Wang and Wang *Journal of Inequalities and Applications* (2015) 2015:233
DOI 10.1186/s13660-015-0756-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

General L_p-dual Blaschke bodies and the applications

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Abstract

Lutwak defined the dual Blaschke combination of star bodies. In this paper, based on the L_p-dual Blaschke combination of star bodies, we define the general L_p-dual Blaschke bodies and obtain the extremal values of their volume and L_p-dual affine surface area. Further, as the applications, we study two negative forms of the L_p-Busemann-Petty problems.

MSC: 52A20; 52A40

Keywords: general L_p-dual Blaschke body; extremal value; L_p-Busemann-Petty problem

1 Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in the Euclidean space $\mathbb{R}^n$. $\mathcal{K}_c^n$ denotes the set of convex bodies whose centroid lies at the origin in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, its volume is written by $\omega_n = V(B)$.

If K is a compact star shaped (about the origin) in $\mathbb{R}^n$, then its radial function $\rho_K = \rho(K,\cdot)$ is defined on S^{n-1} by letting (see [1, 2])

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, \quad u \in S^{n-1}.$$

If ρ_K is positive and continuous, then K will be called a star body (about the origin). For the set of star bodies containing the origin in their interiors and the set of origin-symmetric star bodies in $\mathbb{R}^n$, we write $\mathcal{S}_o^n$ and $\mathcal{S}_{os}^n$, respectively. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

The notion of dual Blaschke combination was given by Lutwak (see [3]). For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero), $n \geq 2$, the dual Blaschke combination $\lambda \circ K \oplus \mu \circ L \in \mathcal{S}_o^n$ of K and L is defined by

$$\rho(\lambda \circ K \oplus \mu \circ L, \cdot)^{n-1} = \lambda \rho(K,\cdot)^{n-1} + \mu \rho(L,\cdot)^{n-1},$$

where the operation '$\oplus$' is called dual Blaschke addition and $\lambda \circ K$ denotes dual Blaschke scalar multiplication.

Combining with the definition of dual Blaschke combination, Lutwak [3] gave the concept of dual Blaschke body as follows: For $K \in \mathcal{S}_o^n$, take $\lambda = \mu = 1/2$, $L = -K$, the dual Blaschke body $\overline{\nabla} K$ is given by

$$\overline{\nabla} K = \frac{1}{2} \circ K \oplus \frac{1}{2} \circ (-K).$$

In this paper, we define the notion of L_p-dual Blaschke combination as follows: For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero), $n > p > 0$, the L_p-dual Blaschke combination $\lambda \circ K \oplus_p \mu \circ L \in \mathcal{S}_o^n$ of K and L is defined by

$$\rho(\lambda \circ K \oplus_p \mu \circ L, \cdot)^{n-p} = \lambda \rho(K, \cdot)^{n-p} + \mu \rho(L, \cdot)^{n-p}, \tag{1.1}$$

where the operation '$\oplus_p$' is called L_p-dual Blaschke addition and $\lambda \circ K = \lambda^{\frac{1}{n-p}} K$.

Let $\lambda = \mu = \frac{1}{2}$ and $L = -K$ in (1.1), then the L_p-dual Blaschke body $\overline{\nabla}_p K$ of $K \in \mathcal{S}_o^n$ is given by

$$\overline{\nabla}_p K = \frac{1}{2} \circ K \oplus_p \frac{1}{2} \circ (-K). \tag{1.2}$$

Now, by (1.1) we define the general L_p-dual Blaschke bodies as follows: For $K \in \mathcal{S}_o^n$, $n > p > 0$ and $\tau \in [-1, 1]$, the general L_p-dual Blaschke body $\overline{\nabla}_p^\tau K$ of K is defined by

$$\rho\big(\overline{\nabla}_p^\tau K, \cdot\big)^{n-p} = f_1(\tau) \rho(K, \cdot)^{n-p} + f_2(\tau) \rho(-K, \cdot)^{n-p}, \tag{1.3}$$

where

$$f_1(\tau) = \frac{1+\tau}{2}, \qquad f_2(\tau) = \frac{1-\tau}{2}. \tag{1.4}$$

From (1.4), we have that

$$f_1(\tau) + f_2(\tau) = 1, \tag{1.5}$$

$$f_1(-\tau) = f_2(\tau), \qquad f_2(-\tau) = f_1(\tau). \tag{1.6}$$

From (1.3), it easily follows that

$$\overline{\nabla}_p^\tau K = f_1(\tau) \circ K \oplus_p f_2(\tau) \circ (-K). \tag{1.7}$$

Besides, by (1.2), (1.4) and (1.7), we see that if $\tau = 0$, then $\overline{\nabla}_p^0 K = \overline{\nabla}_p K$; if $\tau = \pm 1$, then $\overline{\nabla}_p^{+1} K = K$, $\overline{\nabla}_p^{-1} K = -K$.

