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On generalized García-Falset coefficient in Musielak-Orlicz sequence spaces

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Abstract

We introduce a new geometric coefficient which is related García-Falset coefficient and weak star fixed point property. The García-Falset coefficient that was introduced by Benavides in (Houst. J. Math. 22:835-849, 1996) is calculated in this paper for Musielak-Orlicz sequence spaces equipped with the Luxemburg norm. Specifically, in reflexive Banach spaces, the new geometric coefficient and the García-Falset coefficient are the same.

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Keywords: generality García-Falset coefficient; weak fixed point property; Musielak-Orlicz sequence spaces

1 Introduction and preliminaries

Throughout this paper X is a *Banach* space which is assumed not to have the *Schur* property, *i.e.*, X has a weakly convergent sequence that is not norm convergent. $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of X, respectively and l^0 denotes the set of all real sequences.

A Banach space X is said to have the fixed point property (FPP, for short) if every nonexpansive mapping $T: C \to C$, *i.e.*, the mapping satisfying

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C,$$

and acting on a nonempty bounded closed and convex subset C of X has a fixed point in C. A natural generalization of FPP is the weak fixed point property (WFPP, for short). A Banach space X is said to have the WFPP whenever it satisfies the above condition from the definition of FPP with 'weakly compact' in place of 'bounded closed'. In 1965 Kirk [2] proved that any reflexive Banach space with normal structure has the FPP. In 1989 Prus [3] introduced a property of a Banach space X, called nearly uniformly smoothness. In 1992 Prus [4] also proved that a weakly nearly uniformly smooth Banach space X with the weak Opial property has the FPP. To obtain the weak fixed point property in Banach spaces, García-Falset introduced in [5] the following coefficient:

$$R(X) = \sup\Big\{\liminf_{n\to\infty} \|x_n - x\| : \{x_n\} \subset B(X), x_n \xrightarrow{w} 0, x \in B(X)\Big\}.$$

He has proved that a Banach space X with $R(X) < 2$ has the weak fixed point property, *i.e.*, every nonexpansive mapping T from a weakly compact nonempty and convex set $A \subset X$ into itself has a fixed point in A (see [6] and [7]).

A Banach space X is said to be *NUS* provided that for every $\varepsilon > 0$ there is $\eta > 0$ such that if $t \in (0, \eta)$ and $\{x_n\}$ is a basic sequence in $B(X)$, then there exists $k > 1$ so that $\|x_1 + tx_k\| \leq 1 + t\varepsilon$ (see [3]).

A natural generalization of this notion is said to be *WNUS*. A Banach space X is *WNUS* whenever it satisfies the above condition with 'for some $\varepsilon \in (0,1)$' in place of 'for every $\varepsilon > 0$' (see [6]).

It is well known that a Banach space X is *WNUS if and only if* X is reflexive and $R(X) < 2$ (see [6]).

The coefficient $R(X, a)$ of a Banach space X was defined in 1996 by Benavides [1], as a generalization of the coefficient $R(X)$, which also plays an important role in the fixed point theory for nonexpansive mappings. Benavides defined the coefficient $R(X, a)$ for a Banach space X as follows.

Definition 1.1 For a given $a \geq 0$,

$$R(X,a) = \sup\Bigl\{\liminf_{n\to\infty} \|x_n - x\| : x_n \in B(X), x_n \xrightarrow{w} 0, D\bigl[(x_n)\bigr] \leq 1, \|x\| \leq a\Bigr\},$$

where $D[(x_n)] = \liminf_{n\to\infty}\{\|x_i - x_j\| : i \neq j, i, j \geq n\}$.

He also defined the following coefficient:

$$M(X) = \sup\left\{\frac{1+a}{R(X,a)} : a \geq 0\right\}.$$

Moreover, the coefficient $R(X, a)$ remains unaltered if in the definition we replace lim inf by lim sup. He obtained the following: for a given $a \geq 0$, if $R(X, a) < 1 + a$, then the Banach space X has the fixed point property; the result means that the condition $M(X) > 1$ implies that X has the weak fixed point property for nonexpansive mappings [1].

In this paper, we introduce a new geometric coefficient that is a related García-Falset coefficient and a weak* fixed point property, written $R^*(X^*, a)$. For a given $a \geq 0$, let

$$R^*\bigl(X^*,a\bigr) = \sup\Bigl\{\lim_{n\to\infty}\inf \|x_n - x\| : x_n \in B\bigl(X^*\bigr), x_n \xrightarrow{w^*} 0, D\bigl[(x_n)\bigr] \leq 1, \|x\| \leq a, x \in X\Bigr\},$$

where $D[(x_n)] = \lim_{n\to\infty}\inf\{\|x_i - x_j\| : i \neq j, i, j \geq n\}$.

Similarly to [1], we can prove that if $R^*(X^*, a) < 1 + a$ for some $a \in (0,1]$ then $T : C \to C$ has a fixed points $a \in C$, if C is a weak*-compact and weak* sequentially complete subset of X^* and T is a nonexpansive mapping. It is clear that $R(X, a) = R^*(X^*, a)$ if the Banach space X is reflexive.

A Banach space X is called a *Köthe* sequence space if it is a subspace of l^0 and for every $x \in l^0$ and $y \in X$ satisfying $|x(i)| \leq |y(i)|$ for all $i \in \mathcal{N}$, we have $x \in X$ and $\|x\| \leq \|y\|$ and if there is a $x = (x(i)) \in X$ with $x(i) > 0$ for all $i \in \mathcal{N}$ (see [8, 9] and [10]).

