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Evolution of a geometric constant along the Ricci flow

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Abstract

In this paper, we establish the first variation formula of the lowest constant $\lambda_a^b(g)$ along the Ricci flow and the normalized Ricci flow, such that to the following nonlinear equation there exist positive solutions:

$$-\Delta u + au \log u + bRu = \lambda_a^b u$$

with $\int_M u^2\, dv = 1$, where a is a real constant. In particular, the results proved in this paper generalize partial results in Cao (Proc. Am. Math. Soc. 136:4075-4078, 2008) and Li (Math. Ann. 338:927-946, 2007).

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1 Introduction

Let (M,g) be an n-dimensional compact Riemannian manifold. In [3], Perelman introduced the functional

$$\mathcal{F}(g,f) = \int_M \big(|\nabla f|^2 + R\big)e^{-f}\, dv \tag{1.1}$$

and proved that the $\mathcal{F}$-functional is nondecreasing under the Ricci flow coupled to a backward heat-type equation

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \\ f_t = -\Delta f + |\nabla f|^2 - R, \end{cases} \tag{1.2}$$

where R is the scalar curvature depending on the metric g. More precisely, they proved that under the system (1.2),

$$\frac{d}{dt}\mathcal{F} = 2\int_M |R_{ij} + f_{ij}|^2 e^{-f}\, dv \geq 0. \tag{1.3}$$

If we define

$$\lambda(g) = \inf_f \mathcal{F}(g,f), \tag{1.4}$$

where the infimum is taken over all smooth functions f which satisfy

$$\int_M e^{-f}\,dv = 1, \tag{1.5}$$

then the nondecreasing of the $\mathcal{F}$-functional implies the nondecreasing of $\lambda(g)$. In particular, $\lambda(g)$ defined in (1.4) is the lowest eigenvalue of the operator

$$-4\Delta + R. \tag{1.6}$$

In [4], Cao considered the eigenvalues of the operator $-\Delta + \frac{R}{2}$ on manifolds with nonnegative curvature operator and showed that the eigenvalues are nondecreasing along the Ricci flow. Using the same technique, Li [2] also obtained the same monotonicity of the first eigenvalue of the operator $-\Delta + \frac{R}{2}$ by removing the assumption on a nonnegative curvature operator.

Later, Cao [1] proved the first eigenvalues of the operator $-\Delta + bR$ with the constant $b \geq 1/4$ are nondecreasing along the Ricci flow. That is, they assume $u = u(x,t)$ is the corresponding positive eigenfunction of $\lambda(t)$:

$$(-\Delta + bR)u = \lambda^b u \tag{1.7}$$

with $\int_M u^2\,dv = 1$, then

$$\frac{d}{dt}\lambda^b = \frac{1}{2}\int_M |R_{ij} + f_{ij}|^2 e^{-f}\,dv + \left(2b - \frac{1}{2}\right)\int_M |R_{ij}|^2 e^{-f}\,dv \geq 0 \tag{1.8}$$

by letting $f = -2\log u$. Multiplying both sides of (1.7) with u and integrating on M, we see that the first eigenvalue given in (1.7) satisfies

$$\lambda(t) = \inf \tilde{\mathcal{F}}^b(g,u), \tag{1.9}$$

where

$$\tilde{\mathcal{F}}^b(g,u) = \int_M \left(|\nabla u|^2 + bRu^2\right) dv. \tag{1.10}$$

In particular,

$$\tilde{\mathcal{F}}^b(g,u) = \frac{1}{4}\mathcal{F}^{4b}(g,f), \tag{1.11}$$

where

$$\mathcal{F}^c(g,f) = \int_M \left(|\nabla f|^2 + cR\right)e^{-f}\,dv$$

if we let $f = -2\log u$. It is easy to see from (1.11) that the nondecreasing of the $\tilde{\mathcal{F}}^b$-functional is equivalent to the nondecreasing of $\lambda(t)$.

