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Harmonic functions with varying coefficients

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Abstract

Complex-valued harmonic functions that are univalent and sense preserving in the open unit disk can be written in the form $f = h + \overline{g}$, where h and g are analytic. In this paper we investigate some classes of univalent harmonic functions with varying coefficients related to Janowski functions. By using the extreme points theory we obtain necessary and sufficient convolution conditions, coefficients estimates, distortion theorems, and integral mean inequalities for these classes of functions. The radii of starlikeness and convexity for these classes are also determined.

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1 Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics (*e.g.* see Choquet [1], Dorff [2], Duren [3] or Lewy [4]). A continuous function $f = u + iv$ is said to be complex-valued harmonic in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain, we can write $f = h + \overline{g}$, where h and g are analytic in D. We shall call h the analytic and g the co-analytic part of f. Clunie and Sheil-Small [5] pointed out that a necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D. Note that for $f = h + \overline{g}$, harmonic and sense preserving in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the condition $h'(0) = 1 > |g'(0)|$ implies that the function $(f - \overline{g'(0)f})/(1 - |g'(0)|^2)$ is also harmonic and sense preserving in $\mathbb{D}$. We let $\mathcal{H}$ be the class of functions $f = h + \overline{g}$, harmonic, sense preserving, and univalent in the open unit disk $\mathbb{D}$, for which $f_{\bar{z}}(0) = g'(0) = 0$. Such harmonic and sense-preserving functions $f = h + \overline{g} \in \mathcal{H}$ may be represented by the power series

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=2}^{\infty} b_k z^k, \quad z \in \mathbb{D}. \tag{1}$$

Clunie and Sheil-Small [5] proved that the class $\mathcal{H}$ is a compact family (with respect to the topology of locally uniform convergence). Note that for $g(z) \equiv 0$, the class $\mathcal{H}$ reduces to the class $\mathcal{S}$ of normalized analytic functions univalent in $\mathbb{D}$.

For $0 \leq \alpha < 1$ we let $\mathcal{S}^*_{\mathcal{H}}(\alpha)$ and $\mathcal{S}^c_{\mathcal{H}}(\alpha)$, respectively, denote the subclasses of $\mathcal{S}_{\mathcal{H}}$ consisting of harmonic starlike and harmonic convex functions of order α.

A functions f of the form (1) is in $\mathcal{S}^*_{\mathcal{H}}(\alpha)$ if and only if (*e.g.* see Clunie and Sheil-Small [5] or Duren [3])

$$\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right) > \alpha, \quad |z| = r < 1.$$

Similarly, a function f of the form (1) is in $\mathcal{S}^c_{\mathcal{H}}(\alpha)$ if and only if

$$\frac{\partial}{\partial\theta}\left(\arg \frac{\partial}{\partial\theta} f\left(\rho e^{i\theta}\right)\right) > \alpha, \quad |z| = r < 1.$$

We note that a harmonic function $f \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$ if and only if

$$\operatorname{Re}\frac{J_{\mathcal{H}}f(z)}{f(z)} > \alpha, \quad |z| = r < 1,$$

or

$$\left|\frac{J_{\mathcal{H}}f(z) - (1+\alpha)f(z)}{J_{\mathcal{H}}f(z) + (1-\alpha)f(z)}\right| < 1, \quad |z| = r < 1,$$

where

$$J_{\mathcal{H}}f(z) := zh'(z) - \overline{zg'(z)}.$$

For $0 \le \alpha < 1$, it is easy to verify that

$$f \in \mathcal{S}^c_{\mathcal{H}}(\alpha) \quad \Leftrightarrow \quad J_{\mathcal{H}}f \in \mathcal{S}^*_{\mathcal{H}}(\alpha).$$

For $\lambda \in \{0,1,2,\ldots\}$ and $f = h + \overline{g} \in \mathcal{H}$ of the form (1), we consider the linear operator $J^\lambda_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ defined by $J^0_{\mathcal{H}}f := f = h + \overline{g}$ and

$$\begin{aligned} J^\lambda_{\mathcal{H}}f(z) &:= J_{\mathcal{H}}\left(J^{\lambda-1}_{\mathcal{H}}f(z)\right) \\ &:= z + \sum_{n=2}^{\infty} n^\lambda a_n z^n + (-1)^\lambda \sum_{n=2}^{\infty} n^\lambda \overline{b}_n \overline{z}^n \quad (z \in \mathbb{D}). \end{aligned}$$

For the analytic definition of the above case, see the Sălăgean operator [6].