Let

$$X_a = \Bigl\{x \in X : \lim_{n\to\infty}\bigl\|\bigl(0, 0, \ldots, 0, x(n+1), x(n+2), \ldots\bigr)\bigr\| = 0\Bigr\}.$$

A *Köthe* sequence space X is said to have an absolutely continuous norm if $X_a = X$. A *Köthe* sequence space X is said to have the semi-*Fatou* property if for every sequence $\{x_n\} \subset X$ and $x \in X$ satisfying $|x_n(i)| \uparrow |x(i)|$ for all $i \in N$ we have $\|x_n\| \to \|x\|$.

A mapping $\Phi : \mathcal{R} \to \mathcal{R}_+$ is said to be an *Orlicz function* if Φ vanishes only at 0, Φ is even and convex on the whole line R. For every *Orlicz* function Φ we define *its complementary function* $\Psi : \mathcal{R} \to [0,\infty)$ by the formula

$$\Psi(v) = \sup_{u>0}\{u|v| - \Phi(u)\}$$

for every $v \in \mathcal{R}$.

Denote by $\Phi = \{\Phi_i\}_{i=1}^{\infty}$ a sequence Orlicz function. Such a sequence is called a Musielak-Orlicz function on $N \times R$.

We say that Φ satisfies the δ_2-condition ($\Phi \in \delta_2$ for short) if there exist $k > 0$, $u_0 > 0$,and $c_i \geq 0$, with $\sum_{i\geq 1} c_i < \infty$ such that we have the inequality

$$\Phi_i(2u) \leq k\Phi_i(u) + c_i \quad \bigl(i \geq 1, \Phi_i(u) \leq u_0\bigr).$$

We now introduce a new definition, namely $\overline{\delta}_2(k)$. We say that Φ satisfies the $\overline{\delta}_2(k)$-condition ($\Phi \in \overline{\delta}_2(k)$ for short) if there exist $\varepsilon \in (0,1)$ and $i_\varepsilon \in \mathcal{N}$ such that

$$I_\Phi\left(\frac{x}{2}\right) \leq \frac{1-\varepsilon}{2} I_\Phi(x)$$

whenever $I_\Phi(x) = k$ and $N(x) \geq i_\varepsilon$, where $N(x) = \{i \in \mathcal{N} : x(i) \neq 0\}$ and $N(x) \geq i_\varepsilon$, which means that $\min\{i : i \in N(x)\} \geq i_\varepsilon$.

Proposition 1.2 *The following are equivalent* (*see* [11]):

(1) $\Phi \in \delta_2$;

(2) *for any* $\varepsilon > 0$, *there exist* $k_\varepsilon > 0$, $u_\varepsilon > 0$, *and* $c_i \geq 0$ $(i \geq 1)$, $\sum_{i\geq 1} c_i < \infty$ *such that*

$$\Phi_i\left(\frac{u}{\varepsilon}\right) \leq k_\varepsilon \Phi_i(u) + c_i \quad \bigl(i \geq 1, \Phi_i(u) \leq u_\varepsilon\bigr);$$

(3) *there exist* $\varepsilon \in (0,1)$, $i_\varepsilon \in M$, $c_i \geq 0$ $(i \geq 1)$, $\sum_{i\geq 1} c_i < \infty$ *and* $v_\varepsilon > 0$ *such that*

$$\Psi_i\left(\frac{v}{2}\right) \leq \frac{1-\varepsilon}{2}\Psi_i(v) + c_i \quad \bigl(i \geq 1, \Psi_i(v) \leq v_\varepsilon\bigr).$$

The Musielak-Orlicz sequence space l_Φ is defined to be the set $\{x \in l_0 : I_\Phi(\lambda x) = \sum_{i=1}^{\infty} \Phi_i(\lambda x(i)) < \infty$ for some $\lambda > 0\}$ and its subspace h_Φ is defined to be the set $\{x \in l_0 : I_\Phi(\lambda x) = \sum_{i=1}^{\infty} \Phi_i(\lambda x(i)) < \infty$ for any $\lambda > 0\}$ both equipped with the *Luxemburg* norm

$$\|x\| = \inf\left\{k > 0 : I_\Phi\left(\frac{x}{k}\right) \leq 1\right\}.$$

To simplify notations, we put $l_\Phi = (l_\Phi, \| \|_\Phi)$ and $h_\Phi = (h_\Phi, \| \|_\Phi)$.

We say that a Musielak-Orlicz function Φ satisfies the $\overline{\delta}_2$-condition if its complementary function Ψ satisfies the δ_2-condition.

The basic information on Musielak-Orlicz spaces can be found in [11–13], and [14].

2 Results

The idea of Theorem 2.1 is similar to Corollary 8.2 in [15]. In order to keep the consistency of this paper, we accept it in the following.

Theorem 2.1 *Let X be a Köthe sequence space with the Fatou property. If X has no absolute continuous norm, then $R(X,a) = 1 + a$.*

Proof Suppose that X does not have an absolutely continuous norm. Then there exists $\varepsilon_0 > 0$ and $x_0 \in S(X)$ such that

$$\lim_{n\to\infty}\left\|\sum_{i=n+1}^{\infty} x_0(i)e_i\right\| = \varepsilon_0,$$

where $e_i = (0,0,\ldots,\overset{i\text{th}}{1},0,\ldots)$.

Take a sequence of positive numbers $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$. By $\lim_{n\to\infty}\|\sum_{i=n+1}^{\infty} x_0(i)e_i\| = \varepsilon_0$, there exists $n_1 \in N$ such that

$$\left\|\sum_{i=n_1}^{\infty} x_0(i)e_i\right\| \le (1+\varepsilon_1)\varepsilon_0.$$

Notice that

$$\lim_{m\to\infty}\left\|\sum_{i=n_1+1}^{m} x_0(i)e_i\right\| = \varepsilon_0,$$

so there exists $n_2 > n_1$ such that

$$(1-\varepsilon_1)\varepsilon_0 \le \left\|\sum_{i=n_1+1}^{n_2} x_0(i)e_i\right\| \le (1+\varepsilon_1)\varepsilon_0.$$

In this way, we get by induction a sequence $\{n_i\}$ of natural numbers such that

$$(1-\varepsilon_i)\varepsilon_0 \le \left\|\sum_{j=n_i+1}^{n_{i+1}} x_0(i)e_i\right\| \le (1+\varepsilon_i)\varepsilon_0, \quad i = 1,2,\ldots.$$

Put $x_i = \sum_{j=n_i+1}^{n_{i+1}} x_0(i)e_i$ and $v_k = \sum_{j=n_k+1}^{\infty} x_0(i)e_i$. Then:

(a) $\|x_i\| \to \varepsilon_0$ as $i \to \infty$.