In this paper, we consider the monotonicity along the Ricci flow of lowest constant $\lambda_a^b(g)$ such that to the following nonlinear equation there exist positive solutions:

$$-\Delta u + au\log u + bRu = \lambda_a^b u \tag{1.12}$$

with

$$\int_M u^2\,dv = 1, \tag{1.13}$$

where a is a real constant. In particular, (1.7) can be seen a special case of (1.12) when $a = 0$. For the lowest constant $\lambda_a^b(g)$ such that to the nonlinear equation (1.12) there exist positive solutions, we prove the following.

Theorem 1.1 *Let $g(t)$, $t \in [0,T)$ be a solution to the Ricci flow*

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} \tag{1.14}$$

on a compact Riemannian manifold M. Then for $b \geq \frac{1}{4}$, the lowest constant $\lambda_a^b(g)$ such that to the nonlinear equation (1.12) *with* (1.13) *there exist positive solutions satisfies*

$$\begin{aligned}\frac{d}{dt}\left(\lambda_a^b(t) + \frac{na^2}{8}t\right) &= \frac{1}{2}\int_M \left|R_{ij} + f_{ij} + \frac{a}{2}g_{ij}\right|^2 e^{-f}\,dv + \left(2b - \frac{1}{2}\right)\int_M |R_{ij}|^2 e^{-f}\,dv \\ &\geq 0,\end{aligned} \tag{1.15}$$

where $f = -2\log u$.

For the normalized Ricci flow, we can obtain the following.

Theorem 1.2 *Let $g(t)$, $t \in [0,T)$ be a solution to the normalized Ricci flow*

$$\frac{\partial}{\partial t}g_{ij} = -2\left(R_{ij} - \frac{r}{n}g_{ij}\right) \tag{1.16}$$

on a compact Riemannian manifold M, where $r = (\int_M R\,dv)/(\int_M dv)$ is the average scalar curvature. Then the lowest constant $\lambda_a^b(g)$ such that to the nonlinear equation (1.12) *with* (1.13) *there exist positive solutions satisfies*

$$\begin{aligned}\frac{d}{dt}\left(\lambda_a^b + \frac{na^2}{8}t\right) + \frac{2r}{n}\lambda^b = \frac{1}{2}\int_M \left|R_{ij} + f_{ij} + \frac{a}{2}g_{ij}\right|^2 e^{-f}\,dv \\ + \left(2b - \frac{1}{2}\right)\int_M |R_{ij}|^2 e^{-f}\,dv,\end{aligned} \tag{1.17}$$

where $f = -2\log u$ and λ^b is the lowest eigenvalue of (1.7).

In particular, when $n = 2$, we have $R_{ij} = \frac{R}{2}g_{ij}$ and the normalized Ricci flow (1.16) becomes $\frac{\partial}{\partial t}g_{ij} = -(R-r)g_{ij}$. Hence, $\frac{d}{dt}r = 0$, which implies that r is a constant (or see p.455 in [5] for an alternative proof). Then from the estimate (1.17), we obtain the following.

Theorem 1.3 *Let $g(t)$, $t \in [0, T)$ be a solution to the normalized Ricci flow* (1.16) *on a compact surface M^2. Then for $b \geq \frac{1}{4}$, the lowest constant $\lambda_a^b(g)$ such that to the nonlinear equation* (1.12) *with* (1.13) *there exist positive solutions satisfies*

$$\begin{aligned} \frac{d}{dt}\left(\lambda_a^b + \frac{a^2}{4}t + r\int_0^t \lambda^b(s)\,ds\right) &= \frac{1}{2}\int_M \left|R_{ij} + f_{ij} + \frac{a}{2}g_{ij}\right|^2 e^{-f}\,dv \\ &\quad + \left(2b - \frac{1}{2}\right)\int_M |R_{ij}|^2 e^{-f}\,dv \\ &\geq 0, \end{aligned} \tag{1.18}$$

where $f = -2\log u$ and λ^b is the lowest eigenvalue of (1.7).

