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Some generalized Riemann-Liouville k-fractional integral inequalities

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Abstract

The focus of the present study is to prove some new Pólya-Szegö type integral inequalities involving the generalized Riemann-Liouville k-fractional integral operator. These inequalities are used then to establish some fractional integral inequalities of Chebyshev type.

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1 Introduction and motivation

The celebrated functionals were introduced by the Chebyshev in his famous paper [1] and were subsequently rediscovered in various inequalities (for the celebrated functionals) by numerous authors, including Anastassiou [2], Belarbi and Dahmani [3], Dahmani *et al.* [4], Dragomir [5], Kalla and Rao [6], Lakshmikantham and Vatsala [7], Ntouyas *et al.* [8], Öğünmez and Özkan [9], Sudsutad *et al.* [10], Sulaiman [11]; and, for very recent work, see also Wang *et al.* [12]. This type of functionals is usually defined as

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.1}$$

where f and g are two integrable functions which are synchronous on $[a,b]$, *i.e.*,

$$\bigl(f(x)-f(y)\bigr)\bigl(g(x)-g(y)\bigr)\geq 0, \tag{1.2}$$

for any $x,y\in[a,b]$.

The well-known Grüss inequality [13] is defined by

$$\bigl|T(f,g)\bigr|\leq\frac{(M-m)(N-n)}{4}, \tag{1.3}$$

where f and g are two integrable functions which are synchronous on $[a,b]$ and satisfy the following inequalities:

$$m\leq f(x)\leq M \quad\text{and}\quad n\leq g(y)\leq N, \tag{1.4}$$

for all $x,y\in[a,b]$ and for some $m,M,n,N\in\mathbb{R}$.

Pólya and Szegö [14] introduced the following inequality:

$$\frac{\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx}{(\int_a^b f(x)\,dx \int_a^b g(x)\,dx)^2} \le \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2. \tag{1.5}$$

Dragomir and Diamond [15] by using the Pólya and Szegö inequality, proved that

$$\bigl|T(f,g)\bigr| \le \frac{(M-m)(N-n)}{4(b-a)^2\sqrt{mMnN}} \int_a^b f(x)\,dx \int_a^b g(x)\,dx, \tag{1.6}$$

where f and g are two positive integrable functions which are synchronous on $[a,b]$, and

$$0 < m \le f(x) \le M < \infty, \qquad 0 < n \le g(y) \le N < \infty, \tag{1.7}$$

for all $x, y \in [a,b]$ and for some $m, M, n, N \in \mathbb{R}$.

Recently, k-extensions of some familiar fractional integral operator like Riemann-Liouville have been investigated by many authors in interesting and useful manners (see [16–18], and [19]). Here, we begin with the following.

Definition 1.1 Let $k > 0$, then the generalized k-gamma and k-beta functions defined by [20]

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \tag{1.8}$$

where $(x)_{n,k}$, is the Pochhammer k-symbol defined by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots\bigl(x+(n-1)k\bigr) \quad (n \ge 1).$$

Definition 1.2 The k-gamma function is defined by

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}\,dt, \quad \Re(x) > 0.$$

It is well known that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function. Clearly

$$\Gamma(x) = \lim_{k\to 1}\Gamma_k(x), \quad \Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \quad \text{and} \quad \Gamma_k(x+k) = x\Gamma_k(x).$$

Definition 1.3 If $k > 0$, let $f \in L^1(a,b)$, $a \ge 0$, then the Riemann-Liouville k-fractional integral $R^\alpha_{a,k}$ of order $\alpha > 0$ for a real-valued continuous function $f(t)$ is defined by ([21]; see also [22])

$$R^\alpha_{a,k}\bigl\{f(t)\bigr\} = \frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}f(\tau)\,d\tau \quad \bigl(t \in [a,b]\bigr). \tag{1.9}$$

For $k = 1$, (1.9) is reduced to the classical Riemann-Liouville fractional integral.

Definition 1.4 If $k > 0$, let $f \in L^{1,r}[a,b]$, $a \geq 0$, $r \in \mathbb{R} \setminus \{-1\}$ then the generalized Riemann-Liouville k-fractional integral $R_{a,k}^{\alpha,r}$ of order $\alpha > 0$ for a real-valued continuous function $f(t)$ is defined by ([19])

$$R_{a,k}^{\alpha,r}\{f(t)\} = \frac{(1+r)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \left(t^{r+1} - \tau^{r+1}\right)^{\frac{\alpha}{k}-1} f(\tau)\,d\tau \quad \left(t \in [a,b]\right), \tag{1.10}$$

where Γ_k is the Euler gamma k-function.

The generalized Riemann-Liouville k-fractional integral (1.10) has the properties

$$R_{a,k}^{\alpha,r}\left\{R_{a,k}^{\beta,r}f(t)\right\} = R_{a,k}^{\alpha+\beta,r}\left\{f(t)\right\} = R_{a,k}^{\beta,r}\left\{R_{a,k}^{\alpha,r}f(t)\right\} \tag{1.11}$$

and

$$R_{a,k}^{\alpha,r}\{1\} = \frac{(t^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}, \quad \alpha > 0. \tag{1.12}$$

In this paper, we derive some new Pólya-Szegö type inequalities by making use of the generalized Riemann-Liouville k-fractional integral operators and then use them to establish some Chebyshev type integral inequalities.

We organize the paper as follows: in Section 2, we prove some generalized Pólya-Szegö type integral inequalities involving the generalized Riemann-Liouville k-fractional integral operators that we need to establish main theorems in the sequel and Section 3 contains some Chebyshev type integral inequalities via generalized Riemann-Liouville k-fractional integral operators.

