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# An application of the inequality for modified Poisson kernel

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**Abstract**

As an application of an inequality for modified Poisson kernel obtained by Qiao and Deng (Bull. Malays. Math. Sci. Soc. (2) 36(2):511-523, 2013), we give the generalized solution of the Dirichlet problem with arbitrary growth data.

**Keywords:** growth property; Dirichlet problem; modified Poisson kernel

## 1 Introduction and results

Let $\mathbf{R}^n$ ($n \geq 2$) be the $n$-dimensional Euclidean space. The boundary and the closure of a set $E$ in $\mathbf{R}^n$ are denoted by $\partial E$ and $\overline{E}$, respectively. The Euclidean distance of two points $P$ and $Q$ in $\mathbf{R}^n$ is denoted by $|P - Q|$. Especially, $|P|$ denotes the distance of two points $P$ and $O$ in $\mathbf{R}^n$, where $O$ is the origin in $\mathbf{R}^n$.

We introduce a system of spherical coordinates $(r, \Theta)$, $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to Cartesian coordinates $(x_1, x_2, \dots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$.

Let $B(P, r)$ denote the open ball with center at $P$ and radius $r$ ($>0$) in $\mathbf{R}^n$. The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. The surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$ is denoted $w_n$. Let $\Omega \subset \mathbf{S}^{n-1}$, a point $(1, \Theta)$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ are denoted $\Theta$ and $\Omega$, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, we denote $\Lambda \times \Omega = \{(r, \Theta) \in \mathbf{R}^n; r \in \Lambda, (1, \Theta) \in \Omega\}$, where $\mathbf{R}_+$ is the set of all positive real numbers.

For the set $\Omega \subset \mathbf{S}^{n-1}$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ by $C_n(\Omega)$, which is called a cone. For the set $I \subset \mathbf{R}$, the sets $I \times \Omega$ and $I \times \partial\Omega$ are denoted $C_n(\Omega; I)$ and $S_n(\Omega; I)$, respectively, where $\mathbf{R}$ is the set of all real numbers. Especially, the set $S_n(\Omega; \mathbf{R}_+)$ is denoted $S_n(\Omega)$.

Given a continuous function $f$ on $S_n(\Omega)$, we say that $h$ is a solution of the Dirichlet problem in $C_n(\Omega)$ with $f$, if $h$ is a harmonic function in $C_n(\Omega)$ and

$$\lim_{P \to Q \in S_n(\Omega), P \in C_n(\Omega)} h(P) = f(Q).$$

Let $\Omega \subset \mathbf{S}^{n-1}$ and $\Delta^*$ be a Laplace-Beltrami on the unit sphere. Consider the Dirichlet problem (see, *e.g.* [2], p.41)

$$\Delta^*\varphi(\Theta) + \lambda\varphi(\Theta) = 0 \quad \text{in } \Omega,$$

$$\varphi(\Theta) = 0 \quad \text{in } \partial\Omega.$$

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We denote the non-decreasing sequence of positive eigenvalues of it, repeating accordingly to their multiplicities, and the corresponding eigenfunctions are denoted, respectively, by $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\varphi_i(\Theta)\}_{i=1}^{\infty}$. Especially, we denote the least positive eigenvalue of it $\lambda_1$ and the normalized positive eigenfunction to $\lambda_1$ $\varphi_1(\Theta)$. In the sequel, for the sake of brevity, we shall write $\lambda$ and $\varphi$ instead of $\lambda_1$ and $\varphi_1$, respectively.

The set of sequential eigenfunctions corresponding to the same value of $\{\lambda_i\}_{i=1}^{\infty}$ in the sequence $\{\varphi_i(\Theta)\}_{i=1}^{\infty}$ makes an orthonormal basis for the eigenspace of the eigenvalue $\lambda_i$. Hence for each $\Omega \subset S^{n-1}$ there is a sequence $\{k_j\}$ of positive integers such that $k_1 = 1$, $\lambda_{k_j} < \lambda_{k_{j+1}}$, $\lambda_{k_j} = \lambda_{k_j+1} = \lambda_{k_j+2} = \cdots = \lambda_{k_{j+1}-1}$ and $\{\varphi_{k_j}, \varphi_{k_j+1}, \ldots, \varphi_{k_{j+1}-1}\}$ is an orthonormal basis for the eigenspace of the eigenvalue $\{\lambda_{k_j}\}_{j=1}^{\infty}$. By $I_\Omega(k_m)$ we denote the set of all positive integers less than $\{k_m\}_{m=1}^{\infty}$. In spite of the fact

$$I_\Omega(k_1) = \varnothing,$$

the summation over $I_\Omega(k_1)$ of a function $S(k)$ of a variable $k$ will be used by promising

$$\sum_{k \in I_\Omega(k_1)} S(k) = 0.$$

If we denote the solutions of the equation

$$t^2 + (n-2)t - \lambda_i = 0 \quad (i = 1, 2, 3, \ldots)$$

by $\aleph_i^+$ and $\aleph_i^-$, then the functions

$$r^{\aleph_i^\pm} \varphi_i(\Theta) \quad (i = 1, 2, 3, \ldots)$$

are harmonic functions in $C_n(\Omega)$ and vanish on $S_n(\Omega)$.