Remark 1.1 In particular, when $a = 0$, our estimate (1.15) reduces to Theorem 1.5 of Cao in [1] and the estimate (1.18) reduces to the Corollary 2.4 of Cao in [1], respectively.

On the other hand, under the transformation $f = -2\log u = -\log v$ with $u^2 = v$, equation (1.2) becomes

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \\ v_t = -\Delta v + Rv. \end{cases} \tag{1.19}$$

In particular, the second equation in (1.19) is exactly the conjugate heat equation introduced by Perelman. In [6], Cao and Zhang obtained differential Harnack inequalities for positive solutions of the nonlinear parabolic equation of the type $v_t = \Delta v - v\log v + Rv$. Extending the second equation in (1.19) to the following nonlinear version:

$$v_t = -\Delta v + av\log v + Sv, \tag{1.20}$$

Guo and Ishida [7, 8] studied Harnack inequalities for positive solutions of equation (1.20) on a compact Riemannian manifold with a family of $g(t)$ evolving by a geometric flow $\frac{\partial}{\partial t} g_{ij} = -2S_{ij}$, where S_{ij} is a family of smooth symmetric two-tensor and $S = g^{ij}S_{ij}$. Clearly, there is a one-to-one relation for the following two equations:

$$\frac{\partial}{\partial t} v = -\Delta v + av\log v + Rv \quad \Longleftrightarrow \quad \frac{\partial}{\partial t} f = -\Delta f + |\nabla f|^2 + af - R \tag{1.21}$$

under $f = -\log v$. Therefore, a natural problem is to consider the monotonicity of

$$\overline{\mathcal{F}}_d^c(g, f) = \int_M \left[|\nabla f|^2 + cR + d(f + 1)\right] e^{-f}\,dv \tag{1.22}$$

under the Ricci flow coupled to a nonlinear backward heat-type equation

$$\begin{cases} \frac{d}{dt} g_{ij} = -2R_{ij}, \\ f_t = -\Delta f + |\nabla f|^2 + af - R, \end{cases} \tag{1.23}$$

where c, d are two real constants.

For the functional $\overline{\mathcal{F}}^c_d(g,f)$, we derive the following monotonicity formula.

Theorem 1.4 *Let $g(t)$, $t \in [0,T)$ be a solution to the Ricci flow* (1.14) *on a compact Riemannian manifold M. Then all functionals $\overline{\mathcal{F}}^c_d(g,f)$ defined by* (1.22) *under the system* (1.23) *satisfy*

$$\frac{d}{dt}\overline{\mathcal{F}}^k_{\frac{nak}{8}}(g,f) = 2\int_M \left|R_{ij} + f_{ij} - \frac{a}{4}fg_{ij}\right|^2 e^{-f}\,dv + 2(k-1)\int_M \left|R_{ij} - \frac{a}{4}fg_{ij}\right|^2 e^{-f}\,dv + \frac{na}{8}k\mathcal{F}^1(g,f) + a\mathcal{F}^0(g,f). \tag{1.24}$$

In particular, if $R(t) \geq 0$ for all t and $a \geq 0$, $k \geq 1$, then $\frac{d}{dt}\overline{\mathcal{F}}^k_{\frac{nak}{8}}(g,f) \geq 0$.

Remark 1.2 Choosing $a = 0$ in (1.24), we obtain Theorem 4.2 of Li in [2].

2 Proof of Theorems 1.1 and 1.2

Proof of Theorems 1.1 Let u be a positive solution to the following nonlinear elliptic equation:

$$-\Delta u + au\log u + bRu = \lambda^b_a u. \tag{2.1}$$

Multiplying both sides of (2.1) with u and integrating on M, we have

$$\lambda^b_a = \int_M \left(|\nabla u|^2 + au^2\log u + bRu^2\right)dv. \tag{2.2}$$

If the metric $g(t)$ evolves by (1.14), we have $\frac{\partial}{\partial t}\,dv = -R\,dv$. It follows from (2.2) that

$$\frac{d}{dt}\lambda^b_a = \int_M \left(2R_{ij}u^iu^j + 2(u_t)^iu_i + 2auu_t\log u + auu_t + bR_tu^2 + 2bRuu_t\right)dv - \int_M \left(|\nabla u|^2 + au^2\log u + bRu^2\right)R\,dv. \tag{2.3}$$

