). \tag{2}$$

They include almost all masks of pseudo-splines of Type II [4] when $\alpha = 0$. Furthermore, the properties of the new refinable functions corresponding to the masks (2) are addressed. Convergence of cascade algorithms is implemented, which guarantees the existence of refinable functions. At the same time, we construct their Riesz wavelets whose dilation and translation form a Riesz basis for $L_2(R)$. Finally, stability of the subdivision schemes, regularity, approximation orders, and symmetry are analyzed.

The remainder of this paper is organized as follows: Section 2 collects some notations. Section 3 elaborates the convergence of cascade algorithms based on the masks. Section 4 constructs a Riesz basis for $L_2(\mathbb{R})$. Section 5 analyzes the stability of the subdivision schemes. Section 6 shows the regularity and the influences of the parameters m, l, α on the decay rate. Section 7 gives the approximation order of the new refinable functions. Section 8 illustrates the symmetry of the refinable functions.

2 Preliminaries

For the convenience of the reader, we review some definitions and properties as regards refinement masks in this section.

By $L_2(\mathbb{R})$, we denote all the functions $f(x)$ satisfying

$$\|f(x)\|_{L_2(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^2 \, dx \right)^{\frac{1}{2}} < \infty$$

and by $l_2(\mathbb{Z})$ the set of all sequences c defined on $\mathbb{Z}$ such that

$$\|c\|_{l_2(\mathbb{Z})} := \left(\sum_{i \in \mathbb{Z}} |c(i)|^2 \right)^{\frac{1}{2}} < \infty.$$

A compactly supported function $\phi \in L_2(\mathbb{R})$ is refinable if it satisfies the refinement equation

$$\phi = 2 \sum_{k \in \mathbb{Z}} \tau(k) \phi(2 \cdot -k), \tag{3}$$

where τ, called the refinement mask of ϕ, is a finitely supported sequence. The Fourier transform of ϕ is

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}} \phi(t)e^{-i\xi t}\, dt, \quad \xi \in \mathbb{R}.$$

For a given finitely supported sequence c, its corresponding Laurent polynomial is defined by

$$\tilde{c}(z) := \sum_{i\in\mathbb{Z}} c(i)z^i, \quad \text{for } z \in \mathbb{C}\backslash\{0\}.$$

The corresponding trigonometric polynomial is

$$\hat{c}(\xi) = \tilde{c}\bigl(e^{-i\xi}\bigr), \quad \xi \in \mathbb{R}.$$

With the above, the refinement equation (3) can be written in terms of its Fourier transform as

$$\hat{\phi}(\xi) = \hat{\tau}(\xi/2)\hat{\phi}(\xi/2), \quad \xi \in \mathbb{R}. \tag{4}$$

We call $\hat{\tau}$ the refinement mask for convenience, too.

By the iteration of equation (4), the corresponding refinable function ϕ can be written in terms of its Fourier transform as

$$\hat{\phi}(\omega) := \prod_{j=1}^{\infty} \hat{\tau}\bigl(2^{-j}\omega\bigr). \tag{5}$$

Define $\mathbb{T} := \mathbb{R}/[2\pi\mathbb{Z}]$ and define $L_{2,\infty}(\mathbb{R})$ as the subspace of all $f \in L_2(\mathbb{R})$ such that

$$\|f\|_{L_{2,\infty}(\mathbb{R})} := \biggl\| \sum_{k\in Z} \bigl|\hat{f}(\omega + 2\pi k)\bigr|^2 \biggr\|_{L_\infty(\mathbb{T})}^{\frac{1}{2}} < \infty,$$

where $L_\infty(\mathbb{T})$ denotes the space of all 2π-periodic measurable functions with finite essential upper bound. The notation $\mathbb{T}$ and the space $L_{2,\infty}(\mathbb{R})$ were introduced in [17] and [18], respectively.

A system $X(\psi) = \{\psi_{n,k} = 2^{n/2}\psi(2^n \cdot -k) : n, k \in \mathbb{R}\}$ is a Riesz basis if there exist $1 < C_1 \le C_2 < \infty$ such that, for all sequences $c \in l_2(\mathbb{Z}^2)$,

$$C_1\|c\|_{l_2(\mathbb{Z}^2)} \le \biggl\| \sum_{(n,k)\in\mathbb{Z}^2} c[n,k]\psi_{n,k} \biggr\|_{L_2(\mathbb{R})} \le C_2\|c\|_{l_2(\mathbb{Z}^2)}$$

holds and the span of $\{\psi_{n,k} : n, k \in \mathbb{Z}\}$ is dense for $L_2(\mathbb{R})$. The function ψ is called a Riesz wavelet if the $X(\psi)$ form a Riesz basis for $L_2(\mathbb{R})$, which is also called a Riesz wavelet system.

A function $\phi \in L_2(\mathbb{R})$ is called stable if there exist two positive constants A_1, B_1, such that for any sequence $c \in l_2(\mathbb{Z})$

$$A_1\|c\|_{l^2(\mathbb{Z})} \le \biggl\| \sum_{i\in\mathbb{Z}} c(i)\phi(\cdot - i) \biggr\|_{L_2(\mathbb{R})} \le B_1\|c\|_{l^2(\mathbb{Z})}.$$

It is well known that this condition is equivalent to the existence of two other positive constants A_2, B_2, such that

$$A_2 \leq \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\xi + 2\pi k)\right|^2 \leq B_2,$$

for almost all $\xi \in \mathbb{R}$ [19]. The upper bound always holds if ϕ has compact support and the lower bound is equivalent to

$$\left(\hat{\phi}(\xi + 2\pi k)\right)_{k\in\mathbb{Z}} \neq \mathbf{0}, \tag{6}$$

for all $\xi \in \mathbb{R}$, where $\mathbf{0}$ denotes the zero sequence. In particular, we consider the stability of the compactly supported centered B-splines,

$$\widehat{B}_m(\xi) = e^{-im\frac{\xi}{2}} \left(\frac{\sin(\xi/2)}{\xi/2}\right)^m.$$

Since it is obvious that there is $C_1 > 0$, such that $|\widehat{B}_m(\xi)| > \sqrt{C_1}$ for all $\xi \in [-\pi, \pi]$, we have

$$\left[\widehat{B}_m(\xi), \widehat{B}_m(\xi)\right](\xi) = \left|\widehat{B}_m(\xi)\right|^2 + \sum_{k\in\mathbb{Z}} \left|\widehat{B}_m(\xi + 2k\pi)\right|^2 \geq \left|\widehat{B}_m(\xi)\right|^2 \geq C_1.$$

Hence all B-splines are stable.

We use

$$P_n : f \mapsto \sum_{k\in\mathbb{Z}} \langle f, \phi_{n,k}\rangle \phi_{n,k}$$

to approximate $f \in L_2(\mathbb{R})$. A function ϕ satisfies the Strang-Fix condition of order m if

$$\hat{\phi}(0) \neq 0, \qquad D^j\hat{\phi}(2\pi k) = 0, \quad \forall k \in \mathbb{Z} \setminus \{0\}, \forall |j| < m.$$

Under certain conditions on ϕ (*e.g.*, if it is compactly supported and $\hat{\phi}(0) = 1$), the Strang-Fix condition is equal to the requirement that $\hat{\tau_0}$ has a zero of order m at each of the points in $\{0, \pi\} \setminus 0$. In [2], if ϕ satisfies the Strang-Fix condition of order m and the corresponding mark $\hat{\tau_0}$ satisfies $1 - |\hat{\tau_0}(\cdot)|^2 = O(|\cdot|^{m_1})$ at the origin, then the approximation order is $\min\{m, m_1\}$.

In the following, we will adopt some of the notations from [18]. The transition operator $T_{\hat{a}}$ for 2π-periodic functions $\hat{a}$ and f can be defined as

$$[T_{\hat{a}}f](w) := \left|\hat{a}(\omega/2)\right|^2 f(\omega/2) + \left|\hat{a}(\omega/2 + \pi)\right|^2 f(\omega/2 + \pi), \quad \omega \in \mathbb{R}.$$

For $\tau \in \mathbb{R}$, a quantity is defined by

$$\rho_\tau(\hat{a}, \infty) := \limsup_{n\to\infty} \left\| T_{\hat{a}}^n \left(\left| \sin\left(\frac{\omega}{2}\right) \right|^\tau \right) \right\|_{L_\infty(\mathbb{T})}^{1/n}.$$

The notation $\rho(\hat{a})$ is defined by

$$\rho(\hat{a}) := \inf\left\{\rho_\tau(\hat{a}, \infty) : \left|\hat{a}(\omega + \pi)\right|^2 \left|\sin(\omega/2)\right|^\tau \in L_\infty(\mathbb{T}) \text{ and } \tau \geq 0\right\}.$$

A function f belongs to the Hölder class $C^\beta(\mathbb{T})$ with $\beta > 0$, if f is a 2π-periodic continuous function such that f is n times continuously differentiable and there exists a positive number C satisfying

$$\left|f^{(n)}(x) - f^{(n)}(y)\right| \leq C|x-y|^{\beta-n}$$

for all $x, y \in \mathbb{T}$, where n is the largest integer such that $n \leq \beta$.

3 Convergence of cascade algorithms based on the masks

In this section, a demonstration of the convergence of cascade algorithms in the space $L_{2,\infty}(\mathbb{R})$ is given. For notational simplicity, we will introduce the following three definitions:

$$\begin{aligned}
B_j^{m,l,\alpha}(\omega) &:= \left(\prod_{i=0}^{j-1}\left(\sin^2\left(\frac{\omega}{2}\right)+i\alpha\right)\prod_{i=1}^{m+l-j-1}\left(\cos^2\left(\frac{\omega}{2}\right)+i\alpha\right)\right)\Big/\prod_{i=1}^{m+l-1}(1+i\alpha), \\
G_j^{m,l,\alpha}(\omega) &:= \binom{m+1}{j}\left(\prod_{i=0}^{j-1}\left(\sin^2\left(\frac{\omega}{2}\right)+i\alpha\right)\prod_{i=1}^{m+l-j-1}\left(\cos^2\left(\frac{\omega}{2}\right)+i\alpha\right)\right)\Big/\prod_{i=1}^{m+l-1}(1+i\alpha), \\
T_0^{m,l,\alpha}(\omega) &:= \sum_{j=0}^{l}\binom{m+l}{j}\left(\prod_{i=0}^{j-1}\left(\sin^2\left(\frac{\omega}{2}\right)+i\alpha\right)\prod_{i=1}^{m+l-j-1}\left(\cos^2\left(\frac{\omega}{2}\right)+i\alpha\right)\right) \\
&\qquad \Big/\prod_{i=1}^{m+l-1}(1+i\alpha).
\end{aligned}$$

We will provide three lemmas about the relations of the quantities $\rho_\tau(\hat{a},\infty)$ associated with masks and a condition of the convergence of cascade algorithms which are necessary for the following theorem.

Lemma 1 ([18], Theorem 4.1) *Let $\hat{a}$ be a 2π-periodic measurable function such that $|\hat{a}|^2 \in C^\beta(\mathbb{T})$ with $|\hat{a}|^2(0) \neq 0$ and $\beta > 0$. If $|\hat{a}(\omega)|^2 = |1+e^{-i\omega}|^{2\tau}|\hat{A}(\omega)|^2$ a.e. $\omega \in \mathbb{R}$ for some $\tau \geq 0$ such that $\hat{A}(\omega) \in L_\infty(\mathbb{T})$, then*

$$\rho_{2\tau}(\hat{a},\infty) = \inf_{n\in\mathbb{N}}\left\|T_{\hat{a}}^n 1\right\|_{L_\infty(\mathbb{T})}^{\frac{1}{n}} = \lim_{n\to\infty}\left\|T_{\hat{a}}^n 1\right\|_{L_\infty(\mathbb{T})}^{\frac{1}{n}} = \rho_0(\hat{A},\infty).$$

Lemma 2 ([18], Theorem 4.3) *Let $\hat{a}$ and $\hat{c}$ be 2π-periodic measurable functions such that*

$$\left|\hat{a}(\omega)\right| \leq \left|\hat{c}(\omega)\right|$$

for almost every $\omega \in \mathbb{R}$. Then

$$\rho_\tau(\hat{a},\infty) \leq \rho_\tau(\hat{c},\infty), \quad \tau \in \mathbb{R}.$$

Lemma 3 ([18], Theorem 2.1) *Let $\hat{a} \in C^\beta(\mathbb{T})$ with $\hat{a}(0) = 1$ and $\beta > 0$. If $\rho(\hat{a}) < 1$, then the cascade algorithm associated with the mask $\hat{a}$ converges in the space $L_{2,\infty}(\mathbb{R})$.*

A useful condition of proving the convergence of cascade algorithms is described in the following lemma.