We say that a function $f : \mathbb{D} \to \mathbb{C}$ is *subordinate* to a function $g : \mathbb{D} \to \mathbb{C}$, and write $f(z) \prec g(z)$ (or simply $f \prec g$), if there exists a complex-valued function w which maps $\mathbb{D}$ into itself with $w(0) = 0$, such that $f(z) = g(w(z))$; $z \in \mathbb{D}$. In particular, if g is univalent in $\mathbb{D}$, then $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

The Hadamard product (or convolution) of functions f_1 and f_2 of the form

$$f_k(z) = z + \sum_{n=2}^{\infty} a_{k,n} z^n + \sum_{n=2}^{\infty} \overline{b_{k,n} z^n} \quad \left(z \in \mathbb{D}, k \in \{1,2\}\right) \tag{2}$$

is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1,n} a_{2,n} z^n + \sum_{n=2}^{\infty} \overline{b_{1,n} b_{2,n} z^n} \quad (z \in \mathbb{D}).$$

For nonnegative integer $\lambda \in \{0,1,2,\ldots\}$ and for $-B \le A < B \le 1$ we define $\mathcal{H}^\lambda(A,B)$ to be the class of functions $f \in \mathcal{H}$ so that (also see Dziok [7, 8])

$$\frac{J_{\mathcal{H}}^{\lambda+1} f(z)}{J_{\mathcal{H}}^{\lambda} f(z)} \prec \frac{1+Az}{1+Bz} \tag{3}$$

and $\mathcal{G}^\lambda(A,B)$ to consist of functions $f \in \mathcal{H}$ so that

$$\frac{J_{\mathcal{H}}^{\lambda} f(z)}{z} \prec \frac{1+Az}{1+Bz}.$$

We remark that the classes $\mathcal{H}^0(A,B)$ and $\mathcal{G}^0(A,B)$for the analytic case, *i.e.* $g \equiv 0$, were introduced by Janowski [9] and the classes $\mathcal{S}_{\mathcal{H}}^*(\alpha) = \mathcal{H}^0(2\alpha-1,1)$ and $\mathcal{S}_{\mathcal{H}}^c(\alpha) = \mathcal{H}^1(2\alpha-1,1)$ for the harmonic case were investigated by Jahangiri [10, 11] and Silverman [12]. It is the aim of this paper to obtain necessary and sufficient convolution conditions, coefficient bounds, distortion theorems, radii of starlikeness and convexity, compactness, and extreme points for the above defined classes $\mathcal{H}^\lambda(A,B)$ and $\mathcal{G}^\lambda(A,B)$.

2 Analytic criteria

Our first theorem provides a necessary and sufficient convolution condition for the harmonic functions in $\mathcal{H}^\lambda(A,B)$.

Theorem 1 *A function f belongs to the class $\mathcal{H}^\lambda(A,B)$ if and only if $f \in \mathcal{H}$ and*

$$J_{\mathcal{H}}^{\lambda} f(z) * \phi(z;\zeta) \neq 0 \quad (\zeta \in \mathbb{C}, |\zeta| = 1),$$

where

$$\phi(z;\zeta) = \frac{(B-A)\zeta z + (1+A\zeta)z^2}{(1-z)^2} - (-1)^\lambda \frac{2\overline{z} + (A+B)\zeta\overline{z} - (1+A\zeta)\overline{z}^2}{(1-\overline{z})^2} \quad (z \in \mathbb{D}).$$

Proof Let $f \in \mathcal{H}$. Then $f \in \mathcal{H}^\lambda(A,B)$ if and only if the condition (3) holds or equivalently

$$\frac{J_{\mathcal{H}}^{\lambda+1} f(z)}{J_{\mathcal{H}}^{\lambda} f(z)} \neq \frac{1+A\zeta}{1+B\zeta} \quad (\zeta \in \mathbb{C}, |\zeta| = 1). \tag{4}$$

Now for $J_{\mathcal{H}}^{\lambda+1} h(z) = J_{\mathcal{H}}^{\lambda} h(z) * z/(1-z)^2$ and $J_{\mathcal{H}}^{\lambda} h(z) = J_{\mathcal{H}}^{\lambda} h(z) * z/(1-z)$, the inequality (4) yields

$$\begin{aligned}
&(1+B\zeta) J_{\mathcal{H}}^{\lambda+1} f(z) - (1+A\zeta) J_{\mathcal{H}}^{\lambda} f(z) \\
&\quad = (1+B\zeta) J_{\mathcal{H}}^{\lambda+1} h(z) - (1+A\zeta) J_{\mathcal{H}}^{\lambda} h(z) \\
&\qquad - (-1)^\lambda \left[(1+B\zeta) J_{\mathcal{H}}^{\lambda+1} \overline{g(z)} + (1+A\zeta) J_{\mathcal{H}}^{\lambda} \overline{g(z)} \right] \\
&\quad = J_{\mathcal{H}}^{\lambda} h(z) * \left(\frac{(1+B\zeta)z}{(1-z)^2} - \frac{(1+A\zeta)z}{1-z} \right) \\
&\qquad - (-1)^{\lambda+1} J_{\mathcal{H}}^{\lambda} \overline{g(z)} * \left(\frac{(1+B\zeta)\overline{z}}{(1-\overline{z})^2} + \frac{(1+A\zeta)\overline{z}}{1-\overline{z}} \right) \\
&\quad = J_{\mathcal{H}}^{\lambda} f(z) * \phi(z;\zeta) \neq 0.
\end{aligned}$$

□

A sufficient coefficient bound for the functions in $\mathcal{H}^\lambda(A,B)$ is provided in the following.