2 Some Pólya-Szegö types inequalities

In this section, we prove some Pólya-Szegö type integral inequalities for positive integrable functions involving the generalized Riemann-Liouville k-fractional integral operator (1.10).

Lemma 2.1 *Let f and g be two positive integrable functions on $[a,\infty)$. Assume that there exist four positive integrable functions φ_1, φ_2, ψ_1, and ψ_2 on $[a,\infty)$ such that*:

(H_1) $0 < \varphi_1(\tau) \leq f(\tau) \leq \varphi_2(\tau)$, $0 < \psi_1(\tau) \leq g(\tau) \leq \psi_2(\tau)$ $(\tau \in [a,t], t > a)$.

Then, for $t > a$, $k > 0$, $a \geq 0$, $\alpha > 0$, and $r \in \mathbb{R} \setminus \{-1\}$, the following inequality holds:

$$\frac{R_{a,k}^{\alpha,r}\{\psi_1\psi_2 f^2\}(t) R_{a,k}^{\alpha,r}\{\varphi_1\varphi_2 g^2\}(t)}{(R_{a,k}^{\alpha,r}\{(\varphi_1\psi_1 + \varphi_2\psi_2)fg\}(t))^2} \leq \frac{1}{4}. \tag{2.1}$$

Proof From (H_1), for $\tau \in [a,t]$, $t > a$, we have

$$\frac{f(\tau)}{g(\tau)} \leq \frac{\varphi_2(\tau)}{\psi_1(\tau)}, \tag{2.2}$$

which yields

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right) \geq 0. \tag{2.3}$$

Analogously, we have

$$\frac{\varphi_1(\tau)}{\psi_2(\tau)} \le \frac{f(\tau)}{g(\tau)}, \tag{2.4}$$

from which one has

$$\left(\frac{f(\tau)}{g(\tau)} - \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right) \ge 0. \tag{2.5}$$

Multiplying (2.3) and (2.5), we obtain

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)} - \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right) \ge 0,$$

or

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} + \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right)\frac{f(\tau)}{g(\tau)} \ge \frac{f^2(\tau)}{g^2(\tau)} + \frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\tau)\psi_2(\tau)}. \tag{2.6}$$

The inequality (2.6) can be written as

$$\bigl(\varphi_1(\tau)\psi_1(\tau) + \varphi_2(\tau)\psi_2(\tau)\bigr)f(\tau)g(\tau) \ge \psi_1(\tau)\psi_2(\tau)f^2(\tau) + \varphi_1(\tau)\varphi_2(\tau)g^2(\tau). \tag{2.7}$$

Now, multiplying both sides of (2.7) by $\frac{(1+r)^{1-\frac{\alpha}{k}}(t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}$ and integrating with respect to τ from a to t, we get

$$R_{a,k}^{\alpha,r}\bigl\{(\varphi_1\psi_1 + \varphi_2\psi_2)fg\bigr\}(t) \ge R_{a,k}^{\alpha,r}\bigl\{\psi_1\psi_2 f^2\bigr\}(t) + R_{a,k}^{\alpha,r}\bigl\{\varphi_1\varphi_2 g^2\bigr\}(t).$$

Applying the AM-GM inequality, *i.e.*, $a + b \ge 2\sqrt{ab}$, $a, b \in \mathbb{R}^+$, we have

$$R_{a,k}^{\alpha,r}\bigl\{(\varphi_1\psi_1 + \varphi_2\psi_2)fg\bigr\}(t) \ge 2\sqrt{R_{a,k}^{\alpha,r}\bigl\{\psi_1\psi_2 f^2\bigr\}(t)R_{a,k}^{\alpha,r}\bigl\{\varphi_1\varphi_2 g^2\bigr\}(t)},$$

which leads to

$$R_{a,k}^{\alpha,r}\bigl\{\psi_1\psi_2 f^2\bigr\}(t)R_{a,k}^{\alpha,r}\bigl\{\varphi_1\varphi_2 g^2\bigr\}(t) \le \frac{1}{4}\bigl(R_{a,k}^{\alpha,r}\bigl\{(\varphi_1\psi_1 + \varphi_2\psi_2)fg\bigr\}(t)\bigr)^2.$$

Therefore, we obtain the inequality (2.1) as required. □

Lemma 2.2 *Let f and g be two positive integrable functions on $[a,\infty)$. Assume that there exist four positive integrable functions φ_1, φ_2, ψ_1, and ψ_2 satisfying (H_1) on $[a,\infty)$. Then, for $t > a$, $k > 0$, $a \ge 0$, $\alpha > 0$, $\beta > 0$, and $r \in \mathbb{R} \setminus \{-1\}$, the following inequality holds*:

$$\frac{R_{a,k}^{\alpha,r}\{\varphi_1\varphi_2\}(t)R_{a,k}^{\beta,r}\{\psi_1\psi_2\}(t)R_{a,k}^{\alpha,r}\{f^2\}(t)R_{a,k}^{\beta,r}\{g^2\}(t)}{(R_{a,k}^{\alpha,r}\{\varphi_1 f\}(t)R_{a,k}^{\beta,r}\{\psi_1 g\}(t) + R_{a,k}^{\alpha,r}\{\varphi_2 f\}(t)R_{a,k}^{\beta,r}\{\psi_2 g\}(t))^2} \le \frac{1}{4}. \tag{2.8}$$