Lemma 4 *For two positive integers l, m, $l < m-5$, if*

$$0 \le \alpha < \frac{1}{3(m+l)-7}, \tag{7}$$

then

$$\max_{\omega\in\mathbb{T}} B_j(\omega) \le \left(\frac{1}{2}\right)^{m+l-1}, \quad j = 1,2,\dots,l. \tag{8}$$

Proof For $j = 1,2,\dots,l$, it is obvious that

$$B_j(\omega) = \frac{\cos^2(\frac{\omega}{2}) + (m+l-1-j)\alpha}{\sin^2(\frac{\omega}{2}) + j\alpha} B_{j+1}(\omega).$$

We claim that

$$\frac{B_j(\omega)}{B_{j+1}(\omega)} = \frac{\cos^2(\frac{\omega}{2}) + (m+l-1-j)\alpha}{\sin^2(\frac{\omega}{2}) + j\alpha} > 1. \tag{9}$$

Since $l < m-5$, for $j = 1,2,\dots,l$, we have

$$j < m+l-1-j. \tag{10}$$

There are two cases to consider: Case I: Suppose that $\cos(\omega) \ge 0$. By (7) and (10), it is easy to see that

$$\alpha > 0 > \frac{-\cos(\omega)}{m+l-1-2j}.$$

Then

$$\cos^2\left(\frac{\omega}{2}\right) + (m+l-1-j)\alpha > \sin^2\left(\frac{\omega}{2}\right) + j\alpha. \tag{11}$$

This implies Condition (9). Case II: Suppose that $\cos(\omega) < 0$. In the same way, we get

$$\alpha > \frac{-\cos(\omega)}{m+l-1-2j},$$

for $j = 1,2,\dots,l$. Then (11) holds. This concludes the claim (9). By using (7), one gets

$$\bigl(4(m+l-2) - (m+l-1)\bigr)\alpha < \frac{4(m+l-2) - (m+l-1)}{3(m+l)-7} = 1.$$

Then

$$\frac{(m+l-2)\alpha}{(1+\alpha)(1+(m+l-1)\alpha)} < \frac{(m+l-2)\alpha}{1+(m+l-1)\alpha} < \frac{1}{4}. \tag{12}$$

Thus

$$\bigl(2(m+l-3) - (m+l-2)\bigr)\alpha < \frac{2(m+l-3) - (m+l-2)}{m+l-4} = 1.$$

Similarly, one has

$$\frac{(m+l-3)\alpha}{1+(m+l-2)\alpha} < \frac{1}{2}. \tag{13}$$

For any x, notice that

$$\left(\frac{x}{1+(1+x)}\right)' > 0 \tag{14}$$

and $B_1(\omega)$, which is a continuous function on $[-\pi,\pi]$ and is differentiable on $(-\pi,\pi)$, has the maximum value at $\omega=\pi$. The reason as follows: the equation $[B_1(\omega)]'=0$ has three zeros, at $\omega=0,\pm\pi$. Since $[B_1(\omega)]''>0$, $B_1(0)$ is the minimum of $B_1(\omega)$ on $[-\pi,\pi]$. Thus $B_1(\pm\pi)$ is the maximum of $B_1(\omega)$ on $[-\pi,\pi]$. Therefore, applying (9), (12), (13), (14), and

$$\begin{aligned}
B_1(\omega) &= \left(\sin^2\left(\frac{\omega}{2}\right)\prod_{i=1}^{m+l-2}\left(\cos^2\left(\frac{\omega}{2}\right)+i\alpha\right)\right)\Big/\prod_{i=1}^{m+l-1}(1+i\alpha)\\
&\leq \prod_{i=1}^{m+l-2} i\alpha\Big/\prod_{i=1}^{m+l-1}(1+i\alpha)\\
&\leq \prod_{i=1}^{m+l-3}\frac{i\alpha}{1+(i+1)\alpha}\cdot\frac{(m+l-2)\alpha}{(1+\alpha)(1+(m+l-1)\alpha)}\\
&\leq \left(\frac{(m+l-3)\alpha}{1+(m+l-2)\alpha}\right)^{m+l-3}\cdot\frac{1}{4}\\
&= \left(\frac{1}{2}\right)^{m+l-1},
\end{aligned}$$

we get the inequality (8). □

Theorem 1 *If we let* $\tau_0^{m,l,\alpha}(\omega)$ *be the mask* (2), *then the cascade algorithm associated with the mask* $\tau_0^{m,l,\alpha}(\omega)$ *converges in the space* $L_{2,\infty}(\mathbb{R})$.

Proof For $|\cos(\frac{\omega}{2})| = |\frac{1+e^{-i\omega}}{2}|$, one has

$$\begin{aligned}
\left|\tau_0^{m,l,\alpha}(\omega)\right|^2 &= 2^{-4}\left(1+e^{-i\omega}\right)^4\left|T(\omega)\right|^2\\
&= \left(1+e^{-i\omega}\right)^4\left|2^{-2}T(\omega)\right|^2.
\end{aligned}$$

Applying

$$\begin{aligned}
B_0(\omega) &= \prod_{i=0}^{m+l-1}\left(\cos^2\left(\frac{\omega}{2}\right)+i\alpha\right)\Big/\prod_{i=1}^{m+l-1}(1+i\alpha)\\
&\leq \prod_{i=1}^{m+l-1}(1+i\alpha)\Big/\prod_{i=1}^{m+l-1}(1+i\alpha) = 1
\end{aligned}$$

and Lemma 4, we obtain

$$\begin{aligned}\max_{\omega\in\mathbb{T}} 2\left|2^{-2}T_0^{m,l,\alpha}(\omega)\right|^2 &= \max_{\omega\in\mathbb{T}} 2^{-3}\left|B_0(\omega)+\sum_{j=0}^{l}\binom{m+l}{j}B_j(\omega)\right|^2 \\ &< \max_{\omega\in\mathbb{T}} 2^{-3}\left|1+\Bigl(\max_{\omega\in[-\pi,\pi]} B_j(\omega)\Bigr)\sum_{j=1}^{l}\binom{m+l}{j}\right|^2 \\ &< \max_{\omega\in\mathbb{T}} 2^{-3}\left|\left(1+\left(\frac{1}{2}\right)^{m+l-1}(2)^{m+l-1}\right)\right|^2 = \frac{1}{2}. \end{aligned} \tag{15}$$

Bringing Lemma 1 and Lemma 2 together yields

$$\rho\bigl(\tau_0^{m,l,\alpha}(\omega)\bigr) \le \rho_4\bigl(\tau_0^{m,l,\alpha}(\omega),\infty\bigr) = \rho_0\bigl(2^{-2}T(\omega),\infty\bigr) < 1.$$

Thus, by Lemma 3, the cascade algorithm associated with the mask $\tau_0^{m,l,\alpha}(\omega)$ converges in the space $L_{2,\infty}(\mathbb{R})$. □

4 Riesz wavelets

In this section, we shall construct Riesz wavelets based on the masks (2). The following two lemmas analyze recurrence relations of $\tau_0^{m,l,\alpha}(\omega)$, which are useful for the construction of Riesz wavelets.

Lemma 5 *If let $\tau_0^{m,l,\alpha}(\omega)$ be the mask* (2), *then $\tau_0^{m,l,\alpha}(\omega)$ satisfies*

$$\tau_0^{m+1,l,\alpha}(\omega) = \tau_0^{m+1,l-1,\alpha}(\omega) + \left(\frac{\cos^2(\frac{\omega}{2})+m\alpha}{1+(m+l)\alpha}\right)G_l^{m,l,\alpha}(\omega) \tag{16}$$

and

$$\tau_0^{m+1,l,\alpha}(\omega) = \tau_0^{m,l,\alpha}(\omega) - \left(\frac{\sin^2(\frac{\omega}{2})+l\alpha}{1+(m+l)\alpha}\right)G_l^{m,l,\alpha}(\omega). \tag{17}$$

Proof By the summation of $l+1$ terms of generalized Bernstein polynomials satisfying the recurrence [16]

$$S_k^{(n+1)}(t) = \left(\frac{1-t+(n-k)\alpha}{1+n\alpha}\right)S_k^{(n)}(t) + \left(\frac{t+(k-1)\alpha}{1+n\alpha}\right)S_{k-1}^{(n)}(t),$$

we can derive that

$$\begin{aligned}\sum_{j=0}^{l} S_j^{(m+l+1)}(t) &= \sum_{j=0}^{l}\left(\frac{1-t+(m+l-j)\alpha}{1+(m+l)\alpha}\right)S_j^{(m+l)}(t) \\ &\quad + \sum_{j=0}^{l}\left(\frac{t+(j-1)\alpha}{1+(m+l)\alpha}\right)S_{j-1}^{(m+l)}(t). \end{aligned} \tag{18}$$

Substituting $t = \sin^2(\frac{\omega}{2})$ in (18), one gets

$$\begin{aligned}
&\sum_{j=0}^{l} S_j^{(m+l+1)}\left(\sin^2\left(\frac{\omega}{2}\right)\right) \\
&\quad= \sum_{j=0}^{l}\left(\frac{\cos^2(\frac{\omega}{2}) + (m+l-j)\alpha}{1+(m+l)\alpha}\right) S_j^{(m+l)}\left(\sin^2\left(\frac{\omega}{2}\right)\right) \\
&\qquad+ \sum_{j=0}^{l}\left(\frac{\sin^2(\frac{\omega}{2}) + (j-1)\alpha}{1+(m+l)\alpha}\right) S_{j-1}^{(m+l)}\left(\sin^2\left(\frac{\omega}{2}\right)\right) \\
&\quad= \sum_{j=0}^{l-1} S_j^{(m+l)}\left(\sin^2\left(\frac{\omega}{2}\right)\right) + \left(\frac{\cos^2(\frac{\omega}{2}) + m\alpha}{1+(m+l)\alpha}\right) G_l^{m,l,\alpha}(\omega).
\end{aligned} \tag{19}$$

By using (19), one obtains

$$\begin{aligned}
\sum_{j=0}^{l} S_j^{(m+l+1)}\left(\sin^2\left(\frac{\omega}{2}\right)\right) &= \sum_{j=0}^{l-1} S_j^{(m+l)}\left(\sin^2\left(\frac{\omega}{2}\right)\right) + \left(\frac{\cos^2(\frac{\omega}{2}) + m\alpha}{1+(m+l)\alpha}\right) G_l^{m,l,\alpha}(\omega) \\
&\quad+ S_l^{(m+l)}\left(\sin^2\left(\frac{\omega}{2}\right)\right) - S_l^{(m+l)}\left(\sin^2\left(\frac{\omega}{2}\right)\right) \\
&= \sum_{j=0}^{l} S_j^{(m+l)}\left(\sin^2\left(\frac{\omega}{2}\right)\right) - \left(\frac{\sin^2(\frac{\omega}{2}) + l\alpha}{1+(m+l)\alpha}\right) G_l^{m,l,\alpha}(\omega).
\end{aligned}$$

Thus, the conditions (16) and (17) are demonstrated. □

Lemma 6 *If let* $\tau_0^{m,l,\alpha}(\omega)$ *be the mask* (2), *then we derive*

$$\left|\tau_0^{m,l,\alpha}(\omega)\right|^2 + \left|\tau_0^{m,l,\alpha}(\omega+\pi)\right|^2 > \frac{\sqrt{2}}{2}, \quad \omega \in \mathbb{T}. \tag{20}$$