The main results of this paper can be stated as follows: First, we give the extremal values of the volume of general L_p-dual Blaschke bodies.

Theorem 1.1 *If $K \in \mathcal{S}_o^n$, $n > p > 0$, $\tau \in [-1, 1]$, then*

$$V(\overline{\nabla}_p K) \leq V\big(\overline{\nabla}_p^\tau K\big) \leq V(K). \tag{1.8}$$

If $\tau \neq 0$, equality holds in the left inequality if and only if K is origin-symmetric, if $\tau \neq \pm 1$, then equality holds in the right inequality if and only if K is also origin-symmetric.

Moreover, based on the L_p-dual affine surface area $\widetilde{\Omega}_p(K)$ of $K \in \mathcal{S}_o^n$ (see (2.7)), we give another class of extremal values for general L_p-dual Blaschke bodies.

Theorem 1.2 *If* $K \in \mathcal{S}_o^n$, $n > p > 0$, $\tau \in [-1,1]$, *then*

$$\widetilde{\Omega}_p(\overline{\nabla}_p K) \le \widetilde{\Omega}_p(\overline{\nabla}_p^{\tau} K) \le \widetilde{\Omega}_p(K). \tag{1.9}$$

If $\tau \ne 0$, *equality holds in the left inequality if and only if* K *is origin-symmetric, if* $\tau \ne \pm 1$, *then equality holds in the right inequality if and only if* K *is also origin-symmetric*.

Theorems 1.1 and 1.2 belong to a part of new and rapidly evolving asymmetric L_p Brunn-Minkowski theory that has its origins in the work of Ludwig, Haberl and Schuster (see [4–9]). For the studies of asymmetric L_p Brunn-Minkowski theory, also see [10–22].

Haberl and Ludwig [5] defined the L_p-intersection body as follows: For $K \in \mathcal{S}_o^n$, $0 < p < 1$, the L_p-intersection body I_pK of K is the origin-symmetric star body whose radial function is given by

$$\rho_{I_pK}^p(u) = \int_K |u \cdot x|^{-p}\, dx \tag{1.10}$$

for all $u \in S^{n-1}$. Haberl and Ludwig [5] pointed out that the classical intersection body which was introduced by Lutwak (see [3]) IK of K is obtained as a limit of the L_p-intersection body of K, more precisely, for all $u \in S^{n-1}$,

$$\rho(IK, u) = \lim_{p \longrightarrow 1^-} (1-p)\rho(I_pK, u)^p. \tag{1.11}$$

Associated with the L_p-intersection bodies, Haberl [4] obtained a series of results, Berck [23] investigated their convexity. For further results on L_p-intersection bodies, also see [1, 2, 18, 24–27]. In particular, Yuan and Cheung (see [26]) gave the negative solutions of L_p-Busemann-Petty problems as follows.

Theorem 1.A *Let* $K \in \mathcal{S}_o^n$ *and* $0 < p < 1$, *if* K *is not origin-symmetric, then there exists* $L \in \mathcal{S}_{os}^n$ *such that*

$$I_pK \subset I_pL,$$

but

$$V(K) > V(L).$$

As the application of Theorem 1.1, we extend the scope of negative solutions of L_p-Busemann-Petty problems from origin-symmetric star bodies to star bodies.

Theorem 1.3 *Let* $K \in \mathcal{S}_o^n$ *and* $0 < p < 1$, *if* K *is not origin-symmetric, then there exists* $L \in \mathcal{S}_o^n$ *such that*

$$I_pK \subset I_pL,$$

but

$$V(K) > V(L).$$

Similarly, applying Theorem 1.2, we get the form of L_p-dual affine surface areas for the negative solutions of L_p-Busemann-Petty problems.

