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Global L^2 estimates for a class of maximal operators associated to general dispersive equations

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Abstract

For a function ϕ satisfying some suitable growth conditions, consider the general dispersive equation defined by $\begin{cases} i\partial_t u + \phi(\sqrt{-\Delta})u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & f \in \mathcal{S}(\mathbb{R}^n). \end{cases}$ $(*)$ In the present paper, we give some global L^2 estimate for the maximal operator S_ϕ^*, which is defined by $S_\phi^* f(x) = \sup_{0<t<1} |S_{t,\phi} f(x)|$, $x \in \mathbb{R}^n$, where $S_{t,\phi} f$ is a formal solution of the equation $(*)$. Especially, the estimates obtained in this paper can be applied to discuss the properties of solutions of the fractional Schrödinger equation, the fourth-order Schrödinger equation and the beam equation.

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1 Introduction and main results

Suppose $f \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz class on $\mathbb{R}^n$, denote

$$S_t f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it|\xi|^2} \hat{f}(\xi)\, d\xi, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx$. It is well known that $u(x,t) := S_t f(x)$ is the solution of the Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.1}$$

In 1979, Carleson [1] proposed a problem: if $f \in H^s(\mathbb{R}^n)$ for which s does

$$\lim_{t\to 0} u(x,t) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n, \tag{1.2}$$

where $H^s(\mathbb{R}^n)$ $(s \in \mathbb{R})$ denotes the non-homogeneous Sobolev space, which is defined by

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' : \|f\|_{H^s} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s \bigl|\hat{f}(\xi)\bigr|^2\, d\xi \right)^{1/2} < \infty \right\}.$$

Carleson first studied this problem for dimension $n = 1$ in [1]. He proved that the convergence (1.2) holds for $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$. This result is sharp, which was shown

Table 1 Convergence (1.2) holds for $f \in H^s(\mathbb{R}^n)$

Dim.	Range of s	Authors
$n = 1$	$s \geq \frac{1}{4}$	Carleson [1] in 1979
$n \geq 2$	$s > \frac{1}{2}$	Sjölin [3] in 1987 and Vega [4] in 1988, independently
$n = 2$	for some $s < \frac{1}{2}$	Bourgain [5] in 1992
$n = 2$	$s > \kappa$ with $\frac{20}{41} < \kappa < \frac{41}{84}$	Moyua, Vargas and Vega [6] in 1996
$n = 2$	$s > \frac{15}{32}$	Tao and Vargas [7] in 2000
$n = 2$	$s > \frac{2}{5}$	Tao [8] in 2003
$n = 2$	$s > \frac{3}{8}$	Lee [9] in 2006
$n \geq 3$	$s > \frac{1}{2} - \frac{1}{4n}$	Bourgain [10] in 2013

by Dahlberg and Kenig [2]. See Table 1 for the results on the convergence (1.2) when $f \in H^s(\mathbb{R}^n)$.

Moreover, the convergence (1.2) fails if $s < \frac{1}{4}$ (see [2] for $n = 1$ and [4] for $n \geq 2$). Recently, Bourgain [10] showed that the necessary condition of convergence (1.2) is $s \geq \frac{1}{2} - \frac{1}{n}$ when $n > 4$.

It is well known that the pointwise convergence (1.2) is related closely to the local estimate of the local maximal operator S^* defined by

$$S^*f(x) = \sup_{0<t<1} \left|S_t f(x)\right|, \quad x \in \mathbb{R}^n.$$

Naturally, the maximal estimates have been well studied associated with the following oscillatory integral:

$$S_{t,a}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^a} \hat{f}(\xi)\, d\xi, \quad t \in \mathbb{R} \text{ and } a > 0,$$

which is the solution of the fractional Schrödinger equation:

$$\begin{cases} i\partial_t u + (-\Delta)^{a/2} u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.3}$$

Define the local maximal operator associated with the family of operators $\{S_{t,a}\}_{0<t<1}$ by

$$S_a^*f(x) = \sup_{0<t<1} \left|S_{t,a}f(x)\right|, \quad x \in \mathbb{R}^n.$$

Obviously, the following estimate (1.4) can be applied to discuss the pointwise convergence problem on the solution of Schrödinger equation (1.3):

$$\left\|S_a^*f\right\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}, \tag{1.4}$$

which is called the *global L^2 estimate* of the maximal operator S_a^* sometimes. These estimates have also independent interest since they reveal global regularity properties of the corresponding oscillatory integrals. Table 2 shows some main results in studying (1.4).