Theorem 2 *For $z \in \mathbb{D}$, the harmonic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n}$ is in $\mathcal{H}^{\lambda}(A,B)$ if*

$$\sum_{n=2}^{\infty} \bigl(\gamma_n |a_n| + \delta_n |b_n|\bigr) \le B - A, \tag{5}$$

where

$$\gamma_n = n^{\lambda}\bigl(n(1+B) - (1+A)\bigr) \quad \textit{and} \quad \delta_n = n^{\lambda}\bigl(n(1+B) + (1+A)\bigr). \tag{6}$$

Proof Clearly the theorem is true for $f(z) \equiv z$. So, we assume that $a_n \neq 0$ or $b_n \neq 0$ for $n \ge 2$. Since $\gamma_n \ge n(B-A)$ and $\delta_n \ge n(B-A)$ by (5) we have

$$\begin{aligned} \bigl|h'(z)\bigr| - \bigl|g'(z)\bigr| &\ge 1 - \sum_{n=2}^{\infty} n|a_n||z|^n - \sum_{n=2}^{\infty} n|b_n||z|^n \ge 1 - |z| \sum_{n=2}^{\infty} \bigl(n|a_n| + n|b_n|\bigr) \\ &\ge 1 - \frac{|z|}{B-A} \sum_{n=2}^{\infty} \bigl(\gamma_n |a_n| + \delta_n |b_n|\bigr) \ge 1 - |z| > 0 \quad (z \in \mathbb{D}). \end{aligned}$$

Therefore f is sense preserving and locally univalent in $\mathbb{D}$. Further, if $z_1, z_2 \in \mathbb{D}$ and we assume that $z_1 \neq z_2$, then

$$\left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| = \left|\sum_{m=1}^{n} z_1^{m-1} z_2^{n-m}\right| \le \sum_{m=1}^{n} |z_1|^{m-1} |z_2|^{n-m} < n \quad (n = 2, 3, \ldots).$$

Hence

$$\begin{aligned} \bigl|f(z_1) - f(z_2)\bigr| &\ge \bigl|h(z_1) - h(z_2)\bigr| - \bigl|g(z_1) - g(z_2)\bigr| \\ &\ge \left|z_1 - z_2 - \sum_{n=2}^{\infty} a_n\bigl(z_1^n - z_2^n\bigr)\right| - \left|\sum_{n=2}^{\infty} \overline{b_n\bigl(z_1^n - z_2^n\bigr)}\right| \\ &\ge |z_1 - z_2| - \sum_{n=2}^{\infty} |a_n|\bigl|z_1^n - z_2^n\bigr| - \sum_{n=2}^{\infty} |b_n|\bigl|z_1^n - z_2^n\bigr| \\ &= |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} |a_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| - \sum_{n=2}^{\infty} |b_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right|\right) \\ &> |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=2}^{\infty} n|b_n|\right) \ge 0. \end{aligned}$$

This proves that f is univalent in $\mathbb{D}$ *i.e.* $f \in \mathcal{H}$.

On the other hand, $f \in \mathcal{H}^{\lambda}(A,B)$ if and only if there exists a complex-valued function w, $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{D})$ such that

$$\frac{J_{\mathcal{H}}^{\lambda+1} f(z)}{J_{\mathcal{H}}^{\lambda} f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{D}),$$

or equivalently

$$\left|\frac{J_{\mathcal{H}}^{\lambda+1}f(z)-J_{\mathcal{H}}^{\lambda}f(z)}{BD_{\mathcal{H}}^{\lambda+1}f(z)-AD_{\mathcal{H}}^{\lambda}f(z)}\right|<1 \quad (z\in\mathbb{D}). \tag{7}$$