Let $G_\Omega(P, Q)$ be the Green function of $C_n(\Omega)$ for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in C_n(\Omega)$. Then the Poisson kernel in $C_n(\Omega)$ can be defined by

$$PI_\Omega(P, Q) = \frac{1}{c_n} \frac{\partial}{\partial n_Q} G_\Omega(P, Q),$$

where $P \in C_n(\Omega)$, $Q \in S_n(\Omega)$, $\partial/\partial n_Q$ denotes the differentiation at $Q$ along the inward normal into $C_n(\Omega)$ and

$$c_n = \begin{cases} 2\pi & \text{if } n = 2, \\ (n-2)w_n & \text{if } n \geq 3. \end{cases}$$

Let $F(\Theta)$ be a function defined in $\Omega$. We denote $N_i(F)$ by

$$\int_\Omega F(\Theta)\varphi_i(\Theta)\, d\Omega,$$

when it exists.

For any two points $P = (r, \Theta)$ and $Q = (t, \Phi)$ in $C_n(\Omega)$ and $S_n(\Omega)$, respectively, we define

$$\widetilde{K}_{\Omega}^{m}(P,Q) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ K_{\Omega}^{m}(P,Q) & \text{if } 1 \leq t < \infty, \end{cases}$$

where $m$ is a non-negative integer and

$$K_{\Omega}^{m}(P,Q) = \sum_{i \in I_{k_{m+1}}} 2^{\aleph_i^+ + n - 1} N_i\bigl(PI_{\Omega}\bigl((1,\Theta),(2,\Phi)\bigr)\bigr) r^{\aleph_i^+} t^{-\aleph_i^+ - n + 1} \varphi_i(\Theta).$$

To obtain the solution of the Dirichlet problem in a cone, as in [1, 3, 4], we use the modified Poisson kernel defined by

$$PI_{\Omega}^{m}(P,Q) = PI_{\Omega}(P,Q) - \widetilde{K}_{\Omega}^{m}(P,Q),$$

where $P \in C_n(\Omega)$ and $Q \in S_n(\Omega)$, which has the following estimates (see [1]):

$$\bigl|PI_{\Omega}(P,Q) - K_{\Omega}^{m}(P,Q)\bigr| \leq M(2r)^{\aleph_{k_{m+1}}^+} t^{-\aleph_{k_{m+1}}^+ - n + 1} \tag{1}$$

for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in S_n(\Omega)$ satisfying $0 < \frac{r}{t} < \frac{1}{2}$, where $M$ is a constant independent of $P$, $Q$, and $m$. For the construction and applications of a modified Green function in a half space, we refer the reader to the paper by Qiao (see [5]).

Write

$$U_{\Omega}^{m}[f](P) = \int_{S_n(\Omega)} PI_{\Omega}^{m}(P,Q) f(Q)\, d\sigma_Q,$$

where $f(Q)$ is a continuous function on $\partial C_n(\Omega)$ and $d\sigma_Q$ the $(n-1)$-dimensional volume elements induced by the Euclidean metric on $\partial C_n(\Omega)$.

Recently, Qiao and Deng (*cf.* [1]) gave the solution of the Dirichlet problem in a cone. Applications of modified Poisson kernel with respect to the Schrödinger operator, we refer the reader to the papers by Huang and Ychussie (see [6]) and Li and Ychussie (see [7]).

**Theorem A** *If* $\Omega + \aleph^+ - 1 > 0$, $\Omega - n + 1 \leq \aleph_{k_{m+1}}^+ < \Omega - n + 2$ *and* $f(Q)$ $(Q = (t, \Phi))$ *is a continuous function on* $\partial C_n(\Omega)$ *satisfying*

$$\int_{S_n(\Omega)} \frac{|f(Q)|}{1 + t^{\Omega}}\, d\sigma_Q < \infty, \tag{2}$$

*then the function* $U_{\Omega}^{m}[f](P)$ *is a solution of the Dirichlet problem in* $C_n(\Omega)$ *with* $f$ *and*

$$\lim_{r \to \infty, P = (r,\Theta) \in C_n(\Omega)} r^{n - \Omega - 1} \varphi^{n-1}(\Theta) U_{\Omega}^{m}[f](P) = 0.$$

Furthermore, Qiao and Deng (*cf.* [4]) supplemented the above result and proved the following.