Proof To prove (2.8), using the condition (H_1), we obtain

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\rho)} - \frac{f(\tau)}{g(\rho)}\right) \ge 0 \tag{2.9}$$

and

$$\left(\frac{f(\tau)}{g(\rho)} - \frac{\varphi_1(\tau)}{\psi_2(\rho)}\right) \geq 0, \tag{2.10}$$

which imply that

$$\left(\frac{\varphi_1(\tau)}{\psi_2(\rho)} + \frac{\varphi_2(\tau)}{\psi_1(\rho)}\right)\frac{f(\tau)}{g(\rho)} \geq \frac{f^2(\tau)}{g^2(\rho)} + \frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\rho)\psi_2(\rho)}. \tag{2.11}$$

Multiplying both sides of (2.11) by $\psi_1(\rho)\psi_2(\rho)g^2(\rho)$, we have

$$\begin{aligned}&\varphi_1(\tau)f(\tau)\psi_1(\rho)g(\rho) + \varphi_2(\tau)f(\tau)\psi_2(\rho)g(\rho)\\ &\quad\geq \psi_1(\rho)\psi_2(\rho)f^2(\tau) + \varphi_1(\tau)\varphi_2(\tau)g^2(\rho).\end{aligned} \tag{2.12}$$

Multiplying both sides of (2.12) by

$$\frac{(1+r)^{1-\frac{\alpha}{k}}(1+r)^{1-\frac{\beta}{k}}(t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}(t^{r+1}-\rho^{r+1})^{\frac{\beta}{k}-1}}{k\Gamma_k(\alpha)k\Gamma_k(\beta)},$$

and double integrating with respect to τ and ρ from a to t, we have

$$\begin{aligned}&R_{a,k}^{\alpha,r}\{\varphi_1 f\}(t)R_{a,k}^{\beta,r}\{\psi_1 g\}(t) + R_{a,k}^{\alpha,r}\{\varphi_2 f\}(t)R_{a,k}^{\beta,r}\{\psi_2 g\}(t)\\ &\quad\geq R_{a,k}^{\alpha,r}\{f^2\}(t)R_{a,k}^{\beta,r}\{\psi_1\psi_2\}(t) + R_{a,k}^{\alpha,r}\{\varphi_1\varphi_2\}(t)R_{a,k}^{\beta,r}\{g^2\}(t).\end{aligned}$$

Applying the AM-GM inequality, we get

$$\begin{aligned}&R_{a,k}^{\alpha,r}\{\varphi_1 f\}(t)R_{a,k}^{\beta,r}\{\psi_1 g\}(t) + R_{a,k}^{\alpha,r}\{\varphi_2 f\}(t)R_{a,k}^{\beta,r}\{\psi_2 g\}(t)\\ &\quad\geq 2\sqrt{R_{a,k}^{\alpha,r}\{f^2\}(t)R_{a,k}^{\beta,r}\{\psi_1\psi_2\}(t)R_{a,k}^{\alpha,r}\{\varphi_1\varphi_2\}(t)R_{a,k}^{\beta,r}\{g^2\}(t)},\end{aligned}$$

which leads to the desired inequality in (2.8). The proof is completed. □

Lemma 2.3 *Let f and g be two positive integrable functions on $[a,\infty)$. Assume that there exist four positive integrable functions φ_1, φ_2, ψ_1, and ψ_2 satisfying (H_1) on $[a,\infty)$. Then, for $t > a$, $k > 0$, $a \geq 0$, $\alpha > 0$, $\beta > 0$, and $r \in \mathbb{R} \setminus \{-1\}$, the following inequality holds*:

$$R_{a,k}^{\alpha,r}\{f^2\}(t)R_{a,k}^{\beta,r}\{g^2\}(t) \leq R_{a,k}^{\alpha,r}\{(\varphi_2 fg)/\psi_1\}(t)R_{a,k}^{\beta,r}\{(\psi_2 fg)/\varphi_1\}(t). \tag{2.13}$$

Proof From (2.2), we have

$$\begin{aligned}&\frac{(1+r)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}f^2(\tau)\,d\tau\\ &\quad\leq \frac{(1+r)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\frac{\varphi_2(\tau)}{\psi_1(\tau)}f(\tau)g(\tau)\,d\tau,\end{aligned}$$

which implies

$$R_{a,k}^{\alpha,r}\{f^2\}(t) \leq R_{a,k}^{\alpha,r}\{(\varphi_2 fg)/\psi_1\}(t). \tag{2.14}$$

By (2.4), we get

$$\begin{aligned}&\frac{(1+r)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}g^2(\rho)\,d\rho\\&\quad\le\frac{(1+r)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}\frac{\psi_2(\rho)}{\varphi_1(\rho)}f(\rho)g(\rho)\,d\rho,\end{aligned}$$

from which one has

$$R_{0,t}^{\beta}\left\{g^2\right\}(t)\le R_{a,k}^{\beta,r}\left\{(\psi_2 fg)/\varphi_1\right\}(t). \tag{2.15}$$

Multiplying (2.14) and (2.15), we get the desired inequality in (2.13). □

Corollary 2.1 *Let f and g be two positive integrable functions on $[0,\infty)$ satisfying*

(H_2) $0<m\le f(\tau)\le M<\infty$, $0<n\le g(\tau)\le N<\infty$ ($\tau\in[a,t]$, $t>a$).

Then, for $t>a$, $k>0$, $a\ge 0$, $\alpha>0$, $\beta>0$, and $r\in\mathbb{R}\setminus\{-1\}$, we have

$$\frac{(R_{a,k}^{\alpha,r}\{f^2\}(t))(R_{a,k}^{\beta,r}\{g^2\}(t))}{R_{a,k}^{\alpha,r}\{fg\}(t)R_{a,k}^{\beta,r}\{fg\}(t)}\le\frac{MN}{mn}. \tag{2.16}$$

3 Chebyshev type integral inequalities

In the sequel, we establish our main Chebyshev type integral inequalities involving the generalized Riemann-Liouville k-fractional integral operator (1.10), with the help of the Pólya-Szegö fractional integral inequality in Lemma 2.1 as follows.