On the other hand, in 1990, Prestini [16] proved that, if $f \in H^s(\mathbb{R}^n)$ ($n \geq 2$) is a radial function, then the local maximal estimate

$$\left\|S^*f\right\|_{L^1(B)} \leq c_n\|f\|_{H^s} \tag{1.5}$$

Table 2 Global L^2 estimate (1.4) for $f \in H^s(\mathbb{R}^n)$

Dim.	Ran. of a	Ran. of s	Authors
$n \geq 1$	$a > 0$	$s > \frac{a}{2}$	Cowling [11] in 1983 and Carbery [12] in 1985, independently
$n = 1$	$a \geq 2$	$s > \frac{a}{4}$	Kenig, Ponce and Vega [13] in 1991
$n = 1$	$a > 1$	$s > \frac{a}{4}$	Sjölin [14] in 1994
$n = 1$	$0 < a < 1$	$s > \frac{a}{4}$	Walther [15] in 2002

holds if and only if $s \geq \frac{1}{4}$. In 1997, Sjölin [17] proved (1.4) holds for $a > 1$ and $s > \frac{a}{4}$. In 2012, Walther [18] showed (1.4) holds for $0 < a < 1$ and $s > \frac{a}{4}$.

In the present paper, we will discuss some global L^2 maximal estimates like (1.4) for a local maximal operator S^*_ϕ associated with the operator family $\{S_{t,\phi}\}_{t\in\mathbb{R}}$. Let us first give some definitions as follows: Suppose the function $\phi : \mathbb{R}^+ \to \mathbb{R}$ satisfies:

(K1) there exists $l_1 \geq 0$ such that $|\phi(r)| \lesssim r^{l_1}$ for all $0 < r < 1$;
(K2) there exists $m_1 \in \mathbb{R}$ such that $|\phi(r)| \lesssim r^{m_1}$ for all $r \geq 1$;
(K3) there exists $m_2 \in \mathbb{R}$ such that $|\phi'(r)| \lesssim r^{m_2-1}$ for all $r \geq 1$;
(K4) there exists $m_3 \in \mathbb{R}$ such that $|\phi''(r)| \sim r^{m_3-2}$ for all $r \geq 1$;
(K5) there exists $m_4 \in \mathbb{R}$ such that $|\phi^{(3)}(r)| \lesssim r^{m_4-3}$ for all $r \geq 1$.

The operator family $\{S_{t,\phi}\}_{t\in\mathbb{R}}$ is defined by

$$S_{t,\phi}f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it\phi(|\xi|)} \hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n, \tag{1.6}$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and the local maximal operator S^*_ϕ associated with $\{S_{t,\phi}\}_{t\in\mathbb{R}}$ is defined by

$$S^*_\phi f(x) = \sup_{0<t<1} \bigl|S_{t,\phi}f(x)\bigr|, \quad x \in \mathbb{R}^n.$$

Now we state our main results in this paper as follows.

Theorem 1.1 *For $n = 1$ and ϕ satisfies* (K1)-(K5) *with $l_1 \geq 0$, $m_i \in \mathbb{R}$ $(1 \leq i \leq 4)$, and $m_2 = m_3 \geq m_4$. If $f \in H^s(\mathbb{R})$ with $s > \frac{m_2}{4}$ for $m_2 > 0$ or $s > \frac{-m_2}{2}$ for $m_2 \leq 0$, then*

$$\bigl\|S^*_\phi f(x)\bigr\|_{L^2(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})}. \tag{1.7}$$

Theorem 1.2 *For $n \geq 2$ and ϕ satisfying* (K1)-(K5) *with $l_1 \geq 0$, $m_i \in \mathbb{R}$ $(1 \leq i \leq 4)$, and $m_2 = m_3 \geq m_4$. If $f \in H^s(\mathbb{R}^n)$ is radial with $s > \frac{m_2}{4}$ for $m_2 > 0$ or $s > \frac{-m_2}{2}$ for $m_2 \leq 0$, then*

$$\bigl\|S^*_\phi f(x)\bigr\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}. \tag{1.8}$$

Now let us turn to the other result obtained in the present paper, which involves the functions class formed by the radial function and the functions in $\mathscr{A}_k$, the set of all solid spherical harmonics of degree k. It is well known (see [19], p.151) that there exists a direct sum decomposition

$$L^2\bigl(\mathbb{R}^n\bigr) = \sum_{k=0}^{\infty} \oplus \mathfrak{D}_k.$$

The subspace $\mathfrak{D}_k$ is the space of all finite linear combinations of functions of the form $f(|x|)P(x)$, where f ranges over the radial functions and P over $\mathscr{A}_k$ such that $f(|\cdot|)P(\cdot) \in L^2(\mathbb{R}^n)$.