Theorem 1.4 *For $K \in \mathcal{S}_o^n$, $0 < p < 1$, if K is not origin-symmetric, then there exists $L \in \mathcal{S}_o^n$ such that*

$$I_p K \subset I_p L,$$

but

$$\widetilde{\Omega}_p(K) > \widetilde{\Omega}_p(L).$$

In this paper, the proofs of Theorems 1.1-1.4 will be given in Section 4. In Section 3, we obtain some properties of general L_p-dual Blaschke bodies.

2 Preliminaries

2.1 L_p-Dual mixed volume

For $K, L \in S_o^n$, $p > 0$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-radial combination, $\lambda \cdot K \tilde{+}_p \mu \cdot L \in S_o^n$, of K and L is defined by (see [4, 28])

$$\rho(\lambda \cdot K \tilde{+}_p \mu \cdot L, \cdot)^p = \lambda\rho(K, \cdot)^p + \mu\rho(L, \cdot)^p, \tag{2.1}$$

where $\lambda \cdot K$ denotes the L_p-radial scalar multiplication, and we easily know $\lambda \cdot K = \lambda^{\frac{1}{p}} K$.

Associated with (2.1), Haberl in [4] (also see [28]) introduced the notion of L_p-dual mixed volume as follows: For $K, L \in \mathcal{S}_o^n$, $p > 0$, $\varepsilon > 0$, the L_p-dual mixed volume $\widetilde{V}_p(K, L)$ of K and L is defined by

$$\frac{n}{p}\widetilde{V}_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \tilde{+}_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

And he got the following integral form of L_p-dual mixed volume:

$$\widetilde{V}_p(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-p}(u)\rho_L^p(u)\, du, \tag{2.2}$$

where the integration is with respect to spherical Lebesgue measure on S^{n-1}.

From (2.2), we get that

$$\widetilde{V}_p(K, K) = V(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(u)\, du. \tag{2.3}$$

The Minkowski inequality of L_p-dual mixed volume is as follows (see [4, 28]): *If $K, L \in S_o^n$, then for $0 < p < n$,*

$$\widetilde{V}_p(K, L) \leq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}; \tag{2.4}$$

for $p > n$,

$$\widetilde{V}_p(K,L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}. \tag{2.5}$$

In every case, equality holds if and only if K *is a dilate of* L. *For* $p = n$, (2.4) (*or* (2.5)) *is identical.*

From (2.4) and (2.5), we easily get the following result.

Proposition 2.1 *If* $K, L \in \mathcal{S}_o^n$, $p > 0$, *and for any* $Q \in \mathcal{S}_o^n$,

$$\widetilde{V}_p(K,Q) = \widetilde{V}_p(L,Q)$$

or

$$\widetilde{V}_p(Q,K) = \widetilde{V}_p(Q,L),$$

then

$$K = L.$$

2.2 L_p-Dual affine surface area

The notion of L_p-dual affine surface area was given by Wang, Yuan and He (see [29]). For $K \in \mathcal{S}_o^n$, $0 < p < n$, the L_p-dual affine surface area $\widetilde{\Omega}_p(K)$ of K is defined by

$$n^{-\frac{p}{n}} \widetilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\bigl\{n\widetilde{V}_p\bigl(K,Q^*\bigr)V(Q)^{\frac{p}{n}} : Q \in \mathcal{K}_c^n\bigr\}. \tag{2.6}$$

Here E^* is the polar set of a nonempty set E which is defined by (see [1])

$$E^* = \bigl\{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in E\bigr\}.$$

For the sake of convenience of our work, we improve definition (2.6) from $Q \in \mathcal{K}_c^n$ to $Q \in \mathcal{S}_{os}^n$ as follows: For $K \in \mathcal{S}_o^n$, $0 < p < n$, the L_p-dual affine surface area $\widetilde{\Omega}_p(K)$ of K is defined by

$$n^{-\frac{p}{n}} \widetilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\bigl\{n\widetilde{V}_p\bigl(K,Q^*\bigr)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\bigr\}. \tag{2.7}$$

3 Some properties of general L_p-dual Blaschke bodies

In this section, we give some properties of general L_p-dual Blaschke bodies.