Proof When $l = 1$, one has $m > 6$,

$$\begin{aligned}
\left|\tau_0^{m,1,\alpha}(\omega)\right|^2 &= \Biggl(\binom{m+1}{0}\prod_{i=0}^{m}\left(\cos^2\left(\frac{\omega}{2}\right) + i\alpha\right) \Big/ \prod_{i=1}^{m}(1+i\alpha) \\
&\quad+ \binom{m+1}{1}\sin^2\left(\frac{\omega}{2}\right)\prod_{i=0}^{m-1}\left(\cos^2\left(\frac{\omega}{2}\right) + i\alpha\right) \Big/ \prod_{i=1}^{m}(1+i\alpha)\Biggr)^2 \\
&= \left(\left(m\sin^2\left(\frac{\omega}{2}\right) + 1 + m\alpha\right)\prod_{i=0}^{m-1}\left(\cos^2\left(\frac{\omega}{2}\right) + i\alpha\right) \Big/ \prod_{i=1}^{m}(1+i\alpha)\right)^2,
\end{aligned}$$

and

$$\left|\tau_0^{m,1,\alpha}(\omega+\pi)\right|^2 = \left(\left(m\cos^2\left(\frac{\omega}{2}\right) + 1 + m\alpha\right)\prod_{i=0}^{m-1}\left(\sin^2\left(\frac{\omega}{2}\right) + i\alpha\right) \Big/ \prod_{i=1}^{m}(1+i\alpha)\right)^2.$$

It follows from the above two equalities of $|\tau_0^{m,1,\alpha}(\omega)|^2$ and $|\tau_0^{m,1,\alpha}(\omega+\pi)|^2$ that

$$\begin{aligned}
&\left|\tau_0^{m,1,\alpha}(\omega)\right|^2+\left|\tau_0^{m,1,\alpha}(\omega+\pi)\right|^2\\
&\quad=\left(\frac{m\sin^2(\frac{\omega}{2})+1+m\alpha}{1+m\alpha}\right)^2\left(\prod_{i=0}^{m-1}\left(\cos^2\left(\frac{\omega}{2}\right)+i\alpha\right)\Big/\prod_{i=1}^{m}(1+i\alpha)\right)^2\\
&\qquad+\left(\frac{m\cos^2(\frac{\omega}{2})+1+m\alpha}{1+m\alpha}\right)^2\left(\prod_{i=0}^{m-1}\left(\sin^2\left(\frac{\omega}{2}\right)+i\alpha\right)\Big/\prod_{i=1}^{m}(1+i\alpha)\right)^2\\
&\quad=\left(1+\frac{m\sin^2(\frac{\omega}{2})}{1+m\alpha}\right)^2\left(\prod_{j=0}^{m-1}\left(1-\frac{\sin^2(\frac{\omega}{2})}{1+j\alpha}\right)\right)^2\\
&\qquad+\left(1+\frac{m\cos^2(\frac{\omega}{2})}{1+m\alpha}\right)^2\left(\prod_{j=0}^{m-1}\left(1-\frac{\cos^2(\frac{\omega}{2})}{1+j\alpha}\right)\right)^2\\
&\quad\geq\left(1+\frac{m\sin^2(\frac{\omega}{2})}{1+m\alpha}\right)^2\left(1+\sum_{j=0}^{m-1}\frac{-\sin^2(\frac{\omega}{2})}{1+j\alpha}\right)^2\\
&\qquad+\left(1+\frac{m\cos^2(\frac{\omega}{2})}{1+m\alpha}\right)^2\left(1+\sum_{j=0}^{m-1}\frac{-\cos^2(\frac{\omega}{2})}{1+j\alpha}\right)^2\\
&\quad\geq\left(1+\frac{m\sin^2(\frac{\omega}{2})}{1+m\alpha}\right)^2\left(1-m\sin^2\left(\frac{\omega}{2}\right)\right)^2\\
&\qquad+\left(1+\frac{m\cos^2(\frac{\omega}{2})}{1+m\alpha}\right)^2\left(1-m\cos^2\left(\frac{\omega}{2}\right)\right)^2. \qquad (21)
\end{aligned}$$

The first inequality follows from the general formula of the Bernoulli inequality. Substituting $y=\sin^2(\frac{\omega}{2})$ in (21), let

$$f(y)=\left(1+\frac{ym}{1+m\alpha}\right)^2(1-my)^2+\left(1+\frac{(1-y)m}{1+m\alpha}\right)^2\left(1-(1-y)m\right)^2.$$

We claim that

$$f(y)>\frac{\sqrt{2}}{2}. \qquad (22)$$

First of all, $f'(y)$ is derived as follows:

$$\begin{aligned}
&f'(y)\\
&\quad=2\left(1+\frac{ym}{1+m\alpha}\right)\left(\frac{m}{1+m\alpha}\right)(1-my)^2+2\left(1+\frac{ym}{1+m\alpha}\right)^2(1-my)(-m)\\
&\qquad+2\left(1+\frac{(1-y)m}{1+m\alpha}\right)\left(-\frac{m}{1+m\alpha}\right)\left(1-(1-y)m\right)^2\\
&\qquad+2m\left(1+\frac{(1-y)m}{1+m\alpha}\right)^2\left(1-(1-y)m\right)
\end{aligned}$$

$$= \left\{2\left(1+\frac{ym}{1+m\alpha}\right)(1-my)\left[(1-my)\frac{m}{1+m\alpha}-m\left(1+\frac{ym}{1+m\alpha}\right)\right]\right\}-2$$
$$\times\left(1+\frac{(1-y)m}{1+m\alpha}\right)\bigl(1-m(1-y)\bigr)\left[\bigl(1-m(1-y)\bigr)\frac{m}{1+m\alpha}-m\left(1+\frac{(1-y)m}{1+m\alpha}\right)\right].$$

Additionally,

$$\begin{aligned} f''(y) &= 2\left(\frac{m}{1+m\alpha}\right)(1-my)\left[(1-my)\frac{m}{1+m\alpha}-m\left(1+\frac{ym}{1+m\alpha}\right)\right] \\ &\quad + 2(-m)\left(1+\frac{ym}{1+m\alpha}\right)\left[(1-my)\frac{m}{1+m\alpha}-m\left(1+\frac{ym}{1+m\alpha}\right)\right] \\ &\quad + 2\left(1+\frac{ym}{1+m\alpha}\right)(1-my)\left(-\frac{2m^2}{1+m\alpha}\right) \\ &\quad + 2\left(\frac{m}{1+m\alpha}\right)\bigl(1-m(1-y)\bigr)\left[\bigl(1-m(1-y)\bigr)\frac{m}{1+m\alpha}-m\left(1+\frac{(1-y)m}{1+m\alpha}\right)\right] \\ &\quad + 2(-m)\left(1+\frac{(1-y)m}{1+m\alpha}\right)\left[\bigl(1-m(1-y)\bigr)\frac{m}{1+m\alpha}-m\left(1+\frac{(1-y)m}{1+m\alpha}\right)\right] \\ &\quad + 2\left(1+\frac{(1-y)m}{1+m\alpha}\right)\bigl(1-m(1-y)\bigr)\left(-\frac{2m^2}{1+m\alpha}\right). \end{aligned}$$

Combining $f'(\frac{1}{2})=0$ and $f''(\frac{1}{2})>0$ yields

$$f(y)\geq \min\left\{f(0),f\left(\frac{1}{2}\right),f(1)\right\}.$$

Here,

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \left(1+\frac{\frac{1}{2}m}{1+m\alpha}\right)^2\left(1-\frac{1}{2}m\right)^2+\left(1+\frac{\frac{1}{2}m}{1+m\alpha}\right)^2\left(1-\left(1-\frac{1}{2}m\right)\right)^2 \\ &= 2\left(1+\frac{1}{2/m+2\alpha}\right)^2\left(\frac{1}{2}m-1\right)^2>\frac{\sqrt{2}}{2}. \end{aligned} \tag{23}$$

Let

$$g(x)=2\left(1+\frac{1}{2/x+2\alpha}\right)^2\left(\frac{1}{2}x-1\right)^2.$$

The condition (23) follows from the fact that $g(x)$ is increasing on $[5,+\infty]$. It is obviously true for $f(0)>\frac{\sqrt{2}}{2}$ and $f(1)>\frac{\sqrt{2}}{2}$. This concludes the claim (22). Assume (20) holds when $l=k-1$. Consider the case $l=k$. By using (16) in Lemma 5, we have

$$\begin{aligned} \bigl|\tau_0^{m+1,l,\alpha}(\omega)\bigr|^2 &= \bigl|\tau_0^{m+1,l-1,\alpha}(\omega)\bigr|^2+2\bigl(\tau_0^{m+1,l-1,\alpha}(\omega)\bigr)\left(\frac{\cos^2(\frac{\omega}{2})+m\alpha}{1+(m+l)\alpha}\right) \\ &\quad \times S_l^{(m+l)}\left(\sin^2\left(\frac{\omega}{2}\right)\right) \\ &\quad + \left(\left(\frac{\cos^2(\frac{\omega}{2})+(m-1)\alpha}{1+(m+l-1)\alpha}\right)S_l^{(m+l-1)}\left(\sin^2\left(\frac{\omega}{2}\right)\right)\right)^2. \end{aligned}$$

Adding that

$$\begin{aligned}\left|\tau_0^{m,l,\alpha}(\omega+\pi)\right|^2 &= \left|\tau_0^{m,l-1,\alpha}(\omega+\pi)\right|^2 + 2\left(\tau_0^{m,l-1,\alpha}(\omega+\pi)\right)\left(\frac{\sin^2(\frac{\omega}{2})+(m-1)\alpha}{1+(m+l-1)\alpha}\right)\\ &\quad\times S_l^{(m+l-1)}\left(\cos^2\left(\frac{\omega}{2}\right)\right)\\ &\quad+\left(\left(\frac{\sin^2(\frac{\omega}{2})+(m-1)\alpha}{1+(m+l-1)\alpha}\right)S_l^{(m+l-1)}\left(\cos^2\left(\frac{\omega}{2}\right)\right)\right)^2.\end{aligned}$$

Combining together the above equalities so that

$$\left|\tau_0^{m,l,\alpha}(\omega)\right|^2+\left|\tau_0^{m,l,\alpha}(\omega+\pi)\right|^2=\left|\tau_0^{m,l-1,\alpha}(\omega)\right|^2+\left|\tau_0^{m,l-1,\alpha}(\omega+\pi)\right|^2+f(\omega)+f(\omega+\pi),$$

where

$$\begin{aligned}f(\omega) &= 2\left(\tau_0^{m,l-1,\alpha}(\omega)\right)\left(\frac{\cos^2(\frac{\omega}{2})+(m-1)\alpha}{1+(m+l-1)\alpha}\right)S_l^{(m+l-1)}\left(\sin^2\left(\frac{\omega}{2}\right)\right)\\ &\quad+\left(\left(\frac{\cos^2(\frac{\omega}{2})+(m-1)\alpha}{1+(m+l-1)\alpha}\right)S_l^{(m+l-1)}\left(\sin^2\left(\frac{\omega}{2}\right)\right)\right)^2.\end{aligned}$$

The condition (20) follows from the fact that

$$\left|\tau_0^{m,k-1,\alpha}(\omega)\right|^2+\left|\tau_0^{m,k-1,\alpha}(\omega+\pi)\right|^2>\frac{\sqrt{2}}{2}.$$

This concludes the proof of Lemma 6. □

Conditions for constructing a Riesz wavelet basis are given by the following lemma.

Lemma 7 ([18], Theorem 3.2) *Let $\hat{a}\in C^\beta(\mathbb{T})$ with $\hat{a}(0)=1$ and $\beta>0$. Let ϕ be the refinable function corresponding to the refinement mask $\hat{a}$. Denote $\hat{b}(\omega)=e^{-i\omega}\overline{\hat{a}(\omega+\pi)}$. Define a wavelet $\psi(\hat{2\pi}):=\hat{b}(\omega)\hat{\phi}(\omega)$. Then the shifts of ϕ are stable in $L_2(\mathbb{R})$ and ϕ generates a Riesz wavelet basis in $L_2(\mathbb{R})$ if and only if*

(i) $\hat{b}(0)=0$ *and* $d(\omega):=\hat{a}(\omega)\hat{b}(\omega+\pi)-\overline{\hat{a}(\omega+\pi)}\hat{b}(\omega)\neq 0$ *for all* $\omega\in\mathbb{R}$.