The above inequality (7) holds since for $|z|=r$ $(0<r<1)$ we obtain

$$\begin{aligned}
&\left|J_{\mathcal{H}}^{\lambda+1}f(z)-J_{\mathcal{H}}^{\lambda}f(z)\right|-\left|BD_{\mathcal{H}}^{\lambda+1}f(z)-AD_{\mathcal{H}}^{\lambda}f(z)\right| \\
&\quad=\left|\sum_{n=2}^{\infty}n^{\lambda}(n-1)a_nz^n-(-1)^{\lambda}\sum_{n=2}^{\infty}n^{\lambda}(n+1)\overline{b}_n\overline{z}^n\right| \\
&\qquad-\left|(B-A)z+\sum_{n=2}^{\infty}n^{\lambda}(Bn-A)a_nz^n+(-1)^{\lambda}\sum_{n=2}^{\infty}n^{\lambda}(Bn+A)\overline{b}_n\overline{z}^n\right| \\
&\quad\le\sum_{n=2}^{\infty}n^{\lambda}(n-1)|a_n|r^n+\sum_{n=2}^{\infty}n^{\lambda}(n+1)|b_n|r^n-(B-A)r \\
&\qquad+\sum_{n=2}^{\infty}n^{\lambda}(Bn-A)|a_n|r^n+\sum_{n=2}^{\infty}n^{\lambda}(Bn+A)|b_n|r^n \\
&\quad\le r\left\{\sum_{n=2}^{\infty}\bigl(\gamma_n|a_n|+\delta_n|b_n|\bigr)r^{n-1}-(B-A)\right\}<0,
\end{aligned}$$

and therefore $f\in\mathcal{H}^{\lambda}(A,B)$. □

Next we show that the condition (5) is also necessary for the functions $f\in\mathcal{H}$ to be in the class $\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B):=\mathcal{T}^{\lambda}\cap\mathcal{H}^{\lambda}(A,B)$ where $\mathcal{T}^{\lambda}$ is the class of functions $f=h+\overline{g}\in\mathcal{H}$ with varying coefficients (see [13, 14] or [15]) for which there exists a real number ϕ so that

$$f=h+\overline{g}=z-\sum_{n=2}^{\infty}e^{i(1-n)\phi}|a_n|z^n+(-1)^{\lambda}\sum_{n=2}^{\infty}e^{i(n-1)\phi}|b_n|\overline{z}^n \quad (z\in\mathbb{D}). \tag{8}$$

Theorem 3 *Let $f=h+\overline{g}$ be defined by* (8). *Then $f\in\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ if and only if the condition* (5) *holds*.

Proof The 'if' part follows from Theorem 2. For the 'only-if' part, assume that $f\in\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$, then by (7) we have

$$\left|\frac{\sum_{n=2}^{\infty}n^{\lambda}[(n-1)a_nz^{n-1}+(-1)^{\lambda}(n+1)\overline{b}_n\overline{z}^{n-1}]}{B-A-\sum_{n=2}^{\infty}n^{\lambda}[(Bn-A)a_nz^{n-1}+(-1)^{\lambda}(Bn+A)\overline{b}_n\overline{z}^{n-1}]}\right|<1 \quad (z\in\mathbb{D}).$$

For $|z|=r<1$ we obtain

$$\frac{\sum_{n=2}^{\infty}n^{\lambda}[(n-1)|a_n|+(n+1)|b_n|]r^{n-1}}{(B-A)-\sum_{n=2}^{\infty}n^{\lambda}[(Bn-A)|a_n|+(Bn+A)|b_n|]r^{n-1}}<1.$$

Thus, for γ_n and δ_n as defined by (6), we have

$$\sum_{n=2}^{\infty}\bigl(\gamma_n|a_n|+\delta_n|b_n|\bigr)r^{n-1}<B-A \quad (0\le r<1). \tag{9}$$

Let $\{\sigma_n\}$ be the sequence of partial sums of the series $\sum_{n=2}^{\infty}(\gamma_n|a_n| + \delta_n|b_n|)$. Then $\{\sigma_n\}$ is a nondecreasing sequence and by (9) it is bounded above by $B - A$. Thus, it is convergent and

$$\sum_{n=2}^{\infty}\bigl(\gamma_n|a_n| + \delta_n|b_n|\bigr) = \lim_{n\to\infty}\sigma_n \le B - A.$$

This gives the condition (5). □

A similar argument can be used to prove the following.

Theorem 4 *Let $f = h + \overline{g} \in \mathcal{H}$ be a function of the form* (8). *Then $f \in \mathcal{G}_{\mathcal{T}}^{\lambda}(A,B) := \mathcal{T}^{\lambda} \cap \mathcal{G}^{\lambda}(A,B)$ if and only if*

$$\sum_{n=2}^{\infty} n^{\lambda}\bigl(|a_n| + |b_n|\bigr) \le \frac{B - A}{1 + B}.$$

For special cases, Theorems 1, 3, and 4 yield the following corollaries.