Theorem 3.1 *If* $K \in \mathcal{S}_o^n$, $n > p > 0$ *and* $\tau \in [-1,1]$, *then*

$$\overline{\nabla}_p^{-\tau} K = \overline{\nabla}_p^{\tau}(-K) = -\overline{\nabla}_p^{\tau} K.$$

Proof From (1.6) and (1.7), we obtain that for $n > p > 0$ and $\tau \in [-1,1]$,

$$\overline{\nabla}_p^{-\tau} K = f_1(-\tau) \circ K \oplus_p f_2(-\tau) \circ (-K) = f_2(\tau) \circ K \oplus_p f_1(\tau) \circ (-K) = \overline{\nabla}_p^{\tau}(-K).$$

Further, we have that for any $u \in S^{n-1}$,

$$\begin{aligned}\rho\left(-\overline{\nabla}_p^{\tau}K, u\right)^{n-p} &= \rho\left(\overline{\nabla}_p^{\tau}K, -u\right)^{n-p} \\ &= f_1(\tau)\rho(K, -u)^{n-p} + f_2(\tau)\rho(-K, -u)^{n-p} \\ &= f_1(\tau)\rho(-K, u)^{n-p} + f_2(\tau)\rho\left(-(-K), u\right)^{n-p} \\ &= \rho\left(\overline{\nabla}_p^{\tau}(-K), u\right)^{n-p}.\end{aligned}$$

Hence, we get

$$\overline{\nabla}_p^{\tau}(-K) = -\overline{\nabla}_p^{\tau}K. \qquad \square$$

Theorem 3.2 *For $K \in \mathcal{S}_o^n$, $n > p > 0$ and $\tau \in [-1,1]$, if $\tau \neq 0$, then $\overline{\nabla}_p^{\tau}K = \overline{\nabla}_p^{-\tau}K$ if and only if $K \in \mathcal{S}_{os}^n$.*

Proof From (1.3) and (1.6), we get that for all $u \in S^{n-1}$,

$$\rho\left(\overline{\nabla}_p^{\tau}K, u\right)^{n-p} = f_1(\tau)\rho(K, u)^{n-p} + f_2(\tau)\rho(-K, u)^{n-p}, \tag{3.1}$$

$$\rho\left(\overline{\nabla}_p^{-\tau}K, u\right)^{n-p} = f_2(\tau)\rho(K, u)^{n-p} + f_1(\tau)\rho(-K, u)^{n-p}. \tag{3.2}$$

Hence, if $K \in \mathcal{S}_{os}^n$, *i.e.*, $K = -K$, then by (3.1), (3.2) and (1.5) we get, for all $u \in S^{n-1}$,

$$\rho\left(\overline{\nabla}_p^{\tau}K, u\right)^{n-p} = \rho\left(\overline{\nabla}_p^{-\tau}K, u\right)^{n-p}.$$

Thus

$$\overline{\nabla}_p^{\tau}K = \overline{\nabla}_p^{-\tau}K.$$

Conversely, if $\overline{\nabla}_p^{\tau}K = \overline{\nabla}_p^{-\tau}K$, then together with (3.1) and (3.2) it yields

$$\left[f_1(\tau) - f_2(\tau)\right]\rho(K, u)^{n-p} = \left[f_1(\tau) - f_2(\tau)\right]\rho(-K, u)^{n-p}.$$

Since $f_1(\tau) - f_2(\tau) \neq 0$ when $\tau \neq 0$, thus it follows that $\rho(K, u) = \rho(-K, u)$ for all $u \in S^{n-1}$, *i.e.*, $K \in \mathcal{S}_{os}^n$. $\square$

From Theorem 3.2, it immediately yields the following corollary.

Corollary 3.1 *For $K \in \mathcal{S}_o^n$, $n > p > 0$ and $\tau \in [-1,1]$, if K is not origin-symmetric, then $\overline{\nabla}_p^{\tau}K = \overline{\nabla}_p^{-\tau}K$ if and only if $\tau = 0$.*

Theorem 3.3 *If $K \in \mathcal{S}_{os}^n$, $n > p > 0$ and $\tau \in [-1,1]$, then*

$$\overline{\nabla}_p^{\tau}K = K.$$

Proof Since $K \in \mathcal{S}_{os}^n$, *i.e.*, $K = -K$, by (1.3) and (1.5) we know that, for any $u \in S^{n-1}$,

$$\rho\left(\overline{\nabla}_p^{\tau}K, u\right)^{n-p} = f_1(\tau)\rho(K, u)^{n-p} + f_2(\tau)\rho(-K, u)^{n-p} = \rho(K, u)^{n-p}.$$

That is,

$$\overline{\nabla}_p^{\tau} K = K. \qquad \square$$

4 Proofs of theorems

In this section, we complete the proofs of Theorems 1.1-1.4.