(ii) $\rho(\hat{a})<1$ *and* $\rho(\hat{\hat{a}})<1$, *where* $\hat{\hat{a}}(\omega):=\overline{\hat{b}(\omega+\pi)}/\overline{d(\omega)}$.

Theorem 2 *Let $\phi_{m,l,\alpha}$ be the refinable functions from generalized Bernstein polynomials with the refinement mask* (2). *Define a wavelet $\Psi_{m,l,\alpha}$ such that*

$$\hat{\Psi}_{m,l,\alpha}(2\omega):=e^{-i\omega}\overline{\tau_0^{m,l,\alpha}(\omega+\pi)}\hat{\phi}_{m,l,\alpha}(\omega).$$

Then $\Psi_{m,l,\alpha}$ generates a Riesz wavelet basis in $L_2(\mathbb{R})$.

Proof Note that $\hat{b}(\omega)=e^{-i\omega}\overline{\tau_0^{m,l,\alpha}(\omega+\pi)}$, thus $\hat{b}(0)=\overline{\tau_0^{m,l,\alpha}(\pi)}=0$, and

$$\begin{aligned}d(\omega) &= \tau_0^{m,l,\alpha}(\omega)\hat{b}(\omega+\pi)-\tau_0^{m,l,\alpha}(\omega+\pi)\hat{b}(\omega)\\ &= \tau_0^{m,l,\alpha}(\omega)\left(-e^{-i\omega}\right)\overline{\tau_0^{m,l,\alpha}(\omega+2\pi)}-\tau_0^{m,l,\alpha}(\omega+\pi)e^{-i\omega}\overline{\tau_0^{m,l,\alpha}(\omega+\pi)}\\ &= -e^{-i\omega}\left(\left|\tau_0^{m,l,\alpha}(\omega)\right|^2+\left|\tau_0^{m,l,\alpha}(\omega+\pi)\right|^2\right)\neq 0,\quad \omega\in\mathbb{R}.\end{aligned}$$

Therefore, condition (i) in Lemma 7 holds true. $\rho(\tau_0^{m,l,\alpha}(\omega)) < 1$ has been proved in Theorem 1. Noting that

$$\widetilde{\tau_0^{m,l,\alpha}(\omega)} = \frac{\overline{\hat{b}(\omega+\pi)}}{\overline{d(\omega)}} \frac{\overline{-e^{-i\omega}\overline{\tau_0^{m,l,\alpha}(\omega+2\pi)}}}{\overline{-e^{-i\omega}(|\tau_0^{m,l,\alpha}(\omega)|^2+|\tau_0^{m,l,\alpha}(\omega+\pi)|^2)}} = \frac{\tau_0^{m,l,\alpha}(\omega)}{|\tau_0^{m,l,\alpha}(\omega)|^2+|\tau_0^{m,l,\alpha}(\omega+\pi)|^2},$$

we obtain

$$\bigl|\widetilde{\tau_0^{m,l,\alpha}(\omega)}\bigr|^2 = \left|\frac{\tau_0^{m,l,\alpha}(\omega)}{|\tau_0^{m,l,\alpha}(\omega)|^2+|\tau_0^{m,l,\alpha}(\omega+\pi)|^2}\right|^2 = \bigl|\bigl(1+e^{-i\omega}\bigr)\bigr|^4 \frac{2^{-4}|T_0^{m,l,\alpha}(\omega)|^2}{(|\tau_0^{m,l,\alpha}(\omega)|^2+|\tau_0^{m,l,\alpha}(\omega+\pi)|^2)^2}.$$

Combining (15) and (20) in Lemma 6 yields

$$\max_{\omega\in\mathbb{T}} \frac{2|2^{-2}T_0^{m,l,\alpha}(\omega)|^2}{(|\tau_0^{m,l,\alpha}(\omega)|^2+|\tau_0^{m,l,\alpha}(\omega+\pi)|^2)^2} < \frac{\frac{1}{2}}{(\frac{\sqrt{2}}{2})^2} = 1.$$

By using Lemma 1 and Lemma 2, it is derived that

$$\rho\bigl(\widetilde{\tau_0^{m,l,\alpha}(\omega)}\bigr) \le \rho_4\bigl(\widetilde{\tau_0^{m,l,\alpha}(\omega)},\infty\bigr) = \rho_0\left(\frac{2^{-2}T_0^{m,l,\alpha}(\omega)}{(|\tau_0^{m,l,\alpha}(\omega)|^2+|\tau_0^{m,l,\alpha}(\omega+\pi)|^2)},\infty\right) < 1.$$

Consequently, condition (ii) in Lemma 7 holds true. □

5 Stability of the subdivision schemes

In this section, we will prove the stability of refinable functions based on the masks (2). The following lemma gives the recurrence relations of $T_0^{m,l,\alpha}(\omega)$.

Lemma 8 *If we let $\tau_0^{m,l,\alpha}(\omega)$ be the mask* (2), *then $T_0^{m,l,\alpha}(\omega)$ satisfies*

$$T_0^{m+1,l,\alpha}(\omega) = T_0^{m+1,l-1,\alpha}(\omega) + \left(\frac{\cos^2(\frac{\omega}{2})+m\alpha}{\cos^2(\frac{\pi}{3})(1+(m+l)\alpha)}\right)G_l^{m,l,\alpha}(\omega), \quad \textit{for } |\omega| \neq \pi, \tag{24}$$

and

$$T_0^{m,l,\alpha}(\pm\pi) = \frac{1}{1+(m+l-1)\alpha} T_0^{m,l-1,\alpha}(\pm\pi) + \sum_{j=0}^{l} \binom{m+l-1}{j} B_j^{m,l,\alpha}(\pm\pi). \tag{25}$$

Proof We consider into two cases. Suppose that $|\omega| \neq \pi$. Equation (16) is applied in Lemma 4 so that

$$T_0^{m+1,l,\alpha}(\omega) = T_0^{m+1,l-1,\alpha}(\omega) + \left(\frac{\cos^2(\frac{\omega}{2})+m\alpha}{\cos^2(\frac{\pi}{3})(1+(m+l)\alpha)}\right)G_l^{m,l,\alpha}(\omega).$$

Suppose, on the other hand, that $|\omega| = \pi$; we check the condition (25). Note that

$$\begin{aligned} T_0^{m,l,\alpha}(\pm\pi) &= \sum_{j=0}^{l} \binom{m+l}{j} B_j^{m,l,\alpha}(\pm\pi) \\ &= \binom{m+l}{0} B_0^{m,l,\alpha}(\pm\pi) + \sum_{j=1}^{l}\left(\binom{m+l-1}{j} + \binom{m+l-1}{j-1}\right) B_j^{m,l,\alpha}(\pm\pi) \\ &= \frac{1}{1+(m+l-1)\alpha} T_0^{m,l-1,\alpha}(\pm\pi) + \sum_{j=0}^{l}\binom{m+l-1}{j} B_j^{m,l,\alpha}(\pm\pi). \end{aligned}$$

This establishes the lemma. □

Theorem 3 *Let $\tau_0^{m,l,\alpha}(\omega)$ be the mask* (2), *then the refinable functions $\phi_0^{m,l,\alpha}$ with the masks $\tau_0^{m,l,\alpha}(\omega)$ are stable.*

Proof We claim that the condition (6) holds. Indeed, the refinable functions with the masks $\tau_0^{m,l,\alpha}(\omega)$, which belong to $L_2(\mathbb{R})$ as shown in Theorem 1, have been proved. They are compactly supported for finite refinement masks. In the following, we derive the inequality

$$\begin{aligned} \left|\tau_0^{m,l,\alpha}(\omega)\right| &> \left|\sum_{j=0}^{l}\binom{m+l}{j} \frac{(\sin^2(\frac{\omega}{2}))^j(\cos^2(\frac{\omega}{2}))^{m+l-j}}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)}\right| \\ &= \frac{1}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)} \\ &\quad\times \left|\sum_{j=0}^{l}\binom{m+l}{j}\left(\sin^2\left(\frac{\omega}{2}\right)\right)^j\left(\cos^2\left(\frac{\omega}{2}\right)\right)^{m+l-j}\right| \\ &= \frac{1}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)}\left|\cos^{2m}\left(\frac{\omega}{2}\right)\right| \\ &\quad\times \left|\sum_{j=0}^{l}\binom{m+l}{j}\left(\sin^2\left(\frac{\omega}{2}\right)\right)^j\left(\cos^2\left(\frac{\omega}{2}\right)\right)^{l-j}\right| \\ &> \frac{1}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)}\left|\cos^{2m}\left(\frac{\omega}{2}\right)\right|. \end{aligned} \tag{26}$$

Applying the stability of B-splines yields the existence of two positive constants A_2, B_2, such that

$$A_2 \le \sum_{k\in Z}\left|\widehat{B}_{2m}(\xi+2\pi k)\right|^2 \le B_2, \quad \text{for almost all } \xi\in\mathbb{R},$$

where B_{2m} denotes the B-spline of order $2m$. It implies that $|\widehat{B}_{2m}(\omega)|$ has the finite functional value. Combining (26) with (27) yields

$$\begin{aligned} \left|\widehat{\phi}_0^{m,l,\alpha}(\omega)\right| &= \prod_{j=1}^{\infty}\left|\tau_0^{m,l,\alpha}\left(2^{-j}\omega\right)\right| \\ &> \prod_{j=1}^{\infty}\frac{1}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)}\left|\cos^{2m}\left(2^{-j-1}\omega\right)\right| \end{aligned}$$

$$\begin{aligned}
&= \lim_{M\to+\infty}\prod_{j=1}^{M}\frac{1}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)}\left|\cos^{2m}\left(2^{-j-1}\omega\right)\right| \\
&= \lim_{M\to+\infty}\left(\left(\frac{1}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)}\right)^{M}\right. \\
&\quad\left.\times\prod_{j=1}^{M}\left|\cos^{2m}\left(2^{-j-1}\omega\right)\right|\right) \\
&\geq \lim_{M\to+\infty}\left(\left(\frac{1}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)}\right)^{M}\right. \\
&\quad\left.\times\left|\cos^{2m}\left(2^{-1}\omega\right)\right|\prod_{j=1}^{M}\left|\cos^{2m}\left(2^{-j-1}\omega\right)\right|\right).
\end{aligned}$$

Since

$$0<\frac{1}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)}<1,$$

this leads to

$$\lim_{M\to+\infty}\left(\frac{1}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)}\right)^{M}=0. \tag{27}$$

Moreover,

$$\left|\widehat{\phi}_0^{m,l,\alpha}(\omega)\right|=\left|\widehat{B}_{2m}(\omega)\right|\left|\left(\lim_{M\to+\infty}\frac{1}{(1+\alpha)(1+2\alpha)\cdots(1+[m+l-1]\alpha)}\right)^{M}\right|=0.$$

Thus, there exist two positive constants A_2, B_2, such that

$$A_2\leq\sum_{k\in Z}\left|\widehat{\phi}_0^{m,l,\alpha}(\xi+2\pi k)\right|^2\leq B_2,\quad \text{for almost all } \xi\in\mathbb{R}.$$

This concludes the theorem. □

6 Regularity

This section is devoted to an analysis of the regularity of refinable functions $\phi_0^{m,l,\alpha}$ with the mask $\tau_0^{m,l,\alpha}(\omega)$ defined by (2). Our primary goal is to obtain the lower bound of the regularity exponents $\gamma_0^{m,l,\alpha}$ of refinable functions $\phi_0^{m,l,\alpha}$ by estimating the decay rates $\beta_0^{m,l,\alpha}$ of their Fourier transform. The relation is expressed by

$$\gamma_0^{m,l,\alpha}\geq\beta_0^{m,l,\alpha}-1-\varepsilon,$$

for any small enough $\varepsilon>0$; see [20]. Consequently, $\phi_0^{m,l,\alpha}\in C^{\gamma_0^{m,l,\alpha}}$. Next, we will give an estimate of the decay rates $\beta_0^{m,l,\alpha}$ of the Fourier transform of refinable functions $\phi_0^{m,l,\alpha}$ with the mask $\tau_0^{m,l,\alpha}(\omega)$. By [20, 21], for any stable, compactly supported refinable functions ϕ in $L_2(\mathbb{R})$ with $\widehat{\phi}(0)=1$, the refinement mask τ must satisfy $\tau(0)=1$ and $\tau(\pi)=0$. Thus, τ

can be factorized as

$$\tau(\omega) = \cos^n\left(\frac{\omega}{2}\right)\mathcal{L}(\omega),$$

where n is the maximal multiplicity of the zeros of τ at π and $\mathcal{L}(\omega)$ is a trigonometric polynomial with $\mathcal{L}(0) = 1$. Therefore, one obtains

$$\widehat{\phi}(\omega) = \prod_{j=1}^{\infty}\tau\left(2^{-j}\omega\right) = \prod_{j=1}^{\infty}\cos^n\left(2^{-j}\frac{\omega}{2}\right)\prod_{j=1}^{\infty}\mathcal{L}\left(2^{-j}\omega\right) = \operatorname{sin}c^n\left(\frac{\omega}{2}\right)\prod_{j=1}^{\infty}\mathcal{L}\left(2^{-j}\omega\right),$$

which shows the decay of $|\phi|$ can be characterized by $|\tau|$ as stated in the following theorem.