Fix $k \geq 0$ and let $P_1, P_2, \ldots, P_{a_k}$ denote an orthonormal basis in $\mathscr{A}_k$. Every element in $\mathfrak{D}_k$ can be written in the following form:

$$f(x) = \sum_{j=1}^{a_k} f_j\big(|x|\big)P_j(x) \tag{1.9}$$

and

$$\int_{\mathbb{R}^n} \big|f(x)\big|^2\,dx = \sum_{j=1}^{a_k} \int_0^\infty \big|f_j(r)\big|^2 r^{n+2k-1}\,dr.$$

Denote by $\mathcal{H}_0(\mathbb{R}^n)$ the class of all radial functions in $\mathcal{S}(\mathbb{R}^n)$, and $\mathcal{H}_k$ $(k \in \mathbb{N})$ the set of functions defined by (1.9) with $f_j \in \mathcal{H}_0(\mathbb{R}^n)$ and $P_j \in \mathscr{A}_k$ for $j = 1, 2, \ldots, a_k$. Sjölin obtained the following result (see [20], p.397).

Theorem A *Suppose that* $n \geq 2$, $a > 1$, *and* $f \in \mathcal{H}_k$ $(k \geq 0)$. *If* $s > \frac{a}{4}$ *then* (1.4) *holds.*

We give the global L^2 estimate of the maximal operator S_ϕ^* for $f \in \mathcal{H}_k$.

Theorem 1.3 *For* $n \geq 2$ *and* ϕ *satisfies* (K1)-(K5) *with* $l_1 \geq 0$, $m_i \in \mathbb{R}$ $(1 \leq i \leq 4)$, *and* $m_2 = m_3 \geq m_4$. *If* $f \in \mathcal{H}_k$ $(k \geq 0)$ *with* $s > \frac{m_2}{4}$ *for* $m_2 > 0$ *or* $s > \frac{-m_2}{2}$ *for* $m_2 \leq 0$, *then* (1.8) *holds.*

Note that

$$u(x,t) = e^{it\phi(\sqrt{-\Delta})}f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it\phi(|\xi|)}\hat{f}(\xi)\,d\xi = S_{t,\phi}f(x)$$

gives a formal solution of the following general dispersive equation with initial data function f:

$$\begin{cases} i\partial_t u + \phi(\sqrt{-\Delta})u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.10}$$

Hence, the inequalities (1.7) and (1.8) imply the convergence almost everywhere of the solution of (1.10) in one dimension and higher dimension, respectively.

The proofs of Theorems 1.1-1.3 are given in Sections 2-4, respectively. In the last section, we will give some examples of (1.10).

2 Proof of Theorem 1.1

2.1 Proof of Theorem 1.1 based on Lemma 2.2

In this subsection, we give the proof of Theorem 1.1 by using Lemma 2.2, which will be proved in the next subsection.

Choose a nonnegative function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset \{\xi : \frac{1}{2} < |\xi| < 2\}$ and

$$\sum_{k=-\infty}^{\infty} \varphi\big(2^{-k}\xi\big) = 1, \quad \xi \neq 0.$$

Set $\varphi_0(\xi) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}\xi)$ and $\psi(\xi) = \sum_{k=1}^{\infty} \varphi(2^{-k}\xi)$. It follows that $\varphi_0 \in C_0^{\infty}(\mathbb{R})$. Rewrite

$$\begin{aligned} S_{t,\phi}f(x) &= (2\pi)^{-1} \int_{\mathbb{R}} e^{ix\cdot\xi + it\phi(|\xi|)} \varphi_0(\xi)\hat{f}(\xi)\,d\xi \\ &\quad + (2\pi)^{-1} \sum_{k=1}^{\infty} \int_{\mathbb{R}} e^{ix\cdot\xi + it\phi(|\xi|)} \varphi\bigl(2^{-k}\xi\bigr)\hat{f}(\xi)\,d\xi \\ &=: S_{t,\phi,0}f(x) + \sum_{k=1}^{\infty} S_{t,\phi,k}f(x). \end{aligned} \tag{2.1}$$

Denote

$$S^*_{\phi,0}f(x) = \sup_{0<t<1} \bigl|S_{t,\phi,0}f(x)\bigr|, \quad x \in \mathbb{R}$$

and

$$S^*_{\phi,k}f(x) = \sup_{0<t<1} \bigl|S_{t,\phi,k}f(x)\bigr|, \quad x \in \mathbb{R}.$$

Therefore, by (2.1), we obtain

$$S^*_{\phi}f(x) \le S^*_{\phi,0}f(x) + \sum_{k=1}^{\infty} S^*_{\phi,k}f(x