Lemma 4.1 *If* $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (*not both zero*), $n > p > 0$, *then*

$$V(\lambda \circ K \oplus_p \mu \circ L)^{\frac{n-p}{n}} \leq \lambda V(K)^{\frac{n-p}{n}} + \mu V(L)^{\frac{n-p}{n}}, \tag{4.1}$$

with equality if and only if K *and* L *are dilates.*

Proof Associated with (1.1), (2.2), (2.3) and inequality (2.4), we know that, for any $Q \in \mathcal{S}_o^n$,

$$\begin{aligned} \widetilde{V}_p(\lambda \circ K \oplus_p \mu \circ L, Q) &= \lambda \widetilde{V}_p(K, Q) + \mu \widetilde{V}_p(L, Q) \\ &\leq \left[\lambda V(K)^{\frac{n-p}{n}} + \mu V(L)^{\frac{n-p}{n}}\right] V(Q)^{\frac{p}{n}}. \end{aligned}$$

Let $Q = \lambda \circ K \oplus_p \mu \circ L$, it yields (4.1). From the equality condition of (2.4), we see that equality holds in (4.1) if and only if K is a dilate of L. $\square$

Proof of Theorem 1.1 By (4.1), (1.5) and (1.7), we get, for any $\tau \in [-1, 1]$,

$$\begin{aligned} V\left(\overline{\nabla}_p^{\tau} K\right)^{\frac{n-p}{n}} &= V\left(f_1(\tau) \circ K \oplus_p f_2(\tau) \circ (-K)\right)^{\frac{n-p}{n}} \\ &\leq f_1(\tau) V(K)^{\frac{n-p}{n}} + f_2(\tau) V(-K)^{\frac{n-p}{n}} \\ &= V(K)^{\frac{n-p}{n}}. \end{aligned}$$

Therefore, we obtain, for $n > p > 0$,

$$V\left(\overline{\nabla}_p^{\tau} K\right) \leq V(K). \tag{4.2}$$

This gives the right inequality of (1.8).

Clearly, equality holds in (4.2) if $\tau = \pm 1$. Besides, if $\tau \neq \pm 1$, then by the condition of equality in (4.1), we see that equality holds in (4.2) if and only if K and $-K$ are dilates, this yields $K = -K$, *i.e.*, K is an origin-symmetric star body. This means that if $\tau \neq \pm 1$, then equality holds in the right inequality of (1.8) if and only if K is origin-symmetric.

Now, we prove the left inequality of (1.8). From (1.2), (1.4) and (1.7), we know that for any $u \in S^{n-1}$,

$$\begin{aligned} \overline{\nabla}_p K &= \frac{1}{2} \circ K \oplus_p \frac{1}{2} \circ (-K) \\ &= \frac{1}{2} \frac{(1+\tau) + (1-\tau)}{2} \circ K \oplus_p \frac{1}{2} \frac{(1-\tau) + (1+\tau)}{2} \circ (-K) \\ &= \frac{1}{2} \circ \overline{\nabla}_p^{\tau} K \oplus_p \frac{1}{2} \circ \overline{\nabla}_p^{-\tau} K. \end{aligned} \tag{4.3}$$

From Theorem 3.1 and (4.3), use (4.1) to yield that for $n > p > 0$,

$$\begin{aligned} V(\overline{\nabla}_p K)^{\frac{n-p}{n}} &= V\left(\frac{1}{2}\circ \overline{\nabla}_p^{\tau} K \oplus_p \frac{1}{2}\circ \overline{\nabla}_p^{-\tau} K\right)^{\frac{n-p}{n}} \\ &\leq \frac{1}{2} V\big(\overline{\nabla}_p^{\tau} K\big)^{\frac{n-p}{n}} + \frac{1}{2} V\big(\overline{\nabla}_p^{-\tau} K\big)^{\frac{n-p}{n}} \\ &= \frac{1}{2} V\big(\overline{\nabla}_p^{\tau} K\big)^{\frac{n-p}{n}} + \frac{1}{2} V\big(-\overline{\nabla}_p^{\tau} K\big)^{\frac{n-p}{n}} \\ &= V\big(\overline{\nabla}_p^{\tau} K\big)^{\frac{n-p}{n}}. \end{aligned}$$

This gives that for $n > p > 0$,

$$V(\overline{\nabla}_p K) \leq V\big(\overline{\nabla}_p^{\tau} K\big). \tag{4.4}$$

This is just the left inequality of (1.8).