(b) $x_i \xrightarrow{w} 0$ as $i \to \infty$. It is well known that for any *Köthe* space X we have

$$X^* = X' \oplus S,$$

where S is the space of all singular functionals over X, *i.e.*, functionals which vanish on the subspace $X_a = \{x \in X : x \text{ has absolutely continuous norm}\}$ and $X' = \{y \in l^0 : \sum_{i=1}^{\infty} x(i)y(i) < \infty \text{ for all } x \in X\}$ (see [11]). This means that every $f \in X^*$ is uniquely represented in the form

$$f = T_y + \varphi,$$

where $\varphi \in S$ and for $y \in X'$ the function T_y is defined by

$$T_y(x) = \sum_{i=1}^{\infty} x(i)y(i)$$

for all $x \in X$.

Taking any $y \in X$, we have

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} x_n(j)y(j) = \lim_{i\to\infty} \sum_{j=n_i+1}^{n_{i+1}} x_i(j)y(j) = 0.$$

(c) Put $z_i = \frac{x_i}{\|x_i\|}$ and $w_k = \frac{v_k}{\|v_k\|}$ for all $i, k \in \mathcal{N}$. It is easy to check $D(z_k) \le 1$. Then

$$\begin{aligned}
&\liminf_{i\to\infty} \|z_i + aw_k\| \\
&\quad = \liminf_{i\to\infty} \left\| \frac{x_i}{\|x_i\|} + a\frac{v_k}{\|v_k\|} \right\| \\
&\quad = \liminf_{i\to\infty} \frac{1}{\|x_i\|\|v_k\|} \big\| \|v_k\|x_i + a\|x_i\|v_k \big\| \\
&\quad \ge \frac{1}{\varepsilon_0(1+\varepsilon_k)\varepsilon_0} \liminf_{i\to\infty} \big\| \|v_k\|x_i + a\|x_i\|v_k \big\| \\
&\quad = \frac{1}{\varepsilon_0(1+\varepsilon_{kj})\varepsilon_0} \liminf_{i\to\infty} \big\| \big(\|v_k\| + a\|x_i\|\big)x_i + a\|x_i\|(v_k - x_i) \big\| \\
&\quad \ge \frac{1}{\varepsilon_0(1+\varepsilon_k)\varepsilon_0} \Big(\liminf_{i\to\infty} \big(\varepsilon_0 + a(1-\varepsilon_i)\varepsilon_0\big)\|x_i\| - \limsup_{i\to\infty} \big(\|v_k\| - \|x_i\|\big)\|x_i\| \Big) \\
&\quad \ge \frac{1}{\varepsilon_0(1+\varepsilon_k)\varepsilon_0} \Big(\liminf_{i\to\infty} \big(\varepsilon_0 + a(1-\varepsilon_i)\varepsilon_0\big)(1-\varepsilon_i)\varepsilon_0 - \limsup_{i\to\infty} \big(\|v_k\| - \|x_i\|\big)(1+\varepsilon_i)\varepsilon_0 \Big) \\
&\quad \ge \frac{1}{\varepsilon_0(1+\varepsilon_k)\varepsilon_0} \Big(\liminf_{i\to\infty} \big(\varepsilon_0 + a(1-\varepsilon_i)\varepsilon_0\big)(1-\varepsilon_i)\varepsilon_0 \\
&\qquad - \limsup_{i\to\infty} \big((1+\varepsilon_k)\varepsilon_0 - (1-\varepsilon_i)\varepsilon_0\big)\varepsilon_0 \Big) \\
&\quad = \frac{1}{\varepsilon_0(1+\varepsilon_k)\varepsilon_0} \big((\varepsilon_0 + a\varepsilon_0)\varepsilon_0 - \big((1+\varepsilon_k)\varepsilon_0 - \varepsilon_0\big)\varepsilon_0 \big) \\
&\quad = \frac{1}{\varepsilon_0(1+\varepsilon_k)} \big((\varepsilon_0 + a\varepsilon_0) - \varepsilon_k\varepsilon_0 \big) \\
&\quad = \frac{1}{(1+\varepsilon_k)} \big((1+a) - \varepsilon_k \big).
\end{aligned}$$

By the arbitrariness of k and $\lim_{k\to\infty} \varepsilon_k = 0$, we get $R(X,a) \ge 1+a$. It is clear that $R(X,a) \le 1+a$. Therefore $R(X,a) = 1+a$. □

Corollary 2.2 *If* $\Phi \notin \delta_2$ *then* $R(l_\Phi, a) = R^*(l_\Phi, a) = 1 + a$ *for any* $0 < a \le 1$.