Theorem 3.1 *Let f and g be two positive integrable functions on $[a,\infty)$, $a\ge 0$. Assume that there exist four positive integrable functions φ_1, φ_2, ψ_1, and ψ_2 satisfying (H_1). Then, for $t>a$, $k>0$, $a\ge 0$, $\alpha>0$, and $r\in\mathbb{R}\setminus\{-1\}$, the following inequality is fulfilled*:

$$\begin{aligned}&\left|\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}R_{a,k}^{\alpha,r}\{fg\}(t)-R_{a,k}^{\alpha,r}\{f\}(t)R_{a,k}^{\alpha,r}\{g\}(t)\right|\\&\quad\le\left|G(f,\varphi_1,\varphi_2)(t)G(g,\psi_1,\psi_2)(t)\right|^{\frac{1}{2}},\end{aligned} \tag{3.1}$$

where

$$G(u,v,w)(t)=\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{4(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\frac{(R_{a,k}^{\alpha,r}\{(v+w)u\}(t))^2}{R_{a,k}^{\alpha,r}\{vw\}(t)}-\left(R_{a,k}^{\alpha,r}\{u\}(t)\right)^2. \tag{3.2}$$

Proof Let f and g be two positive integrable functions on $[a,\infty)$. For $\tau,\rho\in(a,t)$ with $t>a$, we define $A(\tau,\rho)$ as

$$A(\tau,\rho)=\left(f(\tau)-f(\rho)\right)\left(g(\tau)-g(\rho)\right), \tag{3.3}$$

or, equivalently,

$$A(\tau,\rho)=f(\tau)g(\tau)+f(\rho)g(\rho)-f(\tau)g(\rho)-f(\rho)g(\tau). \tag{3.4}$$

Multiplying both sides of (3.4) by $\frac{(1+r)^{2(1-\frac{\alpha}{k})}(t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}(t^{r+1}-\rho^{r+1})^{\frac{\alpha}{k}-1}}{(k\Gamma_k(\alpha))^2}$ and double integrating with respect to τ and ρ from a to t, we get

$$\begin{aligned}&\frac{(1+r)^{2(1-\frac{\alpha}{k})}}{(k\Gamma_k(\alpha))^2}\int_a^t\int_a^t\bigl(t^{r+1}-\tau^{r+1}\bigr)^{\frac{\alpha}{k}-1}\bigl(t^{r+1}-\rho^{r+1}\bigr)^{\frac{\alpha}{k}-1}A(\tau,\rho)\,d\tau\,d\rho\\&\quad=2\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}R_{a,k}^{\alpha,r}\{fg\}(t)-2\bigl(R_{a,k}^{\alpha,r}\{g\}(t)\bigr)\bigl(R_{a,k}^{\alpha,r}\{f\}(t)\bigr).\end{aligned}\tag{3.5}$$

By using the Cauchy-Schwartz inequality for double integrals, we have

$$\begin{aligned}&\left|\frac{(1+r)^{2(1-\frac{\alpha}{k})}}{(k\Gamma_k(\alpha))^2}\int_a^t\int_a^t\bigl(t^{r+1}-\tau^{r+1}\bigr)^{\frac{\alpha}{k}-1}\bigl(t^{r+1}-\rho^{r+1}\bigr)^{\frac{\alpha}{k}-1}A(\tau,\rho)\,d\tau\,d\rho\right|\\&\quad\le\Biggl[\frac{(1+r)^{2(1-\frac{\alpha}{k})}}{(k\Gamma_k(\alpha))^2}\int_a^t\int_a^t\bigl(t^{r+1}-\tau^{r+1}\bigr)^{\frac{\alpha}{k}-1}\bigl(t^{r+1}-\rho^{r+1}\bigr)^{\frac{\alpha}{k}-1}f^2(\tau)\,d\tau\,d\rho\\&\qquad+\frac{(1+r)^{2(1-\frac{\alpha}{k})}}{(k\Gamma_k(\alpha))^2}\int_a^t\int_a^t\bigl(t^{r+1}-\tau^{r+1}\bigr)^{\frac{\alpha}{k}-1}\bigl(t^{r+1}-\rho^{r+1}\bigr)^{\frac{\alpha}{k}-1}f^2(\rho)\,d\tau\,d\rho\\&\qquad-2\frac{(1+r)^{2(1-\frac{\alpha}{k})}}{(k\Gamma_k(\alpha))^2}\int_a^t\int_a^t\bigl(t^{r+1}-\tau^{r+1}\bigr)^{\frac{\alpha}{k}-1}\bigl(t^{r+1}-\rho^{r+1}\bigr)^{\frac{\alpha}{k}-1}f(\tau)f(\rho)\,d\tau\,d\rho\Biggr]^{\frac{1}{2}}\\&\qquad\times\Biggl[\frac{(1+r)^{2(1-\frac{\alpha}{k})}}{(k\Gamma_k(\alpha))^2}\int_a^t\int_a^t\bigl(t^{r+1}-\tau^{r+1}\bigr)^{\frac{\alpha}{k}-1}\bigl(t^{r+1}-\rho^{r+1}\bigr)^{\frac{\alpha}{k}-1}g^2(\tau)\,d\tau\,d\rho\\&\qquad+\frac{(1+r)^{2(1-\frac{\alpha}{k})}}{(k\Gamma_k(\alpha))^2}\int_a^t\int_a^t\bigl(t^{r+1}-\tau^{r+1}\bigr)^{\frac{\alpha}{k}-1}\bigl(t^{r+1}-\rho^{r+1}\bigr)^{\frac{\alpha}{k}-1}g^2(\rho)\,d\tau\,d\rho\\&\qquad-2\frac{(1+r)^{2(1-\frac{\alpha}{k})}}{(k\Gamma_k(\alpha))^2}\int_a^t\int_a^t\bigl(t^{r+1}-\tau^{r+1}\bigr)^{\frac{\alpha}{k}-1}\bigl(t^{r+1}-\rho^{r+1}\bigr)^{\frac{\alpha}{k}-1}g(\tau)g(\rho)\,d\tau\,d\rho\Biggr]^{\frac{1}{2}}.\end{aligned}\tag{3.6}$$