Corollary 1 *Let $f = h + \overline{g} \in \mathcal{H}$. Then $f \in \mathcal{H}^{\lambda} := \mathcal{H}^{\lambda}(-1,1)$ if and only if*

$$J_{\mathcal{H}}^{\lambda} f(z) * \phi(z;\zeta) \ne 0 \quad \bigl(|\zeta| = 1\bigr),$$

where

$$\phi(z;\zeta) = \frac{2\zeta z + (1-\zeta)z^2}{(1-z)^2} - (-1)^{\lambda}\frac{2\overline{z} + (1-\zeta)\overline{z}^2}{(1-\overline{z})^2} \quad (z \in \mathbb{D}).$$

Corollary 2 *Let $f = h + \overline{g} \in \mathcal{H}$ be a function of the form* (8). *Then $f \in \mathcal{H}_{\mathcal{T}}^{\lambda}(\alpha) := \mathcal{H}_{\mathcal{T}}^{\lambda}(2\alpha - 1, 1)$ if and only if*

$$\sum_{n=2}^{\infty} n^{\lambda}\bigl((n-\alpha)|a_n| + (n+\alpha)|b_n|\bigr) \le 1.$$

Corollary 3 *Let $f = h + \overline{g} \in \mathcal{H}$ be a function of the form* (8). *Then $f \in \mathcal{H}_{\mathcal{T}}^{\lambda} := \mathcal{H}_{\mathcal{T}}^{\lambda}(0)$ if and only if*

$$\sum_{n=2}^{\infty} n^{\lambda+1}\bigl(|a_n| + |b_n|\bigr) \le 1,$$

i.e.

$$\mathcal{H}_{\mathcal{T}}^{\lambda} \equiv \mathcal{G}_{\mathcal{T}}^{\lambda+1} := \mathcal{G}_{\mathcal{T}}^{\lambda+1}(-1,1).$$

3 Extreme points

A function $f \in \mathcal{F} \subset \mathcal{H}$ is called *an extreme point of* $\mathcal{F}$ if $f = \mu f_1 + (1-\mu)f_2$ implies $f_1 = f_2 = f$ for all f_1 and f_2 in $\mathcal{F}$ and $0 < \mu < 1$. We shall use the notation $E\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F} \subset \mathcal{F}$.

We say that $\mathcal{F}$ is *locally uniformly bounded* if for each r, $0 < r < 1$, there is a real constant $M = M(r)$ so that $|f(z)| \le M$ where $f \in \mathcal{F}$ and $|z| \le r$.

We say that a class $\mathcal{F}$ is *convex* if $\mu f + (1-\mu)g \in \mathcal{F}$ for all f and g in $\mathcal{F}$ and $0 \le \mu \le 1$. Moreover, we define *the closed convex* hull of $\mathcal{F}$, denoted by $\overline{co}\mathcal{F}$, as the intersection of all closed convex subsets of $\mathcal{H}$ (with respect to the topology of locally uniform convergence) that contain $\mathcal{F}$.

A real-valued functional $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ is called *convex* on a convex class $\mathcal{F} \subset \mathcal{H}$ if $\mathcal{J}(\mu f + (1-\mu)g) \le \mu\mathcal{J}(f) + (1-\mu)\mathcal{J}(g)$ for all f and g in $\mathcal{F}$ and $0 \le \mu \le 1$.

The Krein-Milman theorem (see [16]) is fundamental in the theory of extreme points. In particular, it implies the following.

Lemma 1 *If $\mathcal{F}$ is a non-empty compact subclass of the class $\mathcal{H}$, then $E\mathcal{F}$ is non-empty and $\overline{co}E\mathcal{F} = \overline{co}\mathcal{F}$.*

Lemma 2 [7] *Let $\mathcal{F}$ be a non-empty compact convex subclass of the class $\mathcal{H}$ and $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ be a real-valued, continuous, and convex functional on $\mathcal{F}$. Then*

$$\max\{\mathcal{J}(f) : f \in \mathcal{F}\} = \max\{\mathcal{J}(f) : f \in E\mathcal{F}\}.$$

Since $\mathcal{H}$ is a complete metric space, Montel's theorem [17] implies the following.

Lemma 3 *A class $\mathcal{F} \subset \mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

Now, we are ready to state and prove our next theorem.

Theorem 5 *The class $\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ is a convex and compact subset of $\mathcal{H}$.*

Proof For $0 \le \mu \le 1$, let $f_1, f_2 \in \mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ be defined by (2). Then

$$\mu f_1(z) + (1-\mu)f_2(z) = z + \sum_{n=2}^{\infty}\left\{\left(\mu a_{1,n} + (1-\mu)a_{2,n}\right)z^n + \overline{\left(\mu b_{1,n} + (1-\mu)b_{2,n}\right)z^n}\right\}$$

and

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left\{\gamma_n\left|\mu a_{1,n} + (1-\mu)a_{2,n}\right| + \delta_n\left|\mu b_{1,n} + (1-\mu)b_{2,n}z^n\right|\right\} \\
&\quad\le \mu\sum_{n=2}^{\infty}\left\{\gamma_n|a_{1,n}| + \delta_n|b_{1,n}|\right\} + (1-\mu)\sum_{n=2}^{\infty}\gamma_n|a_{2,n}| + \delta_n|b_{2,n}| \\
&\quad\le \mu(B-A) + (1-\mu)(B-A) = B-A.
\end{aligned}$$

Thus, the function $\phi = \mu f_1 + (1-\mu)f_2$ belongs to the class $\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$. This means that the class $\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ is convex.