Obviously, if $\tau = 0$, then equality holds in (4.4). If $\tau \neq 0$, according to the equality condition of (4.1), we see that equality holds in (4.4) if and only if $\widehat{\nabla}_p^{\tau} K$ and $\overline{\nabla}_p^{-\tau} K$ are dilates, this implies $\overline{\nabla}_p^{\tau} K = \overline{\nabla}_p^{-\tau} K$. Therefore, using Corollary 3.1, we obtain that if K is not an origin-symmetric body, then equality holds in (4.4) if and only if $\tau = 0$. This shows that if $\tau \neq 0$, then equality holds in the left inequality of (1.8) if and only if K is origin-symmetric. □

Proof of Theorem 1.2 From definition (2.7) and (1.7), we have that

$$\begin{aligned} & n^{-\frac{p}{n}} \widetilde{\Omega}_p\big(\overline{\nabla}_p^{\tau} K\big)^{\frac{n+p}{n}} \\ &\quad = \sup\big\{n \widetilde{V}_p\big(\widehat{\nabla}_p^{\tau} K, Q^*\big) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\big\} \\ &\quad = \sup\big\{n \widetilde{V}_p\big(f_1(\tau)\circ K \oplus_p f_2(\tau)\circ(-K), Q^*\big) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\big\} \\ &\quad = \sup\left\{\int_{S^{n-1}} \big[\rho\big(f_1(\tau)\circ K \oplus_p f_2(\tau)\circ(-K), u\big)^{n-p} \rho\big(Q^*, u\big)^p\big]\, du V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\right\} \\ &\quad = \sup\left\{\int_{S^{n-1}} \big[f_1(\tau)\rho(K,u)^{n-p} + f_2(\tau)\rho(-K,u)^{n-p}\big]\rho\big(Q^*,u\big)^p\, du V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\right\} \\ &\quad = \sup\big\{n f_1(\tau)\widetilde{V}_p\big(K, Q^*\big) V(Q)^{\frac{p}{n}} + n f_2(\tau)\widetilde{V}_p\big(-K, Q^*\big) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\big\} \\ &\quad \leq f_1(\tau)\sup\big\{n \widetilde{V}_p\big(K, Q^*\big) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\big\} \\ &\qquad + f_2(\tau)\sup\big\{n \widetilde{V}_p\big(-K, Q^*\big) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\big\}. \end{aligned} \tag{4.5}$$

Since $Q \in \mathcal{S}_{os}^n$, thus use $\rho(Q,u) = \rho(-Q,u) = \rho(Q,-u)$ for all $u \in S^{n-1}$ to get

$$\widetilde{V}_p\big(-K, Q^*\big) = \widetilde{V}_p\big(K, Q^*\big),$$

by (2.7) we know $\widetilde{\Omega}_p(K) = \widetilde{\Omega}_p(-K)$. This combining with (4.5) and (1.5), we know

$$\widetilde{\Omega}_p\big(\overline{\nabla}_p^{\tau} K\big) \leq \widetilde{\Omega}_p(K), \tag{4.6}$$

i.e., the right inequality of (1.9) is obtained.

If $\tau \neq \pm 1$, equality of (4.5) holds if and only if K and $-K$ are dilates. This yields $K = -K$, thus K is an origin-symmetric star body. Since (4.5) and (4.6) are equivalent, hence equality holds in (4.6) if and only if K is an origin-symmetric star body when $\tau \neq \pm 1$. Therefore, if $\tau \neq \pm 1$, equality holds in the right inequality of (1.9) if and only if K is origin-symmetric.

Further, we complete the proof of the left inequality of (1.9). From Theorem 3.1, we know that

$$\overline{\nabla}_p^{-\tau} K = -\overline{\nabla}_p^{\tau} K.$$

Thus, (4.3) can be written as

$$\overline{\nabla}_p K = \frac{1}{2} \circ \overline{\nabla}_p^{\tau} K \oplus_p \frac{1}{2} \circ \left(-\overline{\nabla}_p^{\tau} K\right).$$

Similar to the proof of inequality (4.6), we have

$$\widetilde{\Omega}_p(\overline{\nabla}_p K) \leq \widetilde{\Omega}_p\left(\overline{\nabla}_p^{\tau} K\right). \tag{4.7}$$

This yields the left inequality of (1.9).