Applying

$$2\int_M R_{ij}u^iu^j\,dv = \int_M \left(-R_{,i}u^iu - 2R_{ij}u^{ij}u\right)dv \tag{2.4}$$

and

$$-\int_M |\nabla u|^2R\,dv = \int_M \left(R\Delta u + R_{,i}u^i\right)u\,dv \tag{2.5}$$

into (2.3) yields

$$\frac{d}{dt}\lambda^b_a = \int_M \big[-2R_{ij}u^{ij}u + bR_tu^2 + auu_t + 2u_t(-\Delta u + au\log u + bRu) - Ru(-\Delta u + au\log u + bRu)\big]\,dv$$

$$= \int_M \left[-2R_{ij}u^{ij}u + bR_t u^2 + \frac{a}{2}\left(u^2\right)_t \right] dv + \lambda \left(\int_M u^2\, dv \right)_t$$
$$= \int_M \left[-2R_{ij}u^{ij}u + bR_t u^2 + \frac{a}{2}Ru^2 \right] dv, \tag{2.6}$$

where the last equality used

$$\int_M \left[\left(u^2\right)_t - Ru^2 \right] dv = 0 \tag{2.7}$$

from (1.13). Noticing $R_t = \Delta R + 2|R_{ij}|^2$ for the Ricci flow, hence from (2.6) we have

$$\frac{d}{dt}\lambda_a^b = \int_M \left[-2R_{ij}u^{ij}u + bu^2\left(\Delta R + 2|R_{ij}|^2\right) + \frac{a}{2}Ru^2 \right] dv$$
$$= \int_M \left[-2R_{ij}u^{ij}u + bR\Delta\left(u^2\right) + 2b|R_{ij}|^2u^2 + \frac{a}{2}Ru^2 \right] dv. \tag{2.8}$$

Taking a transformation $f = -2\log u$, which is equivalent to $u^2 = e^{-f}$, then

$$u^{ij} = \left(-\frac{1}{2}f^{ij} + \frac{1}{4}f^i f^j \right) e^{-\frac{f}{2}}. \tag{2.9}$$

Thus, (2.8) can be written as

$$\frac{d}{dt}\lambda_a^b = \int_M \left[R_{ij}f^{ij} - \frac{1}{2}R_{ij}f^i f^j - bR\Delta f + bR|\nabla f|^2 + 2b|R_{ij}|^2 + \frac{a}{2}R \right] e^{-f}\, dv. \tag{2.10}$$

Using the second Bianchi identity $R_{,i} = 2R_{ij,}{}^{j}$ again, we have

$$-b\int_M R\Delta f e^{-f}\, dv = \int_M \left(bR_{,i}f^i - bR|\nabla f|^2 \right) e^{-f}\, dv$$
$$= \int_M \left(-2bR_{ij}f^{ij} + 2bR_{ij}f^i f^j - bR|\nabla f|^2 \right) e^{-f}\, dv. \tag{2.11}$$

Therefore, inserting (2.11) into (2.10) yields

$$\frac{d}{dt}\lambda_a^b = (1-2b)\int_M R_{ij}f^{ij}e^{-f}\, dv + \left(2b - \frac{1}{2} \right) \int_M R_{ij}f^i f^j e^{-f}\, dv$$
$$+ 2b\int_M |R_{ij}|^2 e^{-f}\, dv + \frac{a}{2}\int_M Re^{-f}\, dv. \tag{2.12}$$