Therefore, we obtain

$$\begin{aligned}&\left|\frac{(1+r)^{2(1-\frac{\alpha}{k})}}{(k\Gamma_k(\alpha))^2}\int_a^t\int_a^t\bigl(t^{r+1}-\tau^{r+1}\bigr)^{\frac{\alpha}{k}-1}\bigl(t^{r+1}-\rho^{r+1}\bigr)^{\frac{\alpha}{k}-1}A(\tau,\rho)\,d\tau\,d\rho\right|\\&\quad\le2\biggl[\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}R_{a,k}^{\alpha,r}\{f^2\}(t)-\bigl(R_{a,k}^{\alpha,r}\{f\}(t)\bigr)^2\biggr]^{\frac{1}{2}}\\&\qquad\times\biggl[\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}R_{a,k}^{\alpha,r}\{g^2\}(t)-\bigl(R_{a,k}^{\alpha,r}\{g\}(t)\bigr)^2\biggr]^{\frac{1}{2}}.\end{aligned}\tag{3.7}$$

By applying Lemma 2.1, for $\psi_1(t)=\psi_2(t)=g(t)=1$, we get

$$R_{a,k}^{\alpha,r}\{f^2\}(t)\le\frac{1}{4}\frac{(R_{a,k}^{\alpha,r}\{(\varphi_1+\varphi_2)f\}(t))^2}{R_{a,k}^{\alpha,r}\{\varphi_1\varphi_2\}(t)},$$

which leads to

$$\begin{aligned}
&\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} R_{a,k}^{\alpha,r}\{f^2\}(t) - \left(R_{a,k}^{\alpha,r}\{f\}(t)\right)^2 \\
&\quad \le \frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{4(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} \frac{(R_{a,k}^{\alpha,r}\{(\varphi_1+\varphi_2)f\}(t))^2}{R_{a,k}^{\alpha,r}\{\varphi_1\varphi_2\}(t)} - \left(R_{a,k}^{\alpha,r}\{f\}(t)\right)^2 \\
&\quad = G(f,\varphi_1,\varphi_2)(t).
\end{aligned} \tag{3.8}$$

Similarly, we get

$$\begin{aligned}
&\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} R_{a,k}^{\alpha,r}\{g^2\}(t) - \left(R_{a,k}^{\alpha,r}\{g\}(t)\right)^2 \\
&\quad \le \frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{4(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} \frac{(R_{a,k}^{\alpha,r}\{(\psi_1+\psi_2)g\}(t))^2}{R_{a,k}^{\alpha,r}\{\psi_1\psi_2\}(t)} - \left(R_{a,k}^{\alpha,r}\{g\}(t)\right)^2 \\
&\quad = G(g,\psi_1,\psi_2)(t).
\end{aligned} \tag{3.9}$$

Finally, combining (3.5), (3.7), (3.8), and (3.9), we arrive at the desired result in (3.1). This completes the proof. □

Remark 3.2 If $\varphi_1 = m$, $\varphi_2 = M$, $\psi_1 = n$, and $\psi_2 = N$, then we have

$$G(f,m,M)(t) = \frac{(M-m)^2}{4mM}\left(R_{a,k}^{\alpha,r}\{f\}(t)\right)^2, \tag{3.10}$$

$$G(g,n,N)(t) = \frac{(N-n)^2}{4nN}\left(R_{a,k}^{\alpha,r}\{g\}(t)\right)^2. \tag{3.11}$$

Theorem 3.3 *Let f and g be two positive integrable functions on $[a,\infty)$, $a \ge 0$. Assume that there exist four positive integrable functions φ_1, φ_2, ψ_1, and ψ_2 satisfying (H_1). Then, for $t > a$, $k > 0$, $a \ge 0$, $\alpha > 0$, $\beta > 0$, and $r \in \mathbb{R} \setminus \{-1\}$, the following inequality is true*:

$$\begin{aligned}
&\Bigg| \frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} R_{a,k}^{\beta,r}\{fg\}(t) + \frac{(t^{r+1}-a^{r+1})^{\frac{\beta}{k}}}{(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)} R_{a,k}^{\alpha,r}\{fg\}(t) \\
&\quad - R_{a,k}^{\alpha,r}\{f\}(t)R_{a,k}^{\beta,r}\{g\}(t) - R_{a,k}^{\alpha,r}\{g\}(t)R_{a,k}^{\beta,r}\{f\}(t) \Bigg| \\
&\quad \le \left|G_1(f,\varphi_1,\varphi_2)(t) + G_2(f,\varphi_1,\varphi_2)(t)\right|^{\frac{1}{2}} \\
&\qquad \times \left|G_1(g,\psi_1,\psi_2)(t) + G_1(g,\psi_1,\psi_2)(t)\right|^{\frac{1}{2}},
\end{aligned} \tag{3.12}$$

where

$$G_1(u,v,w)(t) = \frac{(t^{r+1}-a^{r+1})^{\frac{\beta}{k}}}{4(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)} \frac{(R_{a,k}^{\alpha,r}\{(v+w)u\}(t))^2}{R_{a,k}^{\alpha,r}\{vw\}(t)} - R_{a,k}^{\alpha,r}\{u\}(t)R_{a,k}^{\beta,r}\{u\}(t),$$

$$G_2(u,v,w)(t) = \frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{4(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} \frac{(R_{a,k}^{\beta,r}\{(v+w)u\}(t))^2}{R_{a,k}^{\beta,r}\{vw\}(t)} - R_{a,k}^{\alpha,r}\{u\}(t)R_{a,k}^{\beta,r}\{u\}(t).$$