Proof Since $\Phi \notin \overline{\delta}_2$, we see that l_Φ has no absolutely continuous norm. So we have $R(l_\Phi, a) = 1 + a$. Since h_Ψ is separable and $(h_\Psi)^* = l_\Phi$, we have $R^*(l_\Phi, a) = 1 + a$. □

For any $x \in \ell_\Phi$ with $\|x\| = a$ and $N(x) = \{i \in N : x(i) \neq 0\}$ being finite, we define c_x by the formula

$$c_x = \lim_{n\to\infty} \sup\left\{c_{x,y} > 0 : I_\Phi\left(\frac{x}{c_{x,y}}\right) + I_\Phi\left(\frac{y}{c_{x,y}}\right) = 1 : y \in \ell_\Phi, I_\Phi(y) \le \frac{1}{2}, n \le N(y) < \infty\right\}.$$

Theorem 2.3 *Suppose that $\Phi \in \delta_2$. Then for the Musielak-Orlicz sequence space ℓ_Φ we have*

$$R^*(\ell_\Phi, a) = \sup\{c_x : x \in \ell_\Phi \textit{ with } \|x\| = a \textit{ and } N(x) \textit{ being finite}\}.$$

Proof Let

$$d_\Phi = \sup\{c_x : x \in \ell_\Phi \text{ with } \|x\| = a \text{ and } N(x) \text{ being finite}\}.$$

Then for any $\varepsilon \in (0, d_\Phi)$, there exists $\|x\| = a$ with finite $N(x)$ such that

$$c_x > d_\Phi - \varepsilon.$$

By the definition of c_x there exists $n_1 \in \mathcal{N}$ such that

$$\sup\left\{c_{x,y} > 0 : I_\Phi\left(\frac{x}{c_{x,y}}\right) + I_\Phi\left(\frac{y}{c_{x,y}}\right) = 1 \text{ for } I_\Phi(y) \le \frac{1}{2} \text{ and } N(y) \ge n_1\right\} > d_\Phi - \varepsilon,$$

whenever $n \ge n_1$. By the definition of the supremum, there exists $y_1 \in S(\ell_\Phi)$ with $N(y_1) > n_1$ such that $c_{x,y_1} > d_\Phi - \varepsilon$, *i.e.*, $I_\Phi(\frac{x}{d_\Phi-\varepsilon}) + I_\Phi(\frac{y_1}{d_\Phi-\varepsilon}) > 1$. Hence there exists $n_2 > n_1$ such that $I_\Phi(\frac{x}{d_\Phi-\varepsilon}) + \sum_{i=n_1+1}^{n_2} \Phi_i(\frac{y_1(i)}{d_\Phi-\varepsilon}) > 1$. Since $n_2 > n_1$, we also have

$$\sup\left\{c_{x,y} > 0 : I_\Phi\left(\frac{x}{c_{x,y}}\right) + I_\Phi\left(\frac{y}{c_{x,y}}\right) = 1 \text{ for } I_\Phi(y) \le \frac{1}{2} \text{ and } N(y) \ge n_2\right\} > d_\Phi - \varepsilon.$$

There exists $y_2 \in \ell_\Phi$ with $I_\Phi(y) \le \frac{1}{2}$ and $N(y_2) > n_1$ such that $c_{x,y_2} > d_\Phi - \varepsilon$, *i.e.*, $I_\Phi(\frac{x}{d_\Phi-\varepsilon}) + I_\Phi(\frac{y_2}{d_\Phi-\varepsilon}) > 1$. Hence there exists $n_2 > n_1$ such that $I_\Phi(\frac{x}{d_\Phi-\varepsilon}) + \sum_{i=n_1+1}^{n_2} \Phi_i(\frac{y_2(i)}{d_\Phi-\varepsilon}) > 1$. Furthermore, there exists $n_3 > n_2$ such that $I_\Phi(\frac{x}{d_\Phi-\varepsilon}) + \sum_{i=n_2+1}^{n_3} \Phi_i(\frac{y_2(i)}{d_\Phi-\varepsilon}) > 1$. In such a way, we can prove by induction that there exist a sequence $\{y_k\}_{k=1}^\infty \subset \ell_\Phi$ with $I_\Phi(y_k) \le \frac{1}{2}$ for any natural k and a sequence of natural numbers $n_1 < n_2 < n_3 < \cdots$ such that $I_\Phi(\frac{x}{d_\Phi-\varepsilon}) + \sum_{i=n_k+1}^{n_{k+1}} \Phi_i(\frac{y_k(i)}{d_\Phi-\varepsilon}) > 1$ for all $k \in \mathcal{N}$. It is clear that y_k is weakly star convergent to 0. Since the supports of y_k are pairwise disjoint, we have $I_\Phi(y_i - y_i) = I_\Phi(y_i) + I_\Phi(y_i) \le \frac{1}{2} + \frac{1}{2} = 1$ for $i, j \in N$ with $i \neq j$. Therefore, $D[(y_k)] \le 1$.

For any $k > i_0$, we have

$$I_\Phi\left(\frac{y_k - x}{d_\Phi - \varepsilon}\right) = \sum_{i=n_k+1}^{n_{k+1}} \Phi_i\left(\frac{y_k(i)}{d_\Phi - \varepsilon}\right) + I_\Phi\left(\frac{x(i)}{d_\Phi - \varepsilon}\right) > 1,$$

i.e., $\|y_k - x\| > d_\Phi - \varepsilon$. Therefore, $R^*(\ell_\Phi, a) \ge d_\Phi - \varepsilon$ and by the arbitrariness of $\varepsilon > 0$, we have $R^*(\ell_\Phi, a) \ge d_\Phi$.

Now, we will prove that $R^*(\ell_\Phi, a) \le d_\Phi$. By the definition of d_Φ, we always have

$$\lim_{n\to\infty} \sup\left\{c_{x,y} > 0 : I_\Phi\left(\frac{x}{c_{x,y}}\right) + I_\Phi\left(\frac{y}{c_{x,y}}\right) = 1 \text{ for } y \in S(\ell_\Phi) \text{ and } N(y) \ge n\right\} \le d_\Phi$$

for any $\|x\| = a$ with finite $N(x)$.