Theorem 4 ([20], Lemma 7.17) *Let τ be the refinement mask of the refinable function ϕ of the form*

$$\left|\tau(\omega)\right| = \cos^n\left(\frac{\omega}{2}\right)\left|\mathcal{L}(\omega)\right|, \quad \omega \in [-\pi, \pi].$$

If

$$\begin{aligned} &\left|\mathcal{L}(\omega)\right| \le \left|\mathcal{L}\left(\frac{2\pi}{3}\right)\right| \quad \textit{for } |\omega| \le \frac{2\pi}{3}, \\ &\left|\mathcal{L}(\omega)\mathcal{L}(2\omega)\right| \le \left|\mathcal{L}\left(\frac{2\pi}{3}\right)\right|^2 \quad \textit{for } \frac{2\pi}{3} \le |\omega| \le \pi, \end{aligned} \tag{28}$$

then $\widehat{\phi}(\omega) \le C(1 + |\omega|)^{-n+\mathcal{K}}$ with $\mathcal{K} = \log(|\mathcal{L}(\frac{2\pi}{3})|)/\log 2$, and this decay is optimal.

By using the following theorem, the decay rate of Fourier transforms of refinable functions $\phi_0^{m,l,\alpha}$ with the mask $\tau_0^{m,l,\alpha}(\omega)$ is analyzed. Especially, the case $\alpha = 0$ was demonstrated in [4], Theorem 3.4. We will give the discussion of the case $0 < \alpha < \frac{1}{3(m+l)-7}$.

Theorem 5 *Let $\phi_0^{m,l,\alpha}$ be refinable functions with the mask $\tau_0^{m,l,\alpha}(\omega)$. Then*

$$\left|\widehat{\phi_0^{m,l,\alpha}}\right| \le C\left(1 + |\omega|\right)^{-2+\mathcal{K}},$$

where $\mathcal{K} = \log(|T_0^{m,l,\alpha}(\frac{2\pi}{3})|)/\log 2$, and the decay rate $\beta_0^{m,l,\alpha} = 2 - \log(|T_0^{m,l,\alpha}(\frac{2\pi}{3})|)/\log 2$ is optimal. As a result, $\phi_0^{m,l,\alpha} \in C^{\gamma_0^{m,l,\alpha}}$, where $\gamma_0^{m,l,\alpha} \ge \beta_0^{m,l,\alpha} - 1 - \varepsilon$, for any small enough $\varepsilon > 0$.

Proof It is obvious that $|\tau_0^{m,l,\alpha}(\omega)| = \cos^2(\frac{\omega}{2})|T_0^{m,l,\alpha}(\omega)|$, $\omega \in [-\pi, \pi]$. We claim that

$$\left|T_0^{m,l,\alpha}(\omega)\right| \le \left|T_0^{m,l,\alpha}\left(\frac{2\pi}{3}\right)\right|, \quad \text{for } |\omega| \in \left[0, \frac{2\pi}{3}\right]. \tag{29}$$

Indeed, $T_0^{m,l,\alpha}(\omega)$ is a continuous function on $[-\frac{2\pi}{3}, \frac{2\pi}{3}]$ and is differentiable on $(-\frac{2\pi}{3}, \frac{2\pi}{3})$. The maximum value of $T_0^{m,l,\alpha}(\omega)$ on $[-\frac{2\pi}{3}, \frac{2\pi}{3}]$ can be derived as follows: We find $\omega = 0$ is the only zero of equation $[T_0^{m,l,\alpha}(\omega)]' = 0$. Since $[T_0^{m,l,\alpha}(0)]'' > 0$, $T_0^{m,l,\alpha}(0)$ is the minimum of $T_0^{m,l,\alpha}(\omega)$ on $[-\frac{2\pi}{3}, \frac{2\pi}{3}]$. Consequently, $T_0^{m,l,\alpha}(0) \leq T_0^{m,l,\alpha}(\pm\frac{2\pi}{3})$. Here, $T_0^{m,l,\alpha}(\omega)$ is an even function. Hence, (29) holds. Next we show

$$\left|T_0^{m,l,\alpha}(\omega)T_0^{m,l,\alpha}(2\omega)\right| \leq \left|T_0^{m,l,\alpha}\left(\frac{2\pi}{3}\right)\right|^2, \quad \text{for } |\omega| \in \left[\frac{2\pi}{3}, \pi\right].$$

Here, $T_0^{m,l,\alpha}(\omega)T_0^{m,l,\alpha}(2\omega)$ is a continuous function on $|\omega| \in [\frac{2\pi}{3}, \pi]$ and is differentiable on $|\omega| \in (\frac{2\pi}{3}, \pi)$. Since the equation $[T_0^{m,l,\alpha}(\omega)T_0^{m,l,\alpha}(2\omega)]' = 0$ has no zeros on $|\omega| \in [\frac{2\pi}{3}, \pi]$, we are required to compare $T_0^{m,l,\alpha}(\pi)T_0^{m,l,\alpha}(2\pi)$ with $T_0^{m,l,\alpha}(\frac{2\pi}{3})T_0^{m,l,\alpha}(\frac{4\pi}{3})$. As a result of

$$T_0^{m,l,\alpha}(\pi)T_0^{m,l,\alpha}(2\pi) = T_0^{m,l,\alpha}(\pi)T_0^{m,l,\alpha}(0) = T_0^{m,l,\alpha}(\pi)$$

and

$$\begin{aligned} T_0^{m,l,\alpha}\left(\frac{2\pi}{3}\right)T_0^{m,l,\alpha}\left(\frac{4\pi}{3}\right) &= T_0^{m,l,\alpha}\left(\frac{2\pi}{3}\right)T_0^{m,l,\alpha}\left(-\frac{2\pi}{3}\right) \\ &= \left(T_0^{m,l,\alpha}\left(\frac{2\pi}{3}\right)\right)^2, \end{aligned}$$

only $T_0^{m,l,\alpha}(\pi)$ and $(T_0^{m,l,\alpha}(\frac{2\pi}{3}))^2$ need to be compared. Let m, α, ω be fixed. For $l = 1$, we claim that

$$\left(T_0^{m,1,\alpha}\left(\frac{2\pi}{3}\right)\right)^2 > T_0^{m,1,\alpha}(\pi).$$

Here,

$$T_0^{m,1,\alpha}\left(\frac{2\pi}{3}\right) = \left(\left(\frac{3}{4}m + 1 + m\alpha\right)\prod_{i=1}^{m-1}\left(\frac{1}{4} + i\alpha\right)\right) \Big/ \prod_{j=1}^{m}(1 + jm)$$

and

$$T_0^{m,l,\alpha}(\pi) = \left((m + 1 + m\alpha)\prod_{i=1}^{m-1} i\alpha\right) \Big/ \prod_{i=1}^{m}(1 + i\alpha).$$

Setting $g(m) = (T_0^{m,1,\alpha}(\frac{2\pi}{3}))^2 - T_0^{m,1,\alpha}(\pi)$, in which α is fixed, then

$$\begin{aligned} g(m) = &\left(\left(\left(\frac{3}{4}m + 1 + m\alpha\right)\prod_{i=1}^{m-1}\left(\frac{1}{4} + i\alpha\right)\right) \Big/ \prod_{j=1}^{m}(1 + jm)\right)^2 \\ &- \left((m + 1 + m\alpha)\prod_{i=1}^{m-1} i\alpha\right) \Big/ \prod_{i=1}^{m}(1 + i\alpha). \end{aligned}$$

Notice that

$$g(m) > 0. \tag{30}$$

Indeed, since $l = 1$ and $l < m - 5$, $m \geq 7$. When $m = 7$, one has

$$g(7) = \left(\left((35/4 + 7 * \alpha)\prod_{i=1}^{6}\left(\frac{1}{4} + i\alpha\right)\right) \Big/ \prod_{i=1}^{7}(i + \alpha)\right)^2 - \left(\left(720(8 + 7 * \alpha)\alpha^6\right) \Big/ \prod_{i=1}^{7}(i + \alpha)\right) > 0,$$

for $0 \geq \alpha < 1/17$. Assuming that $g(m-1) > 0$, we now prove $g(m) > 0$. Let

$$g_1(m) = T_0^{m,1,\alpha}\left(\frac{2\pi}{3}\right) = \left(\left(\frac{3}{4}m + 1 + m\alpha\right)\prod_{i=1}^{m-1}\left(\frac{1}{4} + i\alpha\right)\right) \Big/ \prod_{i=1}^{m}(1 + i\alpha)$$

and

$$g_2(m) = T_0^{m,l,\alpha}(\pi) = \left((m + 1 + m\alpha)\prod_{i=1}^{m-1} i\alpha\right) \Big/ \prod_{i=1}^{m}(1 + i\alpha).$$

Then it is obvious that

$$g_1(m) = \frac{(\frac{1}{4} + (m-1)\alpha)(\frac{3}{4}m + 1 + m\alpha)}{(1 + m\alpha)(\frac{3}{4}(m-1) + 1 + (m-1)\alpha)} g_1(m-1)$$

and

$$g_2(m) = \frac{((m-1)\alpha)(m + 1 + m\alpha)}{(1 + m\alpha)((m-1) + 1 + (m-1)\alpha)} g_2(m-1).$$

Thus,

$$\begin{aligned} g(m) &= \left(g_1(m)\right)^2 - g_2(m) \\ &= \left(g_1(m-1)\right)^2\left(\frac{(\frac{1}{4} + (m-1)\alpha)(\frac{3}{4}m + 1 + m\alpha)}{(1 + m\alpha)(\frac{3}{4}(m-1) + 1 + (m-1)\alpha)}\right)^2 \\ &\quad - g_2(m-1)\frac{((m-1)\alpha)(m + 1 + m\alpha)}{(1 + m\alpha)((m-1) + 1 + (m-1)\alpha)} \\ &> g_2(m-1)g_2(m-1)\left(\left(\frac{(\frac{1}{4} + (m-1)\alpha)(\frac{3}{4}m + 1 + m\alpha)}{(1 + m\alpha)(\frac{3}{4}(m-1) + 1 + (m-1)\alpha)}\right)^2\right. \\ &\quad \left. - \frac{((m-1)\alpha)(m + 1 + m\alpha)}{(1 + m\alpha)((m-1) + 1 + (m-1)\alpha)}\right). \end{aligned}$$

Let

$$h(\alpha) = \left(\left(\frac{\frac{1}{4} + (m-1)\alpha}{1 + m\alpha}\right)\left(\frac{\frac{3}{4}m + 1 + m\alpha}{\frac{3}{4}(m-1) + 1 + (m-1)\alpha}\right)\right)^2 - \left(\frac{(m-1)\alpha}{1 + m\alpha}\right)\left(\frac{m + 1 + m\alpha}{(m-1) + 1 + (m-1)\alpha}\right).$$