Integrating by parts again, one has

$$\int_M R_{ij}f^{ij}e^{-f}\, dv = \int_M R_{ij}f^i f^j e^{-f}\, dv - \frac{1}{2}\int_M R\Delta e^{-f}\, dv \tag{2.13}$$

and

$$\int_M R_{ij}f^{ij}e^{-f}\, dv + \int_M |f_{ij}|^2 e^{-f}\, dv$$
$$= \frac{1}{2}\int_M \Delta|\nabla f|^2 e^{-f}\, dv - \int_M (\Delta f)_i f^i e^{-f}\, dv - \frac{1}{2}\int_M R\Delta e^{-f}\, dv$$

$$
\begin{aligned}
&= -\int_M \left[\Delta f - \frac{1}{2}|\nabla f|^2 + \frac{1}{2}R\right]\Delta e^{-f}\,dv \\
&= \left(2b - \frac{1}{2}\right)\int_M R\Delta e^{-f}\,dv - a\int_M |\nabla f|^2 e^{-f}\,dv,
\end{aligned} \tag{2.14}
$$

where the last equality in (2.14) was used with

$$
2\lambda_a^b = \Delta f - \frac{1}{2}|\nabla f|^2 - af + 2bR. \tag{2.15}
$$

By virtue of (2.14), subtracting (2.13), we obtain

$$
\int_M |f_{ij}|^2 e^{-f}\,dv = 2b\int_M R\Delta e^{-f}\,dv - \int_M R_{ij}f^i f^j e^{-f}\,dv - a\int_M |\nabla f|^2 e^{-f}\,dv. \tag{2.16}
$$

It follows from (2.13) and (2.14) that

$$
\begin{aligned}
\frac{d}{dt}\lambda_a^b &= (1-2b)\int_M R_{ij}f^{ij}e^{-f}\,dv + \left(2b - \frac{1}{2}\right)\int_M R_{ij}f^i f^j e^{-f}\,dv \\
&\quad + 2b\int_M |R_{ij}|^2 e^{-f}\,dv + \frac{a}{2}\int_M Re^{-f}\,dv \\
&= \int_M R_{ij}f^{ij}e^{-f}\,dv - \frac{1}{2}\int_M R_{ij}f^i f^j e^{-f}\,dv \\
&\quad + 2b\int_M |R_{ij}|^2 e^{-f}\,dv + \frac{a}{2}\int_M Re^{-f}\,dv + b\int_M R\Delta e^{-f}\,dv \\
&= \int_M R_{ij}f^{ij}e^{-f}\,dv + 2b\int_M |R_{ij}|^2 e^{-f}\,dv + \frac{a}{2}\int_M Re^{-f}\,dv \\
&\quad + \frac{1}{2}\int_M |f_{ij}|^2 e^{-f}\,dv + \frac{a}{2}\int_M (\Delta f)e^{-f}\,dv \\
&= \frac{1}{2}\int_M \left|R_{ij} + f_{ij} + \frac{a}{2}g_{ij}\right|^2 e^{-f}\,dv + \left(2b - \frac{1}{2}\right)\int_M |R_{ij}|^2 e^{-f}\,dv \\
&\quad - \frac{na^2}{8},
\end{aligned} \tag{2.17}
$$

and the desired estimate (1.15) is achieved. □

Proof of Theorem 1.2 If the metric $g(t)$ evolves by (1.16), we have $\frac{\partial}{\partial t}\,dv = -(R-r)\,dv$. It follows from (2.2) that

$$
\begin{aligned}
\frac{d}{dt}\lambda_a^b &= \int_M \biggl(2R_{ij}u^i u^j - \frac{2r}{n}|\nabla u|^2 + 2(u_t)^i u_i + 2auu_t\log u + auu_t + bR_t u^2 \\
&\quad + 2bRuu_t\biggr)\,dv - \int_M \bigl(|\nabla u|^2 + au^2\log u + bRu^2\bigr)(R-r)\,dv.
\end{aligned} \tag{2.18}
$$