First of all, we want to prove that for any weak star null sequence $\{x_n\} \subset l_\Phi$ and $\varepsilon > 0$ there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $I_\Phi(x_{n_i}) \le \frac{1}{2} + \varepsilon$ for each $i \in N$. Otherwise there exists $\varepsilon_0 > 0$. Without loss of generality, we may assume that $I_\Phi(x_n) > \frac{1}{2} + \varepsilon_0$ for all $n \in \dot{N}$. By $\Phi \in \delta_2$ there exists a $\delta_1 > 0$ such that $\|x\| > 1 + 5\delta_1$ whenever $I_\Phi(x) > 1 + \frac{2\varepsilon_0}{3}$. Using $\Phi \in \delta_2$again, there exists a $\delta_2 > 0$ such that $I_\Phi(x) < \delta_2$ whenever $\|x\| < \frac{\varepsilon_0}{3}$. Set $\delta_0 = \min\{\delta_1, \delta_2\}$.

Put $n_1 = 1$. Then there exists a $i_1 > 1$ such that $\|\sum_{i=i_1+1}^{\infty} x_{n_1}(i)e_i\| < \delta_0 \le \delta_2$. Since the sequence $\{x_n\}$ is a weakly null sequence, there exists $n_2 > n_1$ such that

$$\left\|\sum_{i=1}^{i_1} x_n(i)e_i\right\| < \delta_0 \le \delta_2 \quad \text{whenever } n \ge n_2.$$

Using $\Phi \in \delta_2$ again, we see that there exists a $i_2 > i_1$ such that $\|\sum_{i=i_2+1}^{\infty} x_{n_2}(i)e_i\| < \delta_0 \le \delta_2$. Hence

$$I_\Phi\left(\sum_{i=1}^{i_1} x_{n_2}(i)e_i\right) < \frac{\varepsilon_0}{3} \quad \text{and} \quad I_\Phi\left(\sum_{i=i_2+1}^{\infty} x_{n_2}(i)e_i\right) < \frac{\varepsilon_0}{3}.$$

Therefore $I_\Phi(\sum_{i=i_1+1}^{i_2} x_{n_2}(i)e_i) > \frac{1}{2} + \frac{\varepsilon_0}{3}$.

In such a way, we get a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$I_\Phi\left(\sum_{i=1}^{i_{j-1}} x_{n_j}(i)e_i\right) < \frac{\varepsilon_0}{3}, \qquad I_\Phi\left(\sum_{i=i_j+1}^{\infty} x_{n_2}(i)e_i\right) < \frac{\varepsilon_0}{3}$$

and

$$I_\Phi\left(\sum_{i=i_{j-1}+1}^{i_j} x_{n_2}(i)e_i\right) > \frac{1}{2} + \frac{\varepsilon_0}{3}$$

for all $j \in N$.

So

$$\begin{aligned}
\|x_{n_k} - x_{n_j}\| &= \left\| \sum_{i=1}^{i_{k-1}} x_{n_k}(i)e_i + \sum_{i=i_{k-1}+1}^{i_k} x_{n_k}(i)e_i + \sum_{i=i_k+1}^{\infty} x_{n_k}(i)e_i \right. \\
&\quad \left. - \sum_{i=1}^{i_{j-1}} x_{n_j}(i)e_i - \sum_{i=i_{j-1}+1}^{i_j} x_{n_j}(i)e_i - \sum_{i=i_j+1}^{\infty} x_{n_j}(i)e_i \right\| \\
&\ge \left\| \sum_{i=i_{k-1}+1}^{i_k} x_{n_k}(i)e_i - \sum_{i=i_{j-1}+1}^{i_j} x_{n_j}(i)e_i \right\| - 4\delta_0.
\end{aligned}$$

By

$$\begin{aligned} &I_\Phi\left(\sum_{i=i_{k-1}+1}^{i_k} x_{n_k}(i)e_i - \sum_{i=i_{j-1}+1}^{i_j} x_{n_j}(i)e_i\right) \\ &\quad = I_\Phi\left(\sum_{i=i_{k-1}+1}^{i_k} x_{n_k}(i)e_i\right) + I_\Phi\left(\sum_{i=i_{j-1}+1}^{i_j} x_{n_j}(i)e_i\right) \\ &\quad \geq \frac{1}{2} + \frac{\varepsilon_0}{3} + \frac{1}{2} + \frac{\varepsilon_0}{3} = 1 + \frac{2\varepsilon_0}{3}, \end{aligned}$$

we have

$$\left\|\left(\sum_{i=i_{k-1}+1}^{i_k} x_{n_k}(i)e_i - \sum_{i=i_{j-1}+1}^{i_j} x_{n_j}(i)e_i\right)\right\| \geq 1 + 5\delta_1 \geq 1 + 5\delta_0.$$

Hence

$$\|x_{n_k} - x_{n_j}\| \geq \left\|\sum_{i=i_{k-1}+1}^{i_k} x_{n_k}(i)e_i - \sum_{i=i_{j-1}+1}^{i_j} x_{n_j}(i)e_i\right\| - 4\delta_0 \geq 1 + \delta_0.$$

This contradicts with the inequality $D[(x_n)] \leq 1$.

For convenience, we may assume that $I_\Phi(x_n) \leq \frac{1}{2} + \varepsilon$ for all $n \in N$. Hence

$$I_\Phi\left(\frac{1-\varepsilon}{1+\varepsilon}x_n\right) \leq \frac{1-\varepsilon}{1+\varepsilon}I_\Phi(x_n) \leq \frac{1-\varepsilon}{1+\varepsilon}\left(\frac{1}{2} + \varepsilon\right) = \frac{1}{2}$$

for all $n \in N$.