Proof Multiplying both sides of (3.4) by $\frac{(1+r)^{2-\frac{\alpha+\beta}{k}}(t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}(t^{r+1}-\rho^{r+1})^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$ and double integrating with respect to τ and ρ from a to t, we obtain

$$\begin{aligned}
&\frac{(1+r)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}A(\tau,\rho)\,d\tau\,d\rho\\
&\quad=\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{4(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}R_{a,k}^{\beta,r}\{fg\}(t)+\frac{(t^{r+1}-a^{r+1})^{\frac{\beta}{k}}}{(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)}R_{a,k}^{\alpha,r}\{fg\}(t)\\
&\qquad-R_{a,k}^{\alpha,r}\{f\}(t)R_{a,k}^{\beta,r}\{g\}(t)-R_{a,k}^{\beta,r}\{f\}(t)R_{a,k}^{\alpha,r}\{g\}(t).
\end{aligned}\tag{3.13}$$

By using the Cauchy-Schwartz inequality for double integrals, we have

$$\begin{aligned}
&\left|\frac{(1+r)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}A(\tau,\rho)\,d\tau\,d\rho\right|\\
&\quad\leq\left[\frac{(1+r)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}f^2(\tau)\,d\tau\,d\rho\right.\\
&\qquad+\frac{(1+r)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}f^2(\rho)\,d\tau\,d\rho\\
&\qquad\left.-2\frac{(1+r)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}f(\rho)\,d\tau\,d\rho\right]^{\frac{1}{2}}\\
&\qquad\times\left[\frac{(1+r)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}g^2(\tau)\,d\tau\,d\rho\right.\\
&\qquad+\frac{(1+r)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}g^2(\rho)\,d\tau\,d\rho\\
&\qquad\left.-2\frac{(1+r)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}g(\tau)g(\rho)\,d\tau\,d\rho\right]^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\left|\frac{(1+r)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\beta}{k}-1}A(\tau,\rho)\,d\tau\,d\rho\right|\\
&\quad\leq\left[\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}R_{a,k}^{\beta,r}\left\{f^2(t)\right\}+\frac{(t^{r+1}-a^{r+1})^{\frac{\beta}{k}}}{(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)}R_{a,k}^{\alpha,r}\left\{f^2(t)\right\}\right.\\
&\qquad\left.-2R_{a,k}^{\beta,r}\left\{f(t)\right\}R_{a,k}^{\alpha,r}\left\{f(t)\right\}\right]^{\frac{1}{2}}\\
&\qquad\times\left[\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}R_{a,k}^{\beta,r}\left\{g^2(t)\right\}+\frac{(t^{r+1}-a^{r+1})^{\frac{\beta}{k}}}{(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)}R_{a,k}^{\alpha,r}\left\{g^2(t)\right\}\right.\\
&\qquad\left.-2R_{a,k}^{\beta,r}\left\{g(t)\right\}R_{a,k}^{\alpha,r}\left\{g(t)\right\}\right]^{\frac{1}{2}}.
\end{aligned}\tag{3.14}$$

Applying Lemma 2.1 with $\psi_1(t) = \psi_2(t) = g(t) = 1$, we have

$$\frac{(t^{r+1} - a^{r+1})^{\frac{\beta}{k}}}{(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta + k)} R_{a,k}^{\alpha,r}\{f^2\}(t) \le \frac{(t^{r+1} - a^{r+1})^{\frac{\beta}{k}}}{4(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta + k)} \frac{(R_{a,k}^{\alpha,r}\{(\varphi_1 + \varphi_2)f\}(t))^2}{R_{a,k}^{\alpha,r}\{\varphi_1\varphi_2\}(t)}.$$

This implies that

$$\begin{aligned}
&\frac{(t^{r+1} - a^{r+1})^{\frac{\beta}{k}}}{(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta + k)} R_{a,k}^{\alpha,r}\{f^2\}(t) - R_{a,k}^{\alpha,r}\{f\}(t)R_{a,k}^{\beta,r}\{f\}(t) \\
&\quad\le \frac{(t^{r+1} - a^{r+1})^{\frac{\beta}{k}}}{4(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta + k)} \frac{(R_{a,k}^{\alpha,r}\{(\varphi_1 + \varphi_2)f\}(t))^2}{R_{a,k}^{\alpha,r}\{\varphi_1\varphi_2\}(t)} - R_{a,k}^{\alpha,r}\{f\}(t)R_{a,k}^{\beta,r}\{f\}(t) \\
&\quad= G_1(f, \varphi_1, \varphi_2)(t)
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&\frac{(t^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha + k)} R_{a,k}^{\beta,r}\{f^2\}(t) - R_{a,k}^{\alpha,r}\{f\}(t)R_{a,k}^{\beta,r}\{f\}(t) \\
&\quad\le \frac{(t^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}{4(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha + k)} \frac{(R_{a,k}^{\beta,r}\{(\varphi_1 + \varphi_2)f\}(t))^2}{R_{0,t}^{\beta}\{\varphi_1\varphi_2\}(t)} - R_{a,k}^{\alpha,r}\{f\}(t)R_{a,k}^{\beta,r}\{f\}(t) \\
&\quad= G_2(f, \varphi_1, \varphi_2)(t).
\end{aligned} \tag{3.16}$$