Applying (2.4) and

$$
-\int_M |\nabla u|^2 (R-r)\,dv = \int_M \bigl[(R-r)\Delta u + R_{,i}u^i\bigr]u\,dv \tag{2.19}
$$

to (2.18) yields

$$\begin{aligned}\frac{d}{dt}\lambda_a^b &= \int_M \Big[-2R_{ij}u^{ij}u - \frac{2r}{n}|\nabla u|^2 + bR_t u^2 + auu_t \\ &\quad + 2u_t(-\Delta u + au\log u + bRu) - (R-r)u(-\Delta u + au\log u + bRu)\Big]\,dv \\ &= \int_M \Big[-2R_{ij}u^{ij}u - \frac{2r}{n}|\nabla u|^2 + bR_t u^2 + \frac{a}{2}\big(u^2\big)_t\Big]\,dv + \lambda\Big(\int_M u^2\,dv\Big)_t \\ &= \int_M \Big[-2R_{ij}u^{ij}u - \frac{2r}{n}|\nabla u|^2 + bR_t u^2 + \frac{a}{2}Ru^2\Big]\,dv. \end{aligned} \tag{2.20}$$

Noticing $R_t = \Delta R + 2|R_{ij}|^2 - \frac{2r}{n}R$ for the normalized Ricci flow, we obtain from (2.20)

$$\begin{aligned}\frac{d}{dt}\lambda_a^b &= \int_M \Big[-2R_{ij}u^{ij}u - \frac{2r}{n}|\nabla u|^2 + bR_t u^2 + \frac{a}{2}Ru^2\Big]\,dv \\ &= \int_M \Big[-2R_{ij}u^{ij}u - \frac{2r}{n}|\nabla u|^2 + bu^2\Big(\Delta R + 2|R_{ij}|^2 - \frac{2r}{n}R\Big) + \frac{a}{2}Ru^2\Big]\,dv \\ &= \int_M \Big[-2R_{ij}u^{ij}u + bR\Delta\big(u^2\big) + 2b|R_{ij}|^2u^2 + \frac{a}{2}Ru^2\Big]\,dv \\ &\quad - \frac{2r}{n}\int_M \big(|\nabla u|^2 + bRu^2\big)\,dv. \end{aligned} \tag{2.21}$$

Using (2.9), then (2.21) can be written as

$$\begin{aligned}\frac{d}{dt}\lambda_a^b &= \int_M \Big[R_{ij}f^{ij} - \frac{1}{2}R_{ij}f^if^j - bR\Delta f + bR|\nabla f|^2 + 2b|R_{ij}|^2 + \frac{a}{2}R\Big]e^{-f}\,dv \\ &\quad - \frac{2r}{n}\int_M \Big(\frac{1}{4}|\nabla f|^2 + bR\Big)e^{-f}\,dv. \end{aligned} \tag{2.22}$$

By virtue of a similar computation, we can obtain

$$\begin{aligned}\frac{d}{dt}\lambda_a^b &= \frac{1}{2}\int_M \Big|R_{ij} + f_{ij} + \frac{a}{2}g_{ij}\Big|^2 e^{-f}\,dv + \Big(2b - \frac{1}{2}\Big)\int_M |R_{ij}|^2e^{-f}\,dv \\ &\quad - \frac{na^2}{8} - \frac{2r}{n}\int_M \Big(\frac{1}{4}\Delta f + bR\Big)e^{-f}\,dv, \end{aligned} \tag{2.23}$$

which gives

$$\begin{aligned}\frac{d}{dt}\Big(\lambda_a^b + \frac{na^2}{8}t\Big) + \frac{2r}{n}\lambda^b &= \frac{1}{2}\int_M \Big|R_{ij} + f_{ij} + \frac{a}{2}g_{ij}\Big|^2 e^{-f}\,dv \\ &\quad + \Big(2b - \frac{1}{2}\Big)\int_M |R_{ij}|^2e^{-f}\,dv. \end{aligned} \tag{2.24}$$