Let us take an element $x \in l_\Phi$ with $\|x\| = a$. By $\Phi \in \delta_2$, for any $\varepsilon > 0$ there exists $i_0 > 0$ such that $\|\sum_{i=i_0+1}^{\infty} x(i)e_i\| < \varepsilon$. Put $x_0 = a\frac{\sum_{i=i_0+1}^{\infty} x(i)e_i}{\|\sum_{i=i_0+1}^{\infty} x(i)e_i\|}$. Since $x_n \xrightarrow{w^*} 0$, there is a $n_0 \in N$ such that $\|\sum_{i=1}^{i_0} x_n(i)e_i\| < \varepsilon$ when $n \geq n_0$.

Hence

$$\begin{aligned} \|x_n - x\| &\leq \left\|\sum_{i=1}^{i_0}\bigl(x_n(i) - x(i)\bigr)e_i + \sum_{i=i_0+1}^{\infty}\bigl(x_n(i) - x(i)\bigr)e_i\right\| \\ &\leq \left\|\sum_{i=1}^{i_0} x(i)e_i + \sum_{i=i_0+1}^{\infty} x_n(i)e_i\right\| + 2\varepsilon. \end{aligned}$$

We next estimate the $\|\sum_{i=1}^{i_0} x(i)e_i + \sum_{i=i_0+1}^{\infty} x_n(i)e_i\|$. Put $z_n = \frac{1-\varepsilon}{1+\varepsilon}\sum_{i=i_0+1}^{\infty} x_n(i)e_i$ for $n \geq n_0$. We have

$$\begin{aligned} &I_\Phi\left(\frac{a\frac{\sum_{i=1}^{i_0} x(i)e_i}{\|\sum_{i=1}^{i_0} x(i)e_i\|} - \frac{1-\varepsilon}{1+\varepsilon}\sum_{i=i_0+1}^{\infty} x_n(i)e_i}{d_\Phi + \varepsilon}\right) \\ &\quad = I_\Phi\left(\frac{x_0}{d_\Phi + \varepsilon}\right) + I_\Phi\left(\frac{z_n}{d_\Phi + \varepsilon}\right) \end{aligned}$$

$$\le I_\Phi\left(\frac{x_0}{c_{x_0,z_n}}\right) + I_\Phi\left(\frac{z_n}{c_{x_0,z_n}}\right)$$
$$= 1,$$

whence

$$\left\| a\frac{\sum_{i=1}^{i_0} x(i)e_i}{\|\sum_{i=1}^{i_0} x(i)e_i\|} - \frac{1-\varepsilon}{1+\varepsilon}\sum_{i=i_0+1}^{\infty} x_n(i)e_i \right\| \le d_\Phi + \varepsilon$$

for $n \ge n_0$. Therefore, we obtain the inequalities

$$\begin{aligned}
&\left\| \sum_{i=1}^{i_0} x(i)e_i - \sum_{i=i_0+1}^{\infty} x_n(i)e_i \right\| \\
&\le \left\| x_0\frac{\|\sum_{i=1}^{i_0} x(i)e_i\|}{a} - \frac{1-\varepsilon}{1+\varepsilon}\sum_{i=i_0+1}^{\infty} x_n(i)e_i - \frac{2\varepsilon}{1+\varepsilon}\sum_{i=i_0+1}^{\infty} x_n(i)e_i \right\| \\
&\le a\left(\frac{\|\sum_{i=1}^{i_0} x(i)e_i\|}{a} - 1\right) + \left\| a\frac{\sum_{i=1}^{i_0} x(i)e_i}{\|\sum_{i=1}^{i_0} x(i)e_i\|} - z_n \right\| + \frac{2\varepsilon}{1+\varepsilon}\left\| \sum_{i=i_0+1}^{\infty} x_n(i)e_i \right\| \\
&\le a\left(\frac{a+\varepsilon}{a} - 1\right) + \left\| a\frac{\sum_{i=1}^{i_0} x(i)e_i}{\|\sum_{i=1}^{i_0} x(i)e_i\|} - z_n \right\| + \frac{2\varepsilon}{1+\varepsilon} \\
&\le a\varepsilon + d_\Phi + \varepsilon + \frac{2\varepsilon}{1+\varepsilon} = d_\Phi + \left(\frac{2+(1+a)(1+\varepsilon)}{1+\varepsilon}\right)\varepsilon.
\end{aligned}$$

Therefore, we have $\|x_n - x_0\| \le d_\Phi + (\frac{2+(1+a)(1+\varepsilon)}{1+\varepsilon})\varepsilon + 2\varepsilon$. By the arbitrariness of $\varepsilon > 0$, we get the inequality $R(\ell_\Phi) \le d_\Phi$.

Summing up, we see that the equality $R(\ell_\Phi) = d_\Phi$ holds. □

For any $x \in \ell_\Phi$ with $\|x\| = 1$ and $N(x) = \{i \in N : x(i) \neq 0\}$ being finite we define c_x as follows:

$$\tilde{c}_x = \lim_{n\to\infty} \sup\left\{ c_{x,y} > 0 : I_\Phi\left(\frac{x}{c_{x,y}}\right) + I_\Phi\left(\frac{y}{c_{x,y}}\right) = 1 \text{ for } y \in \ell_\Phi \text{ with } I_\Phi(y) \le 1,\right.$$
$$\left. n \le N(y) < \infty \right\}.$$

Corollary 2.4 *If* $\Phi \in \delta_2$ *then* $R(l_\Phi) = \sup\{\tilde{c}_x : x \in \ell_\Phi$ *with* $\|x\| = 1$ *and* $N(x)$ *being finite*$\}$.

Proof The proof is similar to the proof of Theorem 2.3. □

Corollary 2.5 *If* $\Phi \in \delta_2$ *and* $\Phi \in \overline{\delta}_2$ *then* $R(l_\Phi, a) = R^*(l_\Phi, a)$ *for any* $0 < a \le 1$.