On the other hand, for $f \in \mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$, $|z| \le r$ and $0 < r < 1$, we have

$$\left|f(z)\right| \le r + \sum_{n=2}^{\infty}\left(|a_n| + |b_n|\right)r^n \le r + \sum_{n=2}^{\infty}\left(\gamma_n|a_n| + \delta_n|b_n|\right) \le r + (B-A).$$

Therefore, $\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ is locally uniformly bounded. Let

$$f_k(z) = z + \sum_{n=2}^{\infty} a_{k,n} z^n + \sum_{n=2}^{\infty} \overline{b_{k,n} z^n} \quad (z \in \mathbb{D}, k \in \mathbb{N})$$

and let $f = h + \overline{g}$ be given by (1). Using Theorem 3 we have

$$\sum_{n=2}^{\infty} \bigl(\gamma_n |a_{k,n}| + \delta_n |b_{k,n}|\bigr) \leq B - A \quad (k \in \mathbb{N}). \tag{10}$$

If we assume that $f_k \to f$, then we conclude that $|a_{k,n}| \to |a_n|$ and $|b_{k,n}| \to |b_n|$ as $k \to \infty$ ($n \in \mathbb{N}$). Let $\{\sigma_n\}$ be the sequence of partial sums of the series $\sum_{n=2}^{\infty}(\gamma_n|a_n| + \delta_n|b_n|)$. Then $\{\sigma_n\}$ is a nondecreasing sequence and by (10) it is bounded above by $B - A$. Thus, it is convergent and

$$\sum_{n=2}^{\infty} \bigl(\gamma_n |a_n| + \delta_n |b_n|\bigr) = \lim_{n \to \infty} \sigma_n \leq B - A.$$

Therefore, $f \in \mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$, and therefore the class $\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ is closed. In consequence, by Lemma 3, the class $\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ is compact subset of $\mathcal{H}$, which completes the proof. □

Our next theorem is on the extreme points of $\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$.

Theorem 6 *Extreme points of the class $\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ are the functions f of the form* (1) *where $h = h_n$ and $g = g_n$ are of the form*

$$\begin{aligned} &h_1(z) = z, \qquad h_n(z) = z - \frac{B-A}{\gamma_n} e^{i(1-n)\phi} z^n, \\ &g_n(z) = (-1)^{\lambda} \frac{B-A}{\delta_n} e^{i(n-1)\phi} \overline{z}^n \quad \bigl(z \in \mathbb{D}, n \in \{2,3,\ldots\}\bigr). \end{aligned} \tag{11}$$

Proof Let $g_n = \mu f_1 + (1-\mu) f_2$ where $0 < \mu < 1$ and $f_1, f_2 \in \mathcal{S}_{\mathcal{T}}^{\lambda}(A,B)$ are functions of the form (2). Then, by (5), we have $|b_{1,n}| = |b_{2,n}| = \frac{B-A}{\delta_n}$, and therefore $a_{1,k} = a_{2,k} = 0$ for $k \in \{2,3,\ldots\}$ and $b_{1,k} = b_{2,k} = 0$ for $k \in \{2,3,\ldots\} \setminus \{n\}$. It follows that $g_n = f_1 = f_2$ and consequently $g_n \in E\mathcal{S}_{\mathcal{T}}^{*}(A,B)$. Similarly, we can verify that the functions h_n of the form (11) are the extreme points of the class $\mathcal{S}_{\mathcal{T}}^{\lambda}(A,B)$.

Now, suppose that a function f of the form (1) belongs to the set $E\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ and f is not of the form (11). Then there exists $m \in \{2,3,\ldots\}$ such that

$$0 < |a_m| < \frac{B-A}{\alpha_m} \quad \text{or} \quad 0 < |b_m| < \frac{B-A}{\beta_m}.$$

If $0 < |a_m| < \frac{B-A}{\alpha_m}$, then putting

$$\mu = \frac{|a_m|\alpha_m}{B-A}, \qquad \phi = \frac{1}{1-\mu}(f - \mu h_m),$$

we have $0 < \mu < 1$, $h_m \neq \phi$, and

$$f = \mu h_m + (1-\mu)\phi.$$

Thus, $f \notin E\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$. Similarly, if $0 < |b_m| < \frac{B-A}{\delta_n}$, then putting

$$\mu = \frac{|b_m|\beta_m}{B-A}, \qquad \phi = \frac{1}{1-\mu}(f - \mu g_m),$$

we have $0 < \mu < 1$, $g_m \neq \phi$, and

$$f = \mu g_m + (1-\mu)\phi.$$

It follows that $f \notin E\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$, and so the proof is completed. □

It is clear that if the class $\mathcal{F} = \{f_n \in \mathcal{H} : n \in \mathbb{N}\}$ is locally uniformly bounded, then

$$\overline{co}\mathcal{F} = \left\{ \sum_{n=1}^{\infty} \mu_n f_n : \sum_{n=1}^{\infty} \mu_n = 1, \mu_n \geq 0 \ (n \in \mathbb{N}) \right\}.$$

Thus, by Theorem 6, we have the following.