Similar to the proof of equality in inequality (4.6), we easily know that equality holds in (4.7) if and only if $\overline{\nabla}_p^{\tau} K = \overline{\nabla}_p^{-\tau} K$ when $\tau \neq 0$. Hence, if $\tau \neq 0$, using Theorem 3.2 we get that equality holds in the left inequality of (1.9) if and only if K is origin-symmetric. □

In order to prove Theorems 1.3 and 1.4, the following lemma is required.

Lemma 4.2 *If $K \in S_o^n$, $0 < p < 1$ and $\tau \in [-1, 1]$, then*

$$I_p\left(\overline{\nabla}_p^{\tau} K\right) = I_p K.$$

Proof From definition (1.10), we may obtain the following polar coordinate form:

$$\rho(I_p K, u)^p = \frac{1}{n-p} \int_{S^{n-1}} |u \cdot v|^{-p} \rho(K, v)^{n-p}\, dv.$$

Thus by (1.3) we have that

$$\begin{aligned} \rho\left(I_p\left(\overline{\nabla}_p^{\tau} K\right), u\right)^p &= \frac{1}{n-p} \int_{S^{n-1}} |u \cdot v|^{-p} \rho\left(\overline{\nabla}_p^{\tau} K, v\right)^{n-p}\, dv \\ &= \frac{1}{n-p} \int_{S^{n-1}} |u \cdot v|^{-p} \left[f_1(\tau)\rho(K, v)^{n-p} + f_2(\tau)\rho(-K, v)^{n-p}\right] dv \\ &= f_1(\tau)\rho(I_p K, u)^p + f_2(\tau)\rho\left(I_p(-K), u\right)^p. \end{aligned} \tag{4.8}$$

According to (1.10), we easily know $I_p(-K) = I_p K$, so combining with (4.8) and (1.5), then for any $u \in S^{n-1}$,

$$\rho\left(I_p\left(\overline{\nabla}_p^{\tau} K\right), u\right)^p = \rho(I_p K, u)^p,$$

i.e.,

$$I_p\left(\overline{\nabla}_p^\tau K\right) = I_p K. \qquad \square$$

Proof of Theorem 1.3 Since K is not an origin-symmetric star body, thus from Theorem 1.1, we know that if $\tau \neq \pm 1$, then

$$V\left(\overline{\nabla}_p^\tau K\right) < V(K).$$

Choose $\varepsilon > 0$ such that $V((1+\varepsilon)\overline{\nabla}_p^\tau K) < V(K)$. Therefore, let $L = (1+\varepsilon)\overline{\nabla}_p^\tau K$ (for $\tau = 0$, $L \in S_{os}^n$; for $\tau \neq 0$, $L \in S_o^n$), then

$$V(L) < V(K).$$

But from Lemma 4.2, and notice that $I_p((1+\varepsilon)K) = (1+\varepsilon)^{\frac{n-p}{p}} I_p K$, we can get

$$I_p L = I_p\left((1+\varepsilon)\overline{\nabla}_p^\tau L\right) = (1+\varepsilon)^{\frac{n-p}{p}} I_p\left(\overline{\nabla}_p^\tau K\right) = (1+\varepsilon)^{\frac{n-p}{p}} I_p K \supset I_p K. \qquad \square$$

Proof of Theorem 1.4 Since K is not an origin-symmetric star body, thus by Theorem 1.2, we know that for $\tau \neq \pm 1$,

$$\widetilde{\Omega}_p\left(\overline{\nabla}_p^\tau K\right) < \widetilde{\Omega}_p(K).$$

Choose $\varepsilon > 0$ such that $\widetilde{\Omega}_p((1+\varepsilon)\overline{\nabla}_p^\tau K) < \widetilde{\Omega}_p(K)$. Therefore, let $L = (1+\varepsilon)\overline{\nabla}_p^\tau K$, then $L \in S_o^n$ and

$$\widetilde{\Omega}_p(L) < \widetilde{\Omega}_p(K).$$

But, similar to the proof of Theorem 1.3, we may obtain $I_p L \supset I_p K$. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to sincerely thank the referees for very valuable and helpful comments and suggestions which made the paper more accurate and readable. Research is supported in part by the Natural Science Foundation of China (Grant No. 11371224), Innovation Foundation and Excellent Foundation of Graduate Student of China Three Gorges University (Grant Nos. 2013CX084, 2014PY065).