Theorem 2.6 $R^*(\ell_\Phi, 1) < 2$ *if and only if* $\ell_\Phi \in \delta_2$ *and* $\Phi \in \overline{\delta}_2(1)$.

Proof Necessity. We only need to prove the necessity of $\Phi \in \overline{\delta}_2(1)$. Suppose that $\Phi \notin \overline{\delta}_2(1)$. Then for any natural number k there exists $x_k \in S(\ell_\Phi)$ with $N(x_k) \ge k$ such that

$$I_\Phi\left(\frac{x_k}{2}\right) > \frac{1-\frac{1}{k}}{2} I_\Phi(x_k).$$

For any fixed $k \in N$ and for any $\varepsilon > 0$, there exists $i_k \in N$ such that $\sum_{i=k}^{i_k} \Phi_i(x_k(i)) > 1 - \varepsilon$. Put $\overline{x}_k = \frac{\sum_{i=k}^{i_k} x_k(i)e_i}{\| \sum_{i=k}^{i_k} x_k(i)e_i \|}$. Then

$$\begin{aligned} I_\Phi\left(\frac{\overline{x}_k}{2}\right) + I_\Phi\left(\frac{x_{i_k}}{2}\right) &= I_\Phi\left(\frac{\sum_{i=k}^{i_k} x_k(i)e_i}{2\| \sum_{i=k}^{i_k} x_k(i)e_i \|}\right) + I_\Phi\left(\frac{x_{i_k}}{2}\right) \\ &\geq I_\Phi\left(\frac{\sum_{i=k}^{i_k} x_k(i)e_i}{2}\right) + I_\Phi\left(\frac{x_{i_k}}{2}\right) + I_\Phi\left(\frac{\sum_{i=i_k+1}^{\infty} x_k(i)e_i}{2}\right) - \varepsilon \\ &= I_\Phi\left(\frac{x_k}{2}\right) + I_\Phi\left(\frac{x_{i_k}}{2}\right) - \varepsilon \geq \frac{1-\frac{1}{k}}{2} + \frac{1-\frac{1}{i_k}}{2} - \varepsilon \\ &= 1 - \frac{1}{k} - \varepsilon, \end{aligned}$$

which shows that $\|\overline{x}_k + x_{i_k}\| \geq 2(1 - \frac{1}{k} - \varepsilon)$. Hence $c_{\overline{x}_k, x_{i_k}} \geq 2(1 - \frac{1}{k} - \varepsilon)$. By the arbitrariness of $\varepsilon > 0$ and $k \in N$, we see that $R(\ell_\Phi) = 2$.

Sufficiency. Since $\Phi \in \overline{\delta}_2(1)$, there exist $\varepsilon_0 \in (0,1)$ and $i_\varepsilon \in \mathcal{N}$ such that

$$I_\Phi\left(\frac{x}{2}\right) \leq \frac{1-\varepsilon_0}{2} I_\Phi(x),$$

whenever $I_\Phi(x) = 1$ and $N(x) \geq i_\varepsilon$. For any $x \in S(\ell_\Phi)$ with finite $N(x)$ and $y \in S(\ell_\Phi)$ with $N(y) \geq n$, we may assume without loss of generality that $\max\{i : i \in N(x)\} < n$ and $n \geq i_\varepsilon$. Hence

$$I_\Phi\left(\frac{x}{2}\right) + I_\Phi\left(\frac{y}{2}\right) \leq \frac{1}{2} + \frac{1-\varepsilon}{2} = \frac{2-\varepsilon}{2}.$$

Since $\Phi \in \delta_2$, there exists $0 < \alpha < 1$ such that

$$\|z\| \leq \alpha \quad \text{whenever } I_\Phi(z) \leq \frac{2-\varepsilon}{2}.$$

Therefore $\|\frac{x+y}{2}\| \leq \alpha$, *i.e.*, $\|x + y\| \leq 2\alpha$. Note that $c_{x,y} = \|x + y\|$. Hence $R(\ell_\Phi) \leq 2\alpha < 2$. □

Theorem 2.7 *$R^*(\ell_\Phi, a) < 1 + a$ for $0 < a < 1$ if and only if $\Phi \in \delta_2$.*

Proof We only need to prove the sufficiency. For any $0 < \varepsilon < \frac{1}{2}$, by $\Phi \in \delta_2$, there exists a $d_0 > 0$ such that $\|x\| \leq 1 - d$ whenever $I_\Phi(x) \leq \frac{1}{2} + \varepsilon$. Hence $\|x_n\| \leq 1 - d$ if n large enough for any weakly star null sequence $\{x_n\} \subset B(l_\Phi)$ with $D[(x_n)] \leq 1$. Hence

$$\liminf_{n\to\infty} \|x_n - x\| \leq 1 - d + a < 1 + a,$$

that is, $R^*(l_\Phi, a) < 1 + a$. □

Example 2.8 Let $\Phi_n(u) = \begin{cases} u^2 & \text{if } u \leq \frac{1}{n}, \\ a_n u + b_n & \text{if } \frac{1}{n} \leq u \leq \infty, \end{cases}$ where $a_n = \frac{2}{n}$, $b_n = -\frac{1}{n^2}$. Then Φ_n is an Orlicz function for each $n \in N$.

If $u \leq \frac{1}{n}$ and $2u > \frac{1}{n}$, then $\frac{1}{2n} < u \leq \frac{1}{n}$. Hence

$$\Phi_n(2u) = \frac{2}{n} 2u - \frac{1}{n^2} \leq \frac{4}{n^2} - \frac{1}{n^2} = \frac{3}{n^2} \leq 12 \frac{1}{4n^2} \leq 12u^2 = 12\Phi_n(u).$$

If $0 < u \leq \frac{1}{2n}$ then $\Phi_n(2u) = 4\Phi_n(u)$. If $\frac{1}{n} < u \leq 1$ then $\Phi_n(2u) \leq 2\Phi_n(u)$. If we put $K = 24$ and $u_0 = 1$, then

$$\Phi_n(2u) \leq K\Phi_n(u) \quad \text{for all } n \in N,$$

that is, $\Phi \in \delta_2$.