$h'(\alpha)$ is derived as follows:

$$\begin{aligned} h'(\alpha) = {} & \bigl\{\bigl(-120m - 224m^2 + 454m^3 + 907m^4 - 837m^6 + 108m^7\bigr)\alpha + \bigl(16 + 488m \\ & - 755m^2 - 1{,}328m^3 + 1{,}754m^4 + 1{,}320m^5 - 2{,}079m^6 + 648m^7\bigr)\alpha^2 + \bigl(-128 \\ & - 224m + 1{,}984m^2 - 2{,}228m^3 - 960m^4 + 3{,}976m^5 - 3{,}968m^6 + 1{,}548m^7\bigr)\alpha^3 \\ & + \bigl(256 - 896m - 96m^2 + 2{,}672m^3 - 3{,}584m^4 + 4{,}320m^5 - 4{,}512m^6 \\ & + 1{,}840m^7\bigr)\alpha^4 + \bigl(512m - 1{,}536m^2 + 1{,}088m^3 - 256m^4 + 1{,}920m^5 - 2{,}816m^6 \\ & + 1{,}088m^7\bigr)\alpha^5 + \bigl(256m^2 - 768m^3 + 512m^4 + 512m^5 - 768m^6 + 256m^7\bigr)\alpha^6\bigr\} \\ & /\bigl\{\bigl(8\bigl(m + (-1+m)\alpha\bigr)^2\bigl(1 + 3m + 4(-1+m)\alpha\bigr)^3(1+m\alpha)^3\bigr)\bigr\}. \end{aligned}$$

Note that for any fixed $m > 6$, $h'(\alpha) > 0$ on $\alpha \geq 0$. Thus, $h(\alpha)$ is increasing on $\alpha \geq 0$. This together with $h(0) > 0$ implies that $g(m) > 0$. This concludes the claim (30). Supposing $(T_0^{m,l,\alpha}(\frac{2\pi}{3}))^2 > T_0^{m,l,\alpha}(\pi)$, we verify

$$\left(T_0^{m,l+1,\alpha}\left(\frac{2\pi}{3}\right)\right)^2 > T_0^{m,l+1,\alpha}(\pi). \tag{31}$$

It follows from Lemma 8 that (31) is equivalent to

$$\begin{aligned} & \left(T_0^{m,l,\alpha}\left(\frac{2\pi}{3}\right) + \frac{1+4m\alpha}{1+(m+l)\alpha} G_{l+1}^{m,l,\alpha}\left(\frac{2\pi}{3}\right)\right)^2 \\ & \quad > \frac{1}{1+(m+l)\alpha} T_0^{m,l,\alpha}(\pi) + \sum_{j=0}^{l+1} \binom{m+l}{j} B_j^{m,l+1,\alpha}(\pi). \end{aligned} \tag{32}$$

Since

$$\frac{(m+l)\alpha}{1+(m+l)\alpha} T_0^{m,l,\alpha}\left(\frac{2\pi}{3}\right)^2 > \sum_{j=0}^{l} \binom{m+l}{j} B_j^{m,l+1,\alpha}(\pi),$$

it is obvious that (32) holds. Hence, by Theorem 4 , $\widehat{\phi}_0^{m,l,\alpha}$ satisfies

$$\bigl|\widehat{\phi}_0^{m,l,\alpha}\bigr| \leq C\bigl(1 + |\omega|\bigr)^{-2+\mathcal{K}},$$

where $\mathcal{K} = \log(|T_0^{m,l,\alpha}(\frac{2\pi}{3})|)/\log 2$, and the decay rate $\beta_0^{m,l,\alpha} = 2 - \log(|T_0^{m,l,\alpha}(\frac{2\pi}{3})|)/\log 2$ is optimal. As a result, $\phi_0^{m,l,\alpha} \in C^{\gamma_0^{m,l,\alpha}}$, where $\gamma_0^{m,l,\alpha} \geq \beta_0^{m,l,\alpha} - 1 - \varepsilon$, for any small enough $\varepsilon > 0$. □

The following table gives the decay rates $\beta_0^{m,l,\alpha}$ of the Fourier transform of refinable functions $\phi_0^{m,l,\alpha}$ with the mask $\tau_0^{m,l,\alpha}(\omega)$. To satisfy the constraint condition (2) of m, l, α, we set $\alpha = 0.01$ in Table 1.

Table 1 shows that for fixed l, α, $\beta_0^{m,l,\alpha}$ increases as m increases and for fixed m, α, $\beta_0^{m,l,\alpha}$ decreases as l increases. This is true indeed as presented in the following proposition.

Proposition 6.1 *Let the decay rate* $\beta_0^{m,l,\alpha} = 2 - \log(|T_0^{m,l,\alpha}(\frac{2\pi}{3})|)/\log 2$ *be as given in Theorem* 5, *then*:

Table 1 Decay rates $\beta_0^{m,l,\alpha}$ of refinable functions with the mask $\tau_0^{m,l,\alpha}(\omega)$, for $7 \le m \le 13$, $1 \le l < m - 5$, and $\alpha = 0.01$

(m, l)	**l = 1**	**l = 2**	**l = 3**	**l = 4**	**l = 5**	**l = 6**	**l = 7**
$m = 7$	10.6157						
$m = 8$	12.2067	11.6886					
$m = 9$	13.7834	11.7927	10.2549				
$m = 10$	15.3438	13.2786	11.6088	10.2745			
$m = 11$	16.8868	14.6887	12.9622	11.448	10.3499		
$m = 12$	18.4116	16.1224	14.3121	11.7464	11.506	10.4583	
$m = 13$	19.9179	17.5444	15.6561	14.0890	12.7551	10.8736	10.5870

(1) *For fixed m, α, $\beta_0^{m,l,\alpha}$ decreases as l increases.*
(2) *For fixed l, α, $\beta_0^{m,l,\alpha}$ increases as m increases.*

Proof Applying $\beta_0^{m,l,\alpha} = 2 - \log_2^{(T_0^{m,l,\alpha}(\frac{2\pi}{3}))}$, $T_0^{m,l,\alpha}(\frac{2\pi}{3}) > 0$, yields $2 - \beta_0^{m,l,\alpha} = \log_2^{(T_0^{m,l,\alpha}(\frac{2\pi}{3}))}$, which is equivalent to $2^{2-\beta_0^{m,l,\alpha}} = 2^{\log_2^{(T_0^{m,l,\alpha}(\frac{2\pi}{3}))}} = T_0^{m,l,\alpha}(\frac{2\pi}{3})$. Following the formula given by Lemma 8, one has

$$T_0^{m+1,l,\alpha}\left(\frac{2\pi}{3}\right) = T_0^{m+1,l-1,\alpha}\left(\frac{2\pi}{3}\right) + (1+4m\alpha)\bigl(1+(m+l)\alpha\bigr)G_l^{m,l,\alpha}\left(\frac{2\pi}{3}\right).$$

Thus,

$$2^{2-\beta_0^{m+1,l,\alpha}} = 2^{2-\beta_0^{m+1,l-1,\alpha}} + \frac{4+m\alpha}{4(1+(m+l)\alpha)}G_l^{m,l,\alpha}\left(\frac{2\pi}{3}\right).$$

Let

$$I_0^{m+1,l-1,\alpha} = 2^{2-\beta_0^{m+1,l-1,\alpha}}.$$

For $\frac{4+m\alpha}{4(1+(m+l)\alpha)}S_l^{(m+l)}(\frac{3}{4}) > 0$, it shows that $I_0^{m+1,l,\alpha} > I_0^{m+1,l-1,\alpha}$, which is equivalent to saying for fixed m, α, $I_0^{m,l,\alpha}$ increases as l increases. Hence, for fixed m, α, $\beta_0^{m,l,\alpha}$ decreases as l increases. Applying Lemma 8 to the following expression, one obtains

$$T_0^{m+1,l,\alpha}\left(\frac{2\pi}{3}\right) = T_0^{m,l,\alpha}\left(\frac{2\pi}{3}\right) - \left(\frac{4(\frac{3}{4}+l\alpha)}{1+(m+l)\alpha}\right)G_l^{m,l,\alpha}\left(\frac{2\pi}{3}\right).$$

Furthermore, we have

$$I_0^{m+1,l,\alpha} = I_0^{m,l,\alpha} - \left(\frac{\frac{3}{4}+l\alpha}{\frac{1}{4}(1+(m+l)\alpha)}\right)G_l^{m,l,\alpha}\left(\frac{2\pi}{3}\right).$$

For $(\frac{4(\frac{3}{4}+l\alpha)}{1+(m+l)\alpha})G_l^{m,l,\alpha}(\frac{2\pi}{3}) > 0$, we similarly get $I_0^{m+1,l,\alpha} < I_0^{m,l,\alpha}$, which means that for fixed l, α, $I_0^{m,l,\alpha}$ decreases as m increases. Therefore, for fixed l, α, $\beta_0^{m,l,\alpha}$ increases as m increases. □

7 Approximation orders

In this section, the approximation orders of refinable functions with the mask $\tau_0^{m,l,\alpha}(\omega)$ are analyzed in the following theorem.

Theorem 6 *Let $\phi_0^{m,l,\alpha}$ be refinable functions with the mask $\tau_0^{m,l,\alpha}(\omega)$. Then $\phi_0^{m,l,\alpha}$ provides the approximation orders $2l+2$.*

Proof In fact, the approximation orders of $1-|\tau_0^{m,l,\alpha}(\omega)|^2$ and $\tau_0^{m,l,\alpha}(\omega)$ are independent on α. For convenience, we set $\alpha = 0$. Following [4], let

$$R_{m,l}(y) = \sum_{j=0}^{l} \binom{m+l}{j} y^j (1-y)^{m+l-j},$$

where $y = \sin^2(\frac{\omega}{2})$, then

$$1 - \left| R_{m,l}\left(\sin^2\left(\frac{\omega}{2}\right)\right)\right|^2 = O\bigl(|\omega|^{2l+2}\bigr).$$

Since

$$R'_{m,l}(y) = -(m+l)\binom{m+l-1}{l} y^l (1-y)^{m-1}$$

and $\sin^2(\frac{\omega}{2})$ is equal to 1 when $\omega = \pi$, $\cos^{2(m-1)}(\frac{\omega}{2})$ has a zero of order $4(m-1)$. We conclude that

$$R_{m,l}(y) = O\bigl(|\omega|^{4(m-1)+1}\bigr).$$

For

$$\min\bigl\{2l+2, 4(m-1)\bigr\} = 2l+2,$$

then $\phi_0^{m,l,\alpha}$ provides the approximation orders $2l+2$. □

8 Symmetry

Symmetric coefficients of the mask are of great significance in image processing. In the following, we will give a symmetry proof and provide a graphical illustration.