Corollary 4 *Let h_n, g_n be defined by* (11). *Then*

$$\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B) = \left\{ \sum_{n=1}^{\infty} (\mu_n h_n + \delta_n g_n) : \sum_{n=1}^{\infty} (\mu_n + \delta_n) = 1, \delta_1 = 0, \mu_n, \delta_n \geq 0 \ (n \in \mathbb{N}) \right\}.$$

For all fixed values of $m, n, \lambda \in \mathbb{N}$, $z \in \mathbb{D}$, the following real-valued functionals are continuous and convex on $\mathcal{H}$:

$$\mathcal{J}(f) = |a_n|, \qquad \mathcal{J}(f) = |b_n|, \qquad \mathcal{J}(f) = |f(z)|, \qquad \mathcal{J}(f) = \left|J_{\mathcal{H}}^{\lambda} f(z)\right| \quad (f \in \mathcal{H}).$$

Moreover, for $\mu > 0$, $0 < r < 1$, the real-valued functional

$$\mathcal{J}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^{\mu} d\theta \right)^{1/\mu} \quad (f \in \mathcal{H})$$

is continuous on $\mathcal{H}$. For $\mu \geq 1$, by Minkowski's inequality it is also convex on $\mathcal{H}$.

Therefore, by Lemma 2 and Theorem 6, we have the following corollaries.

Corollary 5 *Let $f \in \mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ be a function of the form* (8). *Then*

$$|a_n| \leq \frac{B-A}{\gamma_n}, \qquad |b_n| \leq \frac{B-A}{\delta_n} \quad (n = 2, 3, \ldots),$$

where γ_n, δ_n are defined by (6). *The result is sharp and the functions h_n, g_n of the form* (11) *are the extremal functions.*

Corollary 6 *Let $f \in \mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ and $|z| = r < 1$. Then*

$$r - \frac{B-A}{2^{\lambda}(1+2B-A)} r^2 \leq \left|f(z)\right| \leq r + \frac{B-A}{2^{\lambda}(1+2B-A)} r^2,$$

$$r-\frac{B-A}{1+2B-A}r^2 \le \left|J_{\mathcal{H}}^{\lambda}f(z)\right| \le r+\frac{B-A}{1+2B-A}r^2 \quad (\lambda=1,2,3,\ldots).$$

The result is sharp and the function h_2 *of the form* (11) *is the extremal function.*

The following covering result follows from Corollary 6.

Corollary 7 *If* $f\in\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ *then* $\mathbb{D}(r)\subset f(\mathbb{D})$ *where*

$$r=1-\frac{B-A}{2^{\lambda}(1+2B-A)}.$$

We also conclude to the following.

Corollary 8 *Let* $0<r<1$ *and* $\mu\ge 1$. *If* $f\in\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ *then*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\mu}d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\left|h_2\left(re^{i\theta}\right)\right|^{\mu}d\theta,$$
$$\frac{1}{2\pi}\int_0^{2\pi}\left|J_{\mathcal{H}}^{\lambda}f(z)\right|^{\mu}d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\left|J_{\mathcal{H}}^{\lambda}h_2\left(re^{i\theta}\right)\right|^{\mu}d\theta \quad (\mu=1,2,\ldots).$$

4 Radii of starlikeness and convexity

Let $\mathcal{B}\subseteq\mathcal{H}$ and let $\mathbb{D}(r):=\{z\in\mathbb{C}:|z|<r\le 1\}$. We define the radius of starlikeness and the radius of convexity of the class $\mathcal{B}$, respectively, by

$$R_{\alpha}^{*}(\mathcal{B}):=\inf_{f\in\mathcal{B}}\left(\sup\left\{r\in(0,1]: f \text{ is starlike of order } \alpha \text{ in } \mathbb{D}(r)\right\}\right),$$
$$R_{\alpha}^{c}(\mathcal{B}):=\inf_{f\in\mathcal{B}}\left(\sup\left\{r\in(0,1]: f \text{ is convex of order } \alpha \text{ in } \mathbb{D}(r)\right\}\right).$$

At this point, for the case $\alpha=0$, it is easy to verify that

$$\mathcal{H}^{\lambda}(A,B)\subset\mathcal{H}^{\lambda}(-1,1)\subset\mathcal{H}^{\lambda-1}(-1,1)\subset\mathcal{S}_{\mathcal{H}}^{*}(0)\subset\mathcal{H},$$

and consequently

$$R_0^{*}\left(\mathcal{H}^{\lambda}(A,B)\right)=R_0^{*}\left(\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)\right)=R_0^{c}\left(\mathcal{H}^{\lambda}(A,B)\right)=R_0^{c}\left(\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)\right)=1 \quad (\lambda=2,3,\ldots).$$

In the following theorem we determine $R_{\alpha}^{*}(\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B))$ for $0\le\alpha<1$.