Received: 26 December 2014 Accepted: 11 July 2015 Published online: 28 July 2015

References
1. Gardner, RJ: Geometric Tomography, 2nd edn. Cambridge University Press, Cambridge (2006)
2. Schneider, R: Convex Bodies: The Brunn-Minkowski Theory, 2nd edn. Cambridge University Press, Cambridge (2014)
3. Lutwak, E: Intersection bodies and dual mixed volumes. Adv. Math. **71**, 232-261 (1988)
4. Haberl, C: L_p-Intersection bodies. Adv. Math. **4**, 2599-2624 (2008)
5. Haberl, C, Ludwig, M: A characterization of L_p intersection bodies. Int. Math. Res. Not. **2006**, Article ID 10548 (2006)
6. Haberl, C, Schuster, FE: General L_p affine isoperimetric inequalities. J. Differ. Geom. **83**, 1-26 (2009)
7. Haberl, C, Schuster, FE: Asymmetric affine L_p Sobolev inequalities. J. Funct. Anal. **257**, 641-658 (2009)
8. Ludwig, M: Minkowski valuations. Trans. Am. Math. Soc. **357**, 4191-4213 (2005)
9. Ludwig, M: Intersection bodies and valuations. Am. J. Math. **128**, 1409-1428 (2006)
10. Feng, YB, Wang, WD: General L_p-harmonic Blaschke bodies. Proc. Indian Acad. Sci. Math. Sci. **124**(1), 109-119 (2014)

11. Feng, YB, Wang, WD, Lu, FH: Some inequalities on general L_p-centroid bodies. Math. Inequal. Appl. **18**(1), 39-49 (2015)
12. Haberl, C, Schuster, FE, Xiao, J: An asymmetric affine Pólya-Szegö principle. Math. Ann. **352**, 517-542 (2012)
13. Parapatits, L: SL(n)-Covariant L_p-Minkowski valuations. J. Lond. Math. Soc. **89**, 397-414 (2014)
14. Parapatits, L: SL(n)-Contravariant L_p-Minkowski valuations. Trans. Am. Math. Soc. **366**, 1195-1211 (2014)
15. Schuster, FE, Wannerer, T: GL(n) Contravariant Minkowski valuations. Trans. Am. Math. Soc. **364**, 815-826 (2012)
16. Schuster, FE, Weberndorfer, M: Volume inequalities for asymmetric Wulff shapes. J. Differ. Geom. **92**, 263-283 (2012)
17. Wang, WD, Feng, YB: A general L_p-version of Petty's affine projection inequality. Taiwan. J. Math. **17**(2), 517-528 (2013)
18. Wang, WD, Li, YN: Busemann-Petty problems for general L_p-intersection bodies. Acta Math. Sin. Engl. Ser. **31**(5), 777-786 (2015)
19. Wang, WD, Ma, TY: Asymmetric L_p-difference bodies. Proc. Am. Math. Soc. **142**(7), 2517-2527 (2014)
20. Wang, WD, Wan, XY: Shephard type problems for general L_p-projection bodies. Taiwan. J. Math. **16**(5), 1749-1762 (2012)
21. Wannerer, T: GL(n) Equivariant Minkowski valuations. Indiana Univ. Math. J. **60**, 1655-1672 (2011)
22. Weberndorfer, M: Shadow systems of asymmetric L_p zonotopes. Adv. Math. **240**, 613-635 (2013)
23. Berck, G: Convexity of L_p-intersection bodies. Adv. Math. **222**(3), 920-936 (2009)
24. Kalton, NJ, Koldobsky, A: Intersection bodies and L_p spaces. Adv. Math. **196**, 257-275 (2005)
25. Lu, FH, Mao, WH: Affine isoperimetric inequalities for L_p-intersection bodies. Rocky Mt. J. Math. **40**, 489-500 (2010)
26. Yuan, J, Cheung, WS: L_p-Intersection bodies. J. Math. Anal. Appl. **339**(2), 1431-1439 (2008)
27. Yu, WY, Wu, DH, Leng, GS: Quasi L_p-intersection bodies. Acta Math. Sin. **23**(11), 1937-1948 (2007)
28. Grinberg, E, Zhang, GY: Convolutions transforms and convex bodies. Proc. Lond. Math. Soc. **78**(3), 77-115 (1999)
29. Wang, W, Yuan, J, He, BW: Large inequalities for L_p-dual affine surface area. Math. Inequal. Appl. **7**, 34-45 (2008)