Lemma 9 *For $m, l \in \mathbb{Z}^+$, $\alpha \geq 0$, $j = 0, 1, \ldots, l$, $z = e^{-i\omega}$, we derive*

$$\prod_{i=0}^{j-1}\left(\frac{1}{2} - \frac{1}{4}\bigl(z^{-1}+z\bigr) + i\alpha\right) \prod_{i=0}^{m+l-j-1}\left(\frac{1}{2} + \frac{1}{4}\bigl(z^{-1}+z\bigr) + i\alpha\right) = \sum_{k=0}^{m+l} b_k^{j,m,l,\alpha}\bigl(z^{-1}+z\bigr)^k,$$

where

$$b_k^{j,m,l,\alpha} = \frac{D_k}{D}, \quad k = 0, 1, \ldots, m+l, \tag{33}$$

$$D = \begin{vmatrix} 1 & \frac{1}{2} & (\frac{1}{2})^2 & \ldots & (\frac{1}{2})^{m+l} \\ 1 & (\frac{1}{2})^2 & ((\frac{1}{2})^2)^2 & \ldots & ((\frac{1}{2})^2)^{m+l} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & (\frac{1}{2})^{m+l+1} & ((\frac{1}{2})^{m+l+1})^2 & \ldots & ((\frac{1}{2})^{m+l+1})^{m+l} \end{vmatrix},$$

and

$$D_k = \begin{vmatrix} 1 & \dots & (\frac{1}{2})^{k-1} & f(\frac{1}{2}) & (1/2)^{k+1} & \dots & (\frac{1}{2})^{m+l} \\ 1 & \dots & ((\frac{1}{2})^2)^{k-1} & f((\frac{1}{2})^2) & ((1/2)^2)^{k+1} & \dots & ((\frac{1}{2})^2)^{m+l} \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 1 & \dots & ((\frac{1}{2})^{m+l+1})^{k-1} & f((\frac{1}{2})^{m+l+1}) & ((1/2)^{m+l+1})^{k+1} & \dots & ((\frac{1}{2})^{m+l+1})^{m+l} \end{vmatrix},$$

where

$$f(x) = \prod_{i=0}^{j-1}\left(\frac{1}{2} - \frac{1}{4}x + i\alpha\right)\prod_{i=0}^{m+l-j-1}\left(\frac{1}{2} + \frac{1}{4}x + i\alpha\right), \quad x \in [-2, 2].$$

Proof Set $x(z) = z^{-1} + z$, then $x \in [-2, 2]$ and

$$\prod_{i=0}^{j-1}\left(\frac{1}{2} - \frac{1}{4}x + i\alpha\right)\prod_{i=0}^{m+l-j-1}\left(\frac{1}{2} + \frac{1}{4}x + i\alpha\right) = \sum_{k=0}^{m+l} b_k^{j,m,l,\alpha} x^k. \tag{34}$$

Fixing $x = \frac{1}{2}, (\frac{1}{2})^2, \dots, (\frac{1}{2})^{m+l+1}$, by using (34), we obtain

$$\begin{cases} (\frac{1}{2})^0 b_0^{j,m,l,\alpha} + (\frac{1}{2})^1 b_1^{j,m,l,\alpha} + \dots + (\frac{1}{2})^{m+l} b_{m+l}^{j,m,l,\alpha} = f(\frac{1}{2}), \\ ((\frac{1}{2})^2)^0 b_0^{j,m,l,\alpha} + ((\frac{1}{2})^2)^1 b_1^{j,m,l,\alpha} + \dots + ((\frac{1}{2})^2)^{m+l} b_{m+l}^{j,m,l,\alpha} = f((\frac{1}{2})^2), \\ \dots \\ ((\frac{1}{2})^{m+l+1})^0 b_0^{j,m,l,\alpha} + ((\frac{1}{2})^{m+l+1})^1 b_1^{j,m,l,\alpha} + \dots + ((\frac{1}{2})^{m+l+1})^{m+l} b_{m+l}^{j,m,l,\alpha} = f((\frac{1}{2})^{m+l+1}). \end{cases} \tag{35}$$

Since the coefficient determinant of (35)

$$D = \begin{vmatrix} 1 & \frac{1}{2} & (\frac{1}{2})^2 & \dots & (\frac{1}{2})^{m+l} \\ 1 & (\frac{1}{2})^2 & ((\frac{1}{2})^2)^2 & \dots & ((\frac{1}{2})^2)^{m+l} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & (\frac{1}{2})^{m+l+1} & ((\frac{1}{2})^{m+l+1})^2 & \dots & ((\frac{1}{2})^{m+l+1})^{m+l} \end{vmatrix} \neq 0,$$

applying Cramer's rule yields

$$b_k^{j,m,l,\alpha} = \frac{D_k}{D}, \quad k = 0, 1, \dots, m+l,$$

where

$$D_k = \begin{vmatrix} 1 & \dots & (\frac{1}{2})^{k-1} & f(\frac{1}{2}) & (1/2)^{k+1} & \dots & (\frac{1}{2})^{m+l} \\ 1 & \dots & ((\frac{1}{2})^2)^{k-1} & f((\frac{1}{2})^2) & ((1/2)^2)^{k+1} & \dots & ((\frac{1}{2})^2)^{m+l} \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 1 & \dots & ((\frac{1}{2})^{m+l+1})^{k-1} & f((\frac{1}{2})^{m+l+1}) & ((1/2)^{m+l+1})^{k+1} & \dots & ((\frac{1}{2})^{m+l+1})^{m+l} \end{vmatrix}.$$

□

Lemma 10 *For $k \in \mathbb{N}$, we have*

$$\left(z + z^{-1}\right)^k = \begin{cases} \sum_{i=0}^{(k-1)/2} \binom{k}{i}(z^{k-2i} + z^{-(k-2i)}), & \textit{if } k \textit{ is an odd number}, \\ \sum_{i=0}^{k/2-1} \binom{k}{i}(z^{k-2i} + z^{-(k-2i)}) + \binom{k}{k/2} z^0, & \textit{else}. \end{cases} \tag{36}$$

Proof We consider into two cases. Suppose that k is an odd number. One gets

$$\left(z+z^{-1}\right)^k=\sum_{j=0}^{k}\binom{k}{j}z^{k-2j}=\sum_{j=0}^{(k-1)/2}\binom{k}{j}z^{k-2j}+\sum_{j=(k+1)/2}^{k}\binom{k}{j}z^{k-2j}.$$

Let $i=k-j$, then

$$\sum_{j=(k+1)/2}^{k}\binom{k}{j}z^{k-2j}=\sum_{i=0}^{(k-1)/2}\binom{k}{i}z^{-(k-2i)}.$$

Thus,

$$\begin{aligned}\left(z+z^{-1}\right)^k&=\sum_{j=0}^{(k-1)/2}\binom{k}{j}z^{k-2j}+\sum_{j=(k+1)/2}^{k}\binom{k}{j}z^{k-2j}\\&=\sum_{i=0}^{(k-1)/2}\binom{k}{i}z^{k-2i}+\sum_{i=0}^{(k-1)/2}\binom{k}{i}z^{-(k-2i)}\\&=\sum_{i=0}^{(k-1)/2}\binom{k}{i}\left(z^{k-2i}+z^{-(k-2i)}\right).\end{aligned}$$

Suppose, on the other hand, that k is an even number. It is clear that

$$\left(z+z^{-1}\right)^k=\sum_{j=0}^{k}\binom{k}{j}z^{k-2j}=\sum_{j=0}^{k/2-1}\binom{k}{j}z^{k-2j}+\sum_{j=k/2+1}^{k}\binom{k}{j}z^{k-2j}+\binom{k}{k/2}z^0.$$

Let $i=k-j$, then

$$\sum_{j=k/2+1}^{k}\binom{k}{j}z^{k-2j}=\sum_{i=0}^{k/2-1}\binom{k}{i}z^{-(k-2i)}.$$

Therefore,

$$\begin{aligned}\left(z+z^{-1}\right)^k&=\sum_{j=0}^{k/2-1}\binom{k}{j}z^{k-2j}+\sum_{j=k/2+1}^{k}\binom{k}{j}z^{k-2j}+\binom{k}{k/2}z^0\\&=\sum_{i=0}^{k/2-1}\binom{k}{i}z^{k-2i}+\sum_{i=0}^{k/2-1}\binom{k}{i}z^{-(k-2i)}+\binom{k}{k/2}z^0\\&=\sum_{i=0}^{k/2-1}\binom{k}{i}\left(z^{k-2i}+z^{-(k-2i)}\right)+\binom{k}{k/2}z^0.\end{aligned}$$

This concludes the proof. □

Theorem 7 *Let $\tau_0^{m,l,\alpha}(\omega)$ be the mask* (2), *then the coefficients of the mask are symmetric.*

Proof Since $\sin^2(\frac{\omega}{2}) = \frac{1}{2} - \frac{1}{4}(e^{i\omega} + e^{-i\omega})$, $\cos^2(\frac{\omega}{2}) = \frac{1}{2} + \frac{1}{4}(e^{i\omega} + e^{-i\omega})$, we set $z = e^{-i\omega}$ and obtain

$$\begin{aligned}
\tau_0^{m,l,\alpha}(\omega) &= \sum_{j=0}^{l} \binom{m+l}{j} \left(\prod_{i=0}^{j-1} \left(\sin^2\left(\frac{\omega}{2}\right) + i\alpha \right) \prod_{i=0}^{m+l-j-1} \left(\cos^2\left(\frac{\omega}{2}\right) + i\alpha \right) \right) \\
&\quad \Big/ \prod_{i=1}^{m+l-1} (1 + i\alpha) \\
&= \sum_{j=0}^{l} \binom{m+l}{j} \left(\prod_{i=0}^{j-1} \left(\frac{1}{2} - \frac{1}{4}\left(e^{i\omega} + e^{-i\omega}\right) + i\alpha \right) \right. \\
&\quad \times \left. \prod_{i=0}^{m+l-j-1} \left(\frac{1}{2} + \frac{1}{4}\left(e^{i\omega} + e^{-i\omega}\right) + i\alpha \right) \right) \Big/ \prod_{i=1}^{m+l-1} (1 + i\alpha) \\
&= \sum_{j=0}^{l} \binom{m+l}{j} \left(\prod_{i=0}^{j-1} \left(\frac{1}{2} - \frac{1}{4}\left(z^{-1} + z\right) + i\alpha \right) \right. \\
&\quad \times \left. \prod_{i=0}^{m+l-j-1} \left(\frac{1}{2} + \frac{1}{4}\left(z^{-1} + z\right) + i\alpha \right) \right) \Big/ \prod_{i=1}^{m+l-1} (1 + i\alpha).
\end{aligned}$$

Let $a_j^{m,l,\alpha} = \binom{m+l}{j} / \prod_{i=1}^{m+l-1}(1 + i\alpha)$, by using Lemma 9, one can obtain

$$\begin{aligned}
\tau_0^{m,l,\alpha}(\omega) &= \sum_{j=0}^{l} \binom{m+l}{j} \left(\prod_{i=0}^{j-1} \left(\frac{1}{2} - \frac{1}{4}\left(z^{-1} + z\right) + i\alpha \right) \right. \\
&\quad \times \left. \prod_{i=0}^{m+l-j-1} \left(\frac{1}{2} + \frac{1}{4}\left(z^{-1} + z\right) + i\alpha \right) \right) \Big/ \prod_{i=1}^{m+l-1} (1 + i\alpha) \\
&= \sum_{j=0}^{l} a_j^{m,l,\alpha} \sum_{k=0}^{m+l} b_k^{j,m,l,\alpha} \left(z + z^{-1}\right)^k \\
&= \sum_{j=0}^{l} \sum_{k=0}^{m+l} a_j^{m,l,\alpha} b_k^{j,m,l,\alpha} \left(z + z^{-1}\right)^k,
\end{aligned}$$

where $\{b_k^{j,m,l,\alpha}\}_{k=0}^{m+l}$ is (33) in Lemma 9. Let $c_k^{j,m,l,\alpha} = a_j^{m,l,\alpha} b_k^{j,m,l,\alpha}$ and $d_k^{m,l,\alpha} = \sum_{j=0}^{l} c_k^{j,m,l,\alpha}$, then

$$\begin{aligned}
\tau_0^{m,l,\alpha}(\omega) &= \sum_{j=0}^{l} \sum_{k=0}^{m+l} a_j^{m,l,\alpha} b_k^{j,m,l,\alpha} \left(z + z^{-1}\right)^k \\
&= \sum_{j=0}^{l} \sum_{k=0}^{m+l} c_k^{j,m,l,\alpha} \left(z + z^{-1}\right)^k \\
&= \sum_{k=0}^{m+l} \left(\sum_{j=0}^{l} c_k^{j,m,l,\alpha} \right) \left(z + z^{-1}\right)^k \\
&= \sum_{k=0}^{m+l} d_k^{m,l,\alpha} \left(z + z^{-1}\right)^k.
\end{aligned}$$