Theorem 7 *Let* $0\le\alpha<1$ *and* γ_n *and* δ_n *be defined by* (6). *Then*

$$R_{\alpha}^{*}\left(\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)\right)=\inf_{n\ge 2}\left(\frac{1-\alpha}{B-A}\min\left\{\frac{\gamma_n}{n-\alpha},\frac{\delta_n}{n+\alpha}\right\}\right)^{\frac{1}{n-1}}. \tag{12}$$

Proof Let $f\in\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)$ be of the form (8). Then, for $|z|=r<1$, we have

$$\left|\frac{J_{\mathcal{H}}f(z)-(1+\alpha)f(z)}{J_{\mathcal{H}}f(z)+(1-\alpha)f(z)}\right| = \left|\frac{-\alpha z+\sum_{n=2}^{\infty}((n-1-\alpha)a_nz^n-(n+1+\alpha)b_n\overline{z}^n)}{(2-\alpha)z+\sum_{n=2}^{\infty}((n+1-\alpha)a_nz^n-(n-1+\alpha)b_n\overline{z}^n)}\right|$$
$$\le\frac{\alpha+\sum_{n=2}^{\infty}((n-1-\alpha)|a_n|+(n+1+\alpha)|b_n|)r^{n-1}}{2-\alpha-\sum_{n=2}^{\infty}((n+1-\alpha)|a_n|+(n-1+\alpha)|b_n|)r^{n-1}}.$$

Note (see Jahangiri [11], Theorem 2) that f is starlike of order α in $\mathbb{D}(r)$ if and only if

$$\left|\frac{J_{\mathcal{H}}f(z)-(1+\alpha)f(z)}{J_{\mathcal{H}}f(z)+(1-\alpha)f(z)}\right|<1 \quad \bigl(z\in\mathbb{D}(r)\bigr)$$

or

$$\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n|+\frac{n+\alpha}{1-\alpha}|b_n|\right)r^{n-1}\leq 1. \tag{13}$$

Also, by Theorem 1, we have

$$\sum_{n=2}^{\infty}\left(\frac{\gamma_n}{B-A}|a_n|+\frac{\delta_n}{B-A}|b_n|\right)\leq 1,$$

where γ_n and δ_n are defined by (6).

Since $\gamma_n<\delta_n$ $(n=2,3,\ldots)$, the condition (13) is true if

$$\frac{n-\alpha}{1-\alpha}r^{n-1}\leq\frac{\gamma_n}{B-A} \quad\text{and}\quad \frac{n+\alpha}{1-\alpha}r^{n-1}\leq\frac{\delta_n}{B-A} \quad (n=2,3,\ldots),$$

or if

$$r\leq\left(\frac{1-\alpha}{B-A}\min\left\{\frac{\gamma_n}{n-\alpha},\frac{\delta_n}{n+\alpha}\right\}\right)^{\frac{1}{n-1}} \quad (n=2,3,\ldots).$$

It follows that the function f is starlike of order α in the disk $U(r^*)$ where

$$r^*:=\inf_{n\geq 2}\left(\frac{1-\alpha}{B-A}\min\left\{\frac{\gamma_n}{n-\alpha},\frac{\delta_n}{n+\alpha}\right\}\right)^{\frac{1}{n-1}}.$$

The function

$$f_n(z)=h_n(z)+\overline{g_n(z)}=z-\frac{B-A}{\gamma_n}e^{i(1-n)\phi}z^n+(-1)^{\lambda}\frac{B-A}{\delta_n}e^{i(n-1)\phi}\overline{z}^n$$

proves that the radius r^* cannot be any larger. Thus we have (12). □

Using a similar argument as above we obtain the following.

Theorem 8 *Let* $0\leq\alpha<1$ *and* γ_n *and* δ_n *be defined by* (6). *Then*

$$R_{\alpha}^{c}\bigl(\mathcal{H}_{\mathcal{T}}^{\lambda}(A,B)\bigr)=\inf_{n\geq 2}\left(\frac{1-\alpha}{B-A}\min\left\{\frac{\gamma_n}{n(n-\alpha)},\frac{\delta_n}{n(n+\alpha)}\right\}\right)^{\frac{1}{n-1}}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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