If $0 < a < 1$ then $R^*(l_\Phi, a) < 1 + a$.

Let us take $x_n = (0, 0, \ldots, 0, \overbrace{\frac{n}{2} + \frac{1}{2n}}^{n\text{th}}, 0, \ldots)$ for any $n \in N$. Then $I_\Phi(x_n) = 1$ and

$$I_\Phi\left(\frac{x_n}{2}\right) = \frac{2}{n}\left(\frac{n}{4} + \frac{1}{4n}\right) - \frac{1}{n^2} = \frac{1}{2} - \frac{1}{2n^2},$$

whence $\lim_{n\to\infty} I_\Phi(\frac{x_n}{2}) = \frac{1}{2}$. Therefore $\Phi \notin \delta_2(1)$, which implies that $R^*(l_\Phi, 1) = 2$.

Let $\{p_k\}_{k=1}^{\infty}$ be a sequence of real increasing numbers with $1 < p_1$ and $\lim_{n\to\infty} p_n = p < \infty$. Then we have the following.

Theorem 2.9 *Let $l^{(p_i)}$ be a Nakano sequence space equipped with the Luxemburg norm. Then $R(l^{(p_i)}) = 2^{\frac{1}{p}}$ and $R(l_\Phi, a) = (\frac{1}{2} + a^p)^{\frac{1}{p}}$, $0 < a \leq 1$.*

Proof Since $1 < \inf\{p_i\} \leq \sup\{p_i\} = p < \infty$, the Nakano space equipped with the Luxemburg norm is reflexive. For any $x, y \in S(l^{(p_i)})$ with $N(x)$, $N(y)$ being finite. We now consider the following equation:

$$I_\Phi\left(\frac{x}{c}\right) + I_\Phi\left(\frac{y}{c}\right) = 1,$$

that is,

$$\sum_{i=1}^{\infty}\left|\frac{x(i)}{c}\right|^{p_i} + \sum_{i=1}^{\infty}\left|\frac{y(i)}{c}\right|^{p_i} = 1.$$

Then

$$\frac{1}{c^p}\sum_{i=1}^{\infty}\left|x(i)\right|^{p_i} + \frac{1}{c^p}\sum_{i=1}^{\infty}\left|x_n(i)\right|^{p_i} = \frac{2}{c^p} \leq 1,$$

i.e., $c \leq 2^{\frac{1}{p}}$. This shows the inequality $R(l^{(p_i)}, 1) \leq 2^{\frac{1}{p}}$. Take the classical basic sequence $\{e_n\} \subset S(l^{(p_i)})$. If $c_{n,m}$ is a solution of the equation

$$I_\Phi\left(\frac{e_n}{c}\right) + I_\Phi\left(\frac{e_m}{c}\right) = 1,$$

and assuming without loss of generality that we may take $n > m$, we have the inequality

$$\frac{2}{c^{p_n}} \geq 1.$$

Hence $R(l^{(p_i)}) \geq 2^{\frac{1}{p}}$. Together with the opposite inequality proved already, we have $R(l^{(p_i)}) = 2^{\frac{1}{p}}$.

If $0 < a < 1$, for $x \in B(l^{(p_i)})$ with finite $N(x)$ and $I_\Phi(x) = \frac{1}{2}$ and $y \in B(l^{(p_i)})$ with finite $N(y)$ and $\|x\| = a$, we consider the following equation:

$$I_\Phi\left(\frac{x}{c_{x,y}}\right) + I_\Phi\left(\frac{y}{c_{x,y}}\right) = 1,$$

that is, the equation

$$\sum_{i=1}^{\infty}\left|\frac{x(i)}{c_{x,y}}\right|^{p_i} + \sum_{i=1}^{\infty}\left|\frac{y(i)}{c_{x,y}}\right|^{p_i} = 1.$$

Hence

$$\frac{1}{c_{x,y}^p}\sum_{i=1}^{\infty}\left|x(i)\right|^{p_i} + \frac{a^{p_{\max\{i:i\in N(y)\}}}}{c_{x,y}^p}\sum_{i=1}^{\infty}\left|\frac{y(i)}{a}\right|^{p_i} = \frac{1}{2c_{x,y}^p} + \frac{a^{p_{\max\{i:i\in N(y)\}}}}{c_{x,y}^p} \geq 1,$$

where $c_{x,y}^p \leq \frac{1}{2} + a^{p_{\max\{i:i\in N(y)\}}}$. Therefore $R(l_\Phi, a) \leq (\frac{1}{2} + a^p)^{\frac{1}{p}}$.

Taking $x_n = (0, \ldots, 0, (\frac{1}{2})^{\frac{1}{p_n}}, 0, \ldots)$ and $x_m = (0, \ldots, 0, \overset{m\text{th}}{a}, 0, \ldots)$, we get $I_\Phi(x_n) = \frac{1}{2}$ and $I_\Phi(\frac{x_m}{a}) = 1$, which implies the equality $\|x_m\| = a$.

For any $n \neq m$, if $c > 0$ is such that

$$I_\Phi\left(\frac{x_n + x_m}{c}\right) = 1,$$

then

$$\left(\frac{1}{2} + a^{p_m}\right)^{\frac{1}{\max\{p_n, p_m\}}} \leq c.$$

Letting $n, m \to \infty$, we get $R(l_\Phi, a) \leq (\frac{1}{2} + a^p)^{\frac{1}{p}}$, that is, $R(l_\Phi, a) = (\frac{1}{2} + a^p)^{\frac{1}{p}}$. □

Competing interests
The author declares that they have no competing interests.

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