We consider into two cases. Suppose that $m + l$ is an even number. Applying Lemma 10 yields

$$\begin{aligned}
\tau_0^{m,l,\alpha}(\omega) &= \sum_{k=0}^{m+l} d_k^{m,l,\alpha}\left(z + z^{-1}\right)^k \\
&= \sum_{j=0}^{(m+l-2)/2} d_{2j+1}^{m,l,\alpha} \sum_{i=0}^{j} \binom{2j+1}{i}\left(z^{2(j-i)+1} + z^{-(2(j-i)+1)}\right) \\
&\quad + \sum_{j=0}^{(m+l)/2} d_{2j}^{m,l,\alpha} \sum_{i=0}^{j-1} \binom{2j}{i}\left(z^{2(j-i)} + z^{-(2(j-i))}\right) + \sum_{j=0}^{(m+l)/2} d_{2j}^{m,l,\alpha}\binom{2j}{j} \\
&= \sum_{j=0}^{(m+l-2)/2} \sum_{i=0}^{j} \binom{2j+1}{i} d_{2j+1}^{m,l,\alpha}\left(z^{2(j-i)+1} + z^{-(2(j-i)+1)}\right) \\
&\quad + \sum_{j=0}^{(m+l)/2} \sum_{i=0}^{j-1} \binom{2j}{i} d_{2j}^{m,l,\alpha}\left(z^{2(j-i)} + z^{-(2(j-i))}\right) + \sum_{j=0}^{(m+l)/2} d_{2j}^{m,l,\alpha}\binom{2j}{j} \\
&= \sum_{k=1}^{m+l} h_k^{m,l,\alpha}\left(z^k + z^{-k}\right) + h_0 z^0,
\end{aligned}$$

where $h_0 = \sum_{j=0}^{(m+l)/2} \binom{2j}{j} d_{2j}^{m,l,\alpha}$,

$$h_k = \begin{cases} \sum_{\substack{j-i=(k-1)/2 \\ j\in\{0,1,\ldots,(m+l-2)/2\}}} \binom{2j+1}{i} d_{2j+1}^{m,l,\alpha}, & \text{if } k \text{ is an odd number,} \\ \sum_{\substack{j-i=k/2 \\ j\in\{0,1,\ldots,(m+l)/2\}}} \binom{2j}{i} d_{2j}^{m,l,\alpha}, & \text{else.} \end{cases} \tag{37}$$

Suppose, on the other hand, that $m + l$ is an odd number. We have

$$\begin{aligned}
\tau_0^{m,l,\alpha}(\omega) &= \sum_{k=0}^{m+l} d_k^{m,l,\alpha}\left(z + z^{-1}\right)^k \\
&= \sum_{j=0}^{(m+l-1)/2} d_{2j+1}^{m,l,\alpha} \sum_{i=0}^{j} \binom{2j+1}{i}\left(z^{2(j-i)+1} + z^{-(2(j-i)+1)}\right) \\
&\quad + \sum_{j=0}^{(m+l-1)/2} d_{2j}^{m,l,\alpha} \sum_{i=0}^{j-1} \binom{2j}{i}\left(z^{2(j-i)} + z^{-(2(j-i))}\right) + \sum_{j=0}^{(m+l-1)/2} d_{2j}^{m,l,\alpha}\binom{2j}{j} \\
&= \sum_{j=0}^{(m+l-1)/2} \sum_{i=0}^{j} \binom{2j+1}{i} d_{2j+1}^{m,l,\alpha}\left(z^{2(j-i)+1} + z^{-(2(j-i)+1)}\right) \\
&\quad + \sum_{j=0}^{(m+l-1)/2} \sum_{i=0}^{j-1} \binom{2j}{i} d_{2j}^{m,l,\alpha}\left(z^{2(j-i)} + z^{-(2(j-i))}\right) + \sum_{j=0}^{(m+l-1)/2} d_{2j}^{m,l,\alpha}\binom{2j}{j} \\
&= \sum_{k=1}^{m+l} h_k^{m,l,\alpha}\left(z^k + z^{-k}\right) + h_0 z^0,
\end{aligned}$$

where $h_0 = \sum_{j=0}^{(m+l-1)/2} \binom{2j}{j} d_{2j}^{m,l,\alpha}$,

$$h_k = \begin{cases} \sum_{\substack{j-i=(k-1)/2 \\ j\in\{0,1,\ldots,(m+l-1)/2\}}} \binom{2j+1}{i} d_{2j+1}^{m,l,\alpha}, & \text{if } k \text{ is an odd number}, \\ \sum_{\substack{j-i=k/2 \\ j\in\{0,1,\ldots,(m+l-1)/2\}}} \binom{2j}{i} d_{2j}^{m,l,\alpha}, & \text{else}. \end{cases} \tag{38}$$

This establishes the proof. □

In the end, we provide two examples of symmetric refinable functions from the mask $\tau_0^{m,l,\alpha}(\omega)$ for $m = 8$, $l = 2$, $\alpha = 0.02$ and $\tau_0^{m,l,\alpha}(\omega)$ for $m = 16$, $l = 3$, $\alpha = 0.01$.

Example 1 Consider the refinable function with the mask $\tau_0^{m,l,\alpha}(\omega)$, where $l = 2$, $m = 8$, $\alpha = 0.02$, *i.e.*

$$\begin{aligned} \tau_0^{8,2,0.02}(\omega) &= \sum_{j=0}^{2} \binom{10}{j} \left(\prod_{i=0}^{j-1} \left(\sin^2\left(\frac{\omega}{2}\right) + i \right) \prod_{i=0}^{9-j} \left(\cos^2\left(\frac{\omega}{2}\right) + i \right) \right) \Big/ \prod_{i=1}^{9} (1+i) \\ &= h_0 + \sum_{k=1}^{10} h_{\pm k} e^{\pm ik\omega}. \end{aligned}$$

In the following, Table 2 shows the symmetric coefficients $\{h_{\pm k}\}_0^{10}$ of the mark $\tau_0^{m,l,\alpha}(\omega)$ and Figure 1 gives the corresponding refinable function when $l = 2$, $m = 8$, $\alpha = 0.02$.

The graph gives the symmetry and compactly supported refinable function corresponding to the mask $\tau_0^{8,2,0.02}(\omega)$, which has approximation order 6 and decay rate 11.6886.

Example 2 Consider the refinable function with the mask $\tau_0^{m,l,\alpha}(\omega)$, where $m = 16$, $l = 3$, and $\alpha = 0.01$, *i.e.*

$$\begin{aligned} \tau_0^{16,3,0.01}(\omega) &= \sum_{j=0}^{3} \binom{19}{j} \left(\prod_{i=0}^{j-1} \left(\sin^2\left(\frac{\omega}{2}\right) + i \right) \prod_{i=0}^{18-j} \left(\cos^2\left(\frac{\omega}{2}\right) + i \right) \right) \Big/ \prod_{i=1}^{18} (1+i) \\ &= h_0 + \sum_{k=1}^{19} h_{\pm k} e^{\pm ik\omega}. \end{aligned}$$

In the following, Table 3 shows the symmetric coefficients $\{h_{\pm k}\}_0^{19}$ of the mark $\tau_0^{m,l,\alpha}(\omega)$ and Figure 2 gives the corresponding refinable function when $m = 16$, $l = 3$, and $\alpha = 0.01$.

Table 2 Symmetric coefficients of the mask $\tau_0^{8,2,0.02}(\omega)$

i	h_k	h_{-k}
0	3.4461×10^{-1}	3.4461×10^{-1}
1	2.6646×10^{-1}	2.6646×10^{-1}
2	1.0703×10^{-1}	1.0703×10^{-1}
3	-5.4041×10^{-3}	-5.4041×10^{-3}
4	-3.0354×10^{-2}	-3.0354×10^{-2}
5	-1.3589×10^{-2}	-1.3589×10^{-2}
6	-3.6648×10^{-6}	-3.6648×10^{-6}
7	2.3424×10^{-3}	2.3424×10^{-3}
8	1.0108×10^{-3}	1.0108×10^{-3}
9	1.9281×10^{-4}	1.9281×10^{-4}
10	1.4706×10^{-5}	1.4706×10^{-5}

Figure 1 **The refinable function with $\tau_0^{8,2,0.02}(\omega)$.**

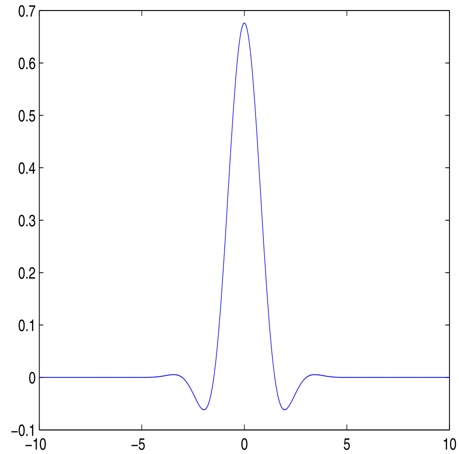


Table 3 **Symmetric coefficients of the mask $\tau_0^{16,3,0.01}(\omega)$**

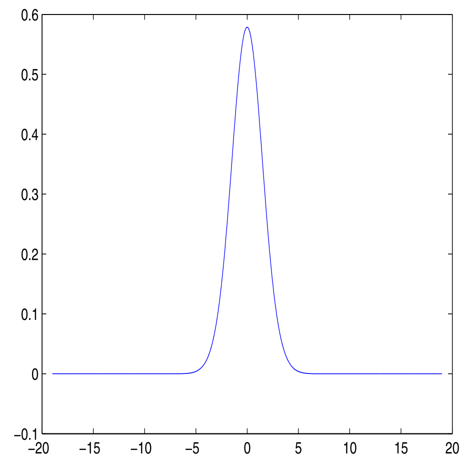
i	h_k	h_{-k}
0	2.2376	2.2376
1	2.0725	2.0725
2	1.6492	1.6492
3	1.1334	1.1334
4	6.8036×10^{-1}	6.8036×10^{-1}
5	3.6390×10^{-1}	3.6390×10^{-1}
6	1.7807×10^{-1}	1.7807×10^{-1}
7	8.1210×10^{-2}	8.1210×10^{-2}
8	3.4068×10^{-2}	3.4068×10^{-2}
9	1.2378×10^{-2}	1.2378×10^{-2}
10	3.4314×10^{-3}	3.4314×10^{-3}
11	4.7361×10^{-4}	4.7361×10^{-4}
12	-1.4408×10^{-4}	-1.4408×10^{-4}
13	1.2584×10^{-4}	1.2584×10^{-4}
14	4.7943×10^{-5}	4.7943×10^{-5}
15	1.2022×10^{-5}	1.2022×10^{-5}
16	2.0920×10^{-6}	2.0920×10^{-6}
17	2.4584×10^{-7}	2.4584×10^{-7}
18	1.7688×10^{-8}	1.7688×10^{-8}
19	5.9141×10^{-10}	5.9141×10^{-10}

The graph gives the symmetry and compactly supported refinable function corresponding to the mark $\tau_0^{16,3,0.01}(\omega)$, which has approximation order 8 and decay rate 20.8651.

9 Results and discussion

In this paper, a further family of refinable functions is constructed, which includes pseudo-splines of Type II, based on refinement masks, which are generalized Bernstein polynomials. They possess beside a compact support also a certain Hölder regularity (Table 1), this implies a suitable decay in Fourier domain and the approximation order in the corresponding shift-invariant space. The integer shifts of the refinable functions form a Riesz basis of their span and the corresponding univariate wavelets can be constructed in the usual manner.

Figure 2 The refinable function with $\tau_0^{16,3,0.01}(\omega)$.



10 Conclusions

We study new masks

$$\tau_0^{m,l,\alpha}(\omega) := \sum_{j=0}^{l} \binom{m+l}{j} \left(\prod_{i=0}^{j-1} \left(\sin^2\left(\frac{\omega}{2}\right) + i\alpha \right) \prod_{i=0}^{m+l-j-1} \left(\cos^2\left(\frac{\omega}{2}\right) + i\alpha \right) \right) \Big/ \prod_{i=1}^{m+l-1} (1 + i\alpha),$$

which include almost all masks of pseudo-splines of Type II [4] when $\alpha = 0$, to provide derived properties. We obtain the convergence of cascade algorithms in Theorem 1, which guarantees the existence of refinable functions. In Theorem 2, Riesz wavelets whose dilation and translation form a Riesz basis for $L_2(R)$ are constructed. Theorem 3 analyzes the stability of the subdivision schemes. Regularity and the influences of parameters m, l, α on the decay rate are showed in Section 6. Section 7 gives the approximation order of the new refinable functions. Finally, the symmetry of the refinable functions, which is of importance, is illustrated in Theorem 7.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this article. They read and approved the final manuscript.

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