**Theorem B** *Let* $0 < p < \infty$, $\gamma > (-\aleph^{+} - n + 2)p + n - 1$ *and*

$$\frac{\gamma - n + 1}{p} < \aleph^{+}_{k_{m+1}} < \frac{\gamma - n + 1}{p} + 1.$$

*If* $f(Q)$ $(Q = (t, \Phi))$ *is a continuous function on* $S_n(\Omega)$ *satisfying*

$$\int_{S_n(\Omega)} \frac{|f(Q)|^p}{1 + t^{\gamma}} \, d\sigma_Q < \infty, \tag{3}$$

*then the function* $U^m_{\Omega}[f](P)$ *satisfies*

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{\frac{n-\gamma-1}{p}} \varphi^{n-1}(\Theta) U^m_{\Omega}[f](P) = 0.$$

It is natural to ask if the continuous function $u$ satisfying (2) and (3) can be replaced by arbitrary continuous function? In this paper, we shall give an affirmative answer to this question. To do this, we first construct a modified Poisson kernel. Let $\phi(l)$ be a positive function of $l \geq 1$ satisfying

$$2^{\aleph^{+}} \phi(1) = 1.$$

Denote the set

$$\left\{ l \geq 1; -\aleph^{+}_{k_i} \log 2 = \log\left(l^{n-1}\phi(l)\right) \right\}$$

by $\pi_{\Omega}(\phi, i)$. Then $1 \in \pi_{\Omega}(\phi, i)$. When there is an integer $N$ such that $\pi_{\Omega}(\phi, N) \neq \Phi$ and $\pi_{\Omega}(\phi, N+1) = \Phi$, denote

$$J_{\Omega}(\phi) = \{i; 1 \leq i \leq N\}$$

of integers. Otherwise, denote the set of all positive integers by $J_{\Omega}(\phi)$. Let $l(i) = l_{\Omega}(\phi, i + 1)$ be the minimum elements $l$ in $\pi_{\Omega}(\phi, i)$ for each $i \in J_{\Omega}(\phi)$. In the former case, we put $l(N+1) = \infty$. Then $l(1) = 1$. The kernel function $\widetilde{K}^{\phi}_{\Omega}(P, Q)$ is defined by

$$\widetilde{K}^{\phi}_{\Omega}(P, Q) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ K^{i}_{\Omega}(P, Q) & \text{if } l(i) \leq t < l(i+2) \text{ and } i \in J_{\Omega}(\phi), \end{cases}$$

where $P \in C_n(\Omega)$ and $Q = (t, \Phi) \in S_n(\Omega)$.

The generalized Poisson kernel $P^{\phi}_{\Omega}(P, Q)$ is defined by

$$PI^{\phi}_{\Omega}(P, Q) = PI_{\Omega}(P, Q) - \widetilde{K}^{\phi}_{\Omega}(P, Q),$$

where $P \in C_n(\Omega)$ and $Q \in S_n(\Omega)$.

As an application of the inequality (1) and the generalized Poisson kernel $PI^{\phi}_{\Omega}(P, Q)$, we have the following.

**Theorem** *Let $g(Q)$ be a continuous function on $S_n(\Omega)$. Then there is a positive continuous function $\phi_g(l)$ of $l \geq 0$ depending on $g$ such that*

$$H_\Omega^{\phi_g}(P) = \int_{S_n(\Omega)} PI_\Omega^{\phi_g}(P,Q)g(Q)\,d\sigma_Q$$

*is a solution of the Dirichlet problem in $C_n(\Omega)$ with $g$.*

## 2 Lemmas

**Lemma 1** *Let $\phi(l)$ be a positive continuous function of $l \geq 1$ satisfying*

$$\phi(1) = 2^{-\aleph^+}.$$

*Then*

$$\left|PI_\Omega(P,Q) - \widetilde{K}_\Omega^\phi(P,Q)\right| \leq M\phi(l)$$

*for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in S_n(\Omega)$ satisfying*

$$t > \max\{1, 4r\}. \tag{4}$$

*Proof* We can choose two points $P = (r,\Theta) \in C_n(\Omega)$ and $Q = (t,\Phi) \in S_n(\Omega)$, which satisfies (4). Moreover, we also can choose an integer $i = i(P,Q) \in J_\Omega(\phi)$ such that

$$l(i-1) \leq t < l(i). \tag{5}$$

Then

$$\widetilde{K}_\Omega^\phi(P,Q) = K_\Omega^{i-1}(P,Q).$$

Hence we have from (1), (4), and (5) that

$$\left|PI_\Omega(P,Q) - \widetilde{K}_\Omega^\phi(P,Q)\right| \leq M2^{-\aleph_{k_i}^+} \leq M\phi(l),$$

which is the conclusion. □

**Lemma 2** (See [4]) *Let $g(Q)$ be a continuous function on $\partial C_n(\Omega)$ and $V(P,Q)$ be a locally integrable function on $\partial C_n(\Omega)$ for any fixed $P \in C_n(\Omega)$, where $Q \in \partial C_n(\Omega)$. Define*

$$W(P,Q) = PI_\Omega(P,Q) - V(P,Q)$$

*for any $P \in C_n(\Omega)$ and any $Q \in \partial C_n(\Omega)$.*

Suppose that the following two conditions are satisfied:

(I) For any $Q' \in \partial C_n(\Omega)$ and any $\epsilon > 0$, there exists a neighborhood $B(Q')$ of $Q'$ such that

$$\int_{S_n(\Omega;[R,\infty))} \left|W(P,Q)\right|\left|u(Q)\right|\,d\sigma_Q < \epsilon \tag{6}$$

for any $P = (r,\Theta) \in C_n(\Omega) \cap B(Q')$, where $R$ is a positive real number.

(II) For any $Q' \in \partial C_n(\Omega)$, we have

$$\limsup_{P\to Q', P\in C_n(\Omega)} \int_{S_n(\Omega;(0,R))} |V(P,Q)||u(Q)|\, d\sigma_Q = 0 \tag{7}$$

for any positive real number $R$.

Then

$$\limsup_{P\to Q', P\in C_n(\Omega)} \int_{S_n(\Omega)} W(P,Q)u(Q')\, d\sigma_Q \le u(Q)$$

for any $Q' \in \partial C_n(\Omega)$.

## 3 Proof of Theorem

Take a positive continuous function $\phi(l)$ ($l \ge 1$) such that

$$\phi(1)2^{\aleph^+} = 1 \tag{8}$$

and

$$\phi(l) \int_{\partial\Omega} |g(l,\Phi)|\, d\sigma_\Phi \le \frac{L}{l^n}$$

for $l > 1$, where

$$L = 2^{-\aleph^+} \int_{\partial\Omega} |g(1,\Phi)|\, d\sigma_\Phi.$$

For any fixed $P = (r,\Theta) \in C_n(\Omega)$, we can choose a number $R$ satisfying $R > \max\{1, 4r\}$. Then we see from Lemma 1 that

$$\begin{aligned} &\int_{S_n(\Omega;(R,\infty))} |PI_\Omega^{\phi_g}(P,Q)||g(Q)|\, d\sigma_Q \\ &\quad \le M \int_R^\infty \left(\int_{\partial\Omega} |g(1,\Phi)|\, d\sigma_\Phi\right)\phi(l) l^{n-2}\, dl \\ &\quad \le \quad ML \int_R^\infty l^{-2}\, dl \\ &\quad < \infty. \end{aligned} \tag{9}$$

Obviously, we have

$$\int_{S_n(\Omega;(0,R))} |PI_\Omega^{\phi_g}(P,Q)||g(Q)|\, d\sigma_Q < \infty,$$

which gives

$$\int_{S_n(\Omega)} |PI_\Omega^{\phi_g}(P,Q)||g(Q)|\, d\sigma_Q < \infty.$$

To see that $H_\Omega^{\phi_g}(P)$ is a harmonic function in $C_n(\Omega)$, we remark that $H_\Omega^{\phi_g}(P)$ satisfies the locally mean-valued property by Fubini's theorem.

Finally we shall show that

$$\lim_{P\in C_n(\Omega),P\to Q'} H_{\Omega}^{\phi_g}(P) = g(Q')$$

for any $Q' = (t', \Phi') \in \partial C_n(\Omega)$. Set

$$V(P,Q) = \widetilde{K}_{\Omega}^{\phi_g}(P,Q)$$

in Lemma 2, which is locally integrable on $S_n(\Omega)$ for any fixed $P \in C_n(\Omega)$. Then we apply Lemma 2 to $g(Q)$ and $-g(Q)$.

For any $\epsilon > 0$ and a positive number $\delta$, by (9) we can choose a number $R$ ($> \max\{1, 2(t' + \delta)\}$) such that (6) holds, where $P \in C_n(\Omega) \cap B(Q', \delta)$.

Since

$$\lim_{\Theta\to\Phi'} \varphi_i(\Theta) = 0 \quad (i = 1, 2, 3 \ldots)$$

as $P = (r, \Theta) \to Q' = (t', \Phi') \in S_n(\Omega)$, we have

$$\lim_{P\in C_n(\Omega),P\to Q'} \widetilde{K}_{\Omega}^{\phi_g}(P,Q) = 0,$$

where $Q \in S_n(\Omega)$ and $Q' \in S_n(\Omega)$. Then (7) holds.

Thus we complete the proof of Theorem.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the manuscript and read and approved the final manuscript.

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