Also, applying the same procedure with $\phi_1(t) = \phi_2(t) = f(t) = 1$, we get

$$\begin{aligned}
&\frac{(t^{r+1} - a^{r+1})^{\frac{\beta}{k}}}{(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta + k)} R_{a,k}^{\alpha,r}\{g^2\}(t) - R_{a,k}^{\alpha,r}\{g\}(t)R_{a,k}^{\beta,r}\{g\}(t) \\
&\quad\le G_1(g, \psi_1, \psi_2)(t)
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&\frac{(t^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha + k)} R_{a,k}^{\beta,r}\{g^2\}(t) - R_{a,k}^{\alpha,r}\{g\}(t)R_{a,k}^{\beta,r}\{g\}(t) \\
&\quad\le G_2(g, \psi_1, \psi_2)(t).
\end{aligned} \tag{3.18}$$

Finally, considering (3.13) to (3.18), we arrive at the desired result in (3.12). This completes the proof of Theorem 3.3. □

Remark 3.4 We conclude the present investigation by remarking that if we follow Sarikaya and Karaca [18] then our main results become the results recently given by Ntouyas *et al.* [8]. Similarly, after some parametric changes our results reduce to numerous well-known results presented in the literature.

4 Examples

In this section, we show some approximations of unknown functions by using four linear functions. Let us define the constants $m_1, m_2, M_1, M_2, n_1, n_2, N_1, N_2 \in \mathbb{R}$ such that

(H_3) $0 < m_1\tau + m_2 \le f(\tau) \le M_1\tau + M_2$, $0 < n_1\tau + n_2 \le g(\tau) \le N_1\tau + N_2$ ($\tau \in [a, t]$, $t > a$).

Proposition 4.1 *Suppose that f and g are two positive integrable functions on $[a,\infty)$, $a\geq 0$ satisfying (H_3). Then, for $t>a$, $k>0$, $a\geq 0$, $\alpha>0$, $\beta>0$, and $r\in\mathbb{R}\setminus\{-1\}$, we have*

$$\begin{aligned}&\bigl(n_1N_1R^{\alpha,r}_{a,k}\{\tau^2f^2\}(t)+(n_1N_2+n_2N_1)R^{\alpha,r}_{a,k}\{\tau f^2\}(t)+n_2N_2R^{\alpha,r}_{a,k}\{f^2\}(t)\bigr)\\&\quad\times\bigl(m_1M_1R^{\alpha,r}_{a,k}\{\tau^2g^2\}(t)+(m_1M_2+m_2M_1)R^{\alpha,r}_{a,k}\{\tau g^2\}(t)+m_2M_2R^{\alpha,r}_{a,k}\{g^2\}(t)\bigr)\\&\quad\leq\frac{1}{4}\bigl((m_1n_1+M_1N_1)R^{\alpha,r}_{a,k}\{\tau^2fg\}(t)+(m_1n_2+m_2n_1+M_1N_2+M_2N_1)R^{\alpha,r}_{a,k}\{\tau fg\}(t)\\&\quad\quad+(m_2n_2+M_2N_2)R^{\alpha,r}_{a,k}\{fg\}(t)\bigr)^2.\end{aligned}\tag{4.1}$$

Proof Setting $\varphi_1(\tau)=m_1\tau+m_2$, $\varphi_2(\tau)=M_1\tau+M_2$, $\psi_1(\tau)=n_1\tau+n_2$, and $\psi_2(\tau)=N_1\tau+N_2$, and applying Lemma 2.1, we obtain (4.1) as desired. □

Corollary 4.1 *Let all assumptions of Proposition 4.1 be fulfilled with $m_1=M_1=n_1=N_1=0$. Then, for $t>a$, $k>0$, $a\geq 0$, $\alpha>0$, $\beta>0$, and $r\in\mathbb{R}\setminus\{-1\}$, the following inequality holds*:

$$\frac{R^{\alpha,r}_{a,k}\{f^2\}(t)R^{\alpha,r}_{a,k}\{g^2\}(t)}{(R^{\alpha,r}_{a,k}\{fg\}(t))^2}\leq\frac{1}{4}\left(\sqrt{\frac{m_2n_2}{M_2N_2}}+\sqrt{\frac{M_2N_2}{m_2n_2}}\right)^2.\tag{4.2}$$

Proposition 4.2 *Suppose that f and g are two positive integrable functions on $[a,\infty)$, $a\geq 0$ satisfying (H_3). Then, for $t>a$, $k>0$, $a\geq 0$, $\alpha>0$, and $r\in\mathbb{R}\setminus\{-1\}$, we get the following inequality*:

$$\begin{aligned}&\left|\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}R^{\alpha,r}_{a,k}\{fg\}(t)-R^{\alpha,r}_{a,k}\{f\}(t)R^{\alpha,r}_{a,k}\{g\}(t)\right|\\&\quad\leq\bigl|G^*(f,m_1,m_2,M_1,M_2)(t)G^*(g,n_1,n_2,N_1,N_2)(t)\bigr|^{\frac{1}{2}},\end{aligned}\tag{4.3}$$

where

$$\begin{aligned}&G^*(u,v,w,x,y)(t)\\&\quad=\frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{4(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\cdot\frac{((v+x)R^{\alpha,r}_{a,k}\{\tau u\}(t)+(w+y)R^{\alpha,r}_{a,k}\{u\}(t))^2}{vxR^{\alpha,r}_{a,k}\{\tau^2\}(t)+(vy+wx)R^{\alpha,r}_{a,k}\{\tau\}(t)+wyR^{\alpha,r}_{a,k}\{1\}(t)}\\&\quad\quad-\bigl(R^{\alpha,r}_{a,k}\{u\}(t)\bigr)^2.\end{aligned}\tag{4.4}$$