Then the desired estimate (1.17) is attained. □

3 Proof of Theorem 1.4

Under the following coupled system (1.23), by a direct computation, we have the following:

$$\begin{aligned}\frac{\partial}{\partial t}\left(e^{-f}\,dv\right) &= -(f_t+R)e^{-f}\,dv = \left[\Delta f - |\nabla f|^2 - af\right]e^{-f}\,dv \\ &= -\left(\Delta e^{-f}\right)dv - afe^{-f}\,dv,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f|^2 &= 2R^{ij}f_if_j + 2f^i(f_t)_i \\ &= 2R^{ij}f_if_j + 2f^i\left(-\Delta f + |\nabla f|^2 + af - R\right)_i \\ &= 2R^{ij}f_if_j - 2f^i(\Delta f)_i + 4f^{ij}f_if_j + 2a|\nabla f|^2 - 2R_if^i.\end{aligned} \tag{3.2}$$

Thus, we have

$$\frac{d}{dt}\int_M e^{-f}\,dv = -a\int_M fe^{-f}\,dv, \tag{3.3}$$

$$\begin{aligned}\frac{d}{dt}\int_M Re^{-f}\,dv &= \int_M\left[\Delta R + 2|R_{ij}|^2 - afR\right]e^{-f}\,dv - \int_M R\left(\Delta e^{-f}\right)dv \\ &= \int_M\left[2|R_{ij}|^2 - afR\right]e^{-f}\,dv,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\frac{d}{dt}\int_M fe^{-f}\,dv &= \int_M(af-R)e^{-f}\,dv - \int_M f\left(\Delta e^{-f}\right)dv - \int_M af^2e^{-f}\,dv \\ &= \int_M\left[af - af^2 - (R+\Delta f)\right]e^{-f}\,dv\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\frac{d}{dt}\int_M |\nabla f|^2e^{-f}\,dv \\ &\quad= \int_M\left[2R^{ij}f_if_j - 2f^i(\Delta f)_i + 4f^{ij}f_if_j + 2a|\nabla f|^2 - 2R_if^i\right]e^{-f}\,dv \\ &\qquad - \int_M\left(\Delta e^{-f}\right)|\nabla f|^2\,dv - \int_M af|\nabla f|^2e^{-f}\,dv \\ &\quad= \int_M\left[-2f_{ij}^2 - 4f^i(\Delta f)_i + 4f^{ij}f_if_j + 2a|\nabla f|^2 - 2R_if^i\right]e^{-f}\,dv \\ &\qquad - \int_M af|\nabla f|^2e^{-f}\,dv.\end{aligned} \tag{3.6}$$

By virtue of the Bochner formula with respect to the f-Laplacian, we have

$$\frac{1}{2}\Delta_f|\nabla u|^2 = u_{ij}^2 + u_i(\Delta_f u)_i + \left(R^{ij} + f^{ij}\right)u_iu_j, \quad \forall u,$$

and hence

$$\begin{aligned}0 &= \int_M\left[f_{ij}^2 + f_i(\Delta_f f)_i + \left(R^{ij} + f^{ij}\right)f_if_j\right]e^{-f}\,dv \\ &= \int_M\left[f_{ij}^2 + f_i(\Delta f)_i + R^{ij}f_if_j - f^{ij}f_if_j\right]e^{-f}\,dv.\end{aligned} \tag{3.7}$$

Therefore, (3.6) becomes

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla f|^2 e^{-f}\,dv &= \int_M \bigl[2f_{ij}^2 + 4R^{ij}f_if_j + 2a|\nabla f|^2 - 2R_if^i\bigr]e^{-f}\,dv - \int_M af|\nabla f|^2 e^{-f}\,dv\\ &= \int_M \bigl[2f_{ij}^2 + 4R^{ij}f_{ij} + 2a|\nabla f|^2\bigr]e^{-f}\,dv - \int_M a(f+1)(\Delta f)e^{-f}\,dv. \end{aligned}\tag{3.8}$$

Therefore, from (3.4) and (3.8), we obtain

$$\begin{aligned}\frac{d}{dt}\int_M \bigl(R+|\nabla f|^2\bigr)e^{-f}\,dv &= 2\int_M \left|R_{ij}+f_{ij}-\frac{a}{4}fg_{ij}\right|^2 e^{-f}\,dv\\ &\quad - \frac{na^2}{8}\int_M f^2e^{-f}\,dv + a\int_M |\nabla f|^2e^{-f}\,dv. \end{aligned}\tag{3.9}$$

Noticing (3.5) tells us that

$$-a\int_M f^2e^{-f}\,dv = \frac{d}{dt}\left(\int_M (f+1)e^{-f}\,dv\right) + \int_M (R+\Delta f)e^{-f}\,dv. \tag{3.10}$$

Thus, (3.9) can be written as

$$\begin{aligned}&\frac{d}{dt}\int_M \left[R+|\nabla f|^2+\frac{na}{8}(f+1)\right]e^{-f}\,dv\\ &\quad = 2\int_M \left|R_{ij}+f_{ij}-\frac{a}{4}fg_{ij}\right|^2 e^{-f}\,dv + \frac{na}{8}\int_M \bigl(R+|\nabla f|^2\bigr)e^{-f}\,dv + a\int_M |\nabla f|^2e^{-f}\,dv. \end{aligned}\tag{3.11}$$

Since (3.4) holds, we have

$$\frac{d}{dt}\int_M Re^{-f}\,dv = 2\int_M \left|R_{ij}-\frac{a}{4}fg_{ij}\right|^2 e^{-f}\,dv - \frac{na^2}{8}\int_M f^2e^{-f}\,dv, \tag{3.12}$$

which gives

$$\begin{aligned}\frac{d}{dt}\int_M \left[R+\frac{na}{8}(f+1)\right]e^{-f}\,dv &= 2\int_M \left|R_{ij}-\frac{a}{4}fg_{ij}\right|^2 e^{-f}\,dv\\ &\quad + \frac{na}{8}\int_M \bigl(R+|\nabla f|^2\bigr)e^{-f}\,dv. \end{aligned}\tag{3.13}$$

Therefore, we have

$$\begin{aligned}&\frac{d}{dt}\int_M \left\{|\nabla f|^2 + k\left[R+\frac{na}{8}(f+1)\right]\right\}e^{-f}\,dv\\ &\quad = 2\int_M \left|R_{ij}+f_{ij}-\frac{a}{4}fg_{ij}\right|^2 e^{-f}\,dv + 2(k-1)\int_M \left|R_{ij}-\frac{a}{4}fg_{ij}\right|^2 e^{-f}\,dv\\ &\qquad + \frac{na}{8}k\int_M \bigl(R+|\nabla f|^2\bigr)e^{-f}\,dv + a\int_M |\nabla f|^2e^{-f}\,dv \end{aligned}\tag{3.14}$$

and the desired estimate (1.24) is obtained.

4 Conclusions

We establish the first variation formula of the lowest constant $\lambda_a^b(g)$ along the Ricci flow and the normalized Ricci flow, such that to the following nonlinear equation there exist positive solutions:

$$-\Delta u + au\log u + bRu = \lambda_a^b u \tag{4.1}$$

with $\int_M u^2\,dv = 1$, where a is a real constant. Equation (4.1) can be seen as a nonlinear version of eigenvalue problem of the operator $-\Delta u + bR$. In particular, when $a = 0$, our estimate (1.15) in Theorem 1.1 reduces to Theorem 1.5 of Cao in [1] and the estimate (1.18) in Theorem 1.3 reduces to the Corollary 2.4 of Cao in [1], respectively.

On the other hand, we obtained the first variation formula (1.24) of the functional

$$\overline{\mathcal{F}}_d^c(g,f) = \int_M \bigl[|\nabla f|^2 + cR + d(f+1)\bigr]e^{-f}\,dv$$

under the Ricci flow coupled to a nonlinear backward heat-type equation

$$\begin{cases} \frac{d}{dt}g_{ij} = -2R_{ij}, \\ f_t = -\Delta f + |\nabla f|^2 + af - R, \end{cases}$$

where c, d are two real constants. In particular, when $a = 0$ in (1.24), we obtain Theorem 4.2 of Li in [2].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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