Proof By setting $\varphi_1(\tau)$, $\varphi_2(\tau)$, $\psi_1(\tau)$, and $\psi_2(\tau)$ as in Proposition 4.1 and using Theorem 3.1, we get the inequality (4.3). □

Remark 4.3 If $m_1=M_1=n_1=N_1=0$, then we have

$$\begin{aligned}&G^*(f,0,m_2,0,M_2)(t)=G(f,m,M)(t),\\&G^*(g,0,n_2,0,N_2)(t)=G(g,n,N)(t),\end{aligned}$$

where $G(f,m,M)(t)$ and $G(g,n,N)(t)$ are defined by (3.10) and (3.11), respectively.

Proposition 4.4 *Assume that f and g are two positive integrable functions on $[a,\infty)$, $a \geq 0$ satisfying (H_3). Then, for $t > a$, $k > 0$, $a \geq 0$, $\alpha > 0$, $\beta > 0$, and $r \in \mathbb{R} \setminus \{-1\}$, we obtain the following estimate*:

$$\begin{aligned} &\left| \frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} R_{a,k}^{\beta,r}\{fg\}(t) + \frac{(t^{r+1}-a^{r+1})^{\frac{\beta}{k}}}{(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)} R_{a,k}^{\alpha,r}\{fg\}(t) \right. \\ &\quad \left. - R_{a,k}^{\alpha,r}\{f\}(t) R_{a,k}^{\beta,r}\{g\}(t) - R_{a,k}^{\alpha,r}\{g\}(t) R_{a,k}^{\beta,r}\{f\}(t) \right| \\ &\quad \leq \left| G_1^*(f,m_1,m_2,M_1,M_2)(t) + G_2^*(f,m_1,m_2,M_1,M_2)(t) \right|^{\frac{1}{2}} \\ &\quad \times \left| G_1^*(g,n_1,n_2,N_1,N_2)(t) + G_2^*(g,n_1,n_2,N_1,N_2)(t) \right|^{\frac{1}{2}}, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} &G_1^*(u,v,w,x,y)(t) \\ &\quad = \frac{(t^{r+1}-a^{r+1})^{\frac{\beta}{k}}}{4(r+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)} \cdot \frac{((v+x)R_{a,k}^{\alpha,r}\{\tau u\}(t) + (w+y)R_{a,k}^{\alpha,r}\{u\}(t))^2}{vxR_{a,k}^{\alpha,r}\{\tau^2\}(t) + (vy+wx)R_{a,k}^{\alpha,r}\{\tau\}(t) + wyR_{a,k}^{\alpha,r}\{1\}(t)} \\ &\quad - R_{a,k}^{\alpha,r}\{u\}(t) R_{a,k}^{\beta,r}\{u\}(t), \\ &G_2^*(u,v,w,x,y)(t) \\ &\quad = \frac{(t^{r+1}-a^{r+1})^{\frac{\alpha}{k}}}{4(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} \cdot \frac{((v+x)R_{a,k}^{\beta,r}\{\tau u\}(t) + (w+y)R_{a,k}^{\beta,r}\{u\}(t))^2}{vxR_{a,k}^{\beta,r}\{\tau^2\}(t) + (vy+wx)R_{a,k}^{\beta,r}\{\tau\}(t) + wyR_{a,k}^{\beta,r}\{1\}(t)} \\ &\quad - R_{a,k}^{\alpha,r}\{u\}(t) R_{a,k}^{\beta,r}\{u\}(t). \end{aligned}$$

Proof By setting the four linear functions as in Proposition 4.1 and using Theorem 3.3, we get the estimate (4.5). □

Corollary 4.2 *If $m_1 = M_1 = n_1 = N_1 = v = x = 0$, then we obtain*

$$\begin{aligned} &G_1^*(u,0,w,0,y)(t) \\ &\quad = \frac{1}{4}\left(\sqrt{\frac{w}{y}} + \sqrt{\frac{y}{w}}\right)^2 \frac{(t^{r+1}-a^{r+1})^{\frac{\beta-\alpha}{k}}\Gamma_k(\alpha+k)}{(r+1)^{\frac{\beta-\alpha}{k}}\Gamma_k(\beta+k)} \cdot \left(R_{a,k}^{\alpha,r}\{u\}(t)\right)^2 \\ &\quad - R_{a,k}^{\alpha,r}\{u\}(t) R_{a,k}^{\beta,r}\{u\}(t), \\ &G_2^*(u,0,w,0,y)(t) \\ &\quad = \frac{1}{4}\left(\sqrt{\frac{w}{y}} + \sqrt{\frac{y}{w}}\right)^2 \frac{(t^{r+1}-a^{r+1})^{\frac{\alpha-\beta}{k}}\Gamma_k(\beta+k)}{(r+1)^{\frac{\alpha-\beta}{k}}\Gamma_k(\alpha+k)} \cdot \left(R_{a,k}^{\beta,r}\{u\}(t)\right)^2 \\ &\quad - R_{a,k}^{\alpha,r}\{u\}(t) R_{a,k}^{\beta,r}\{u\}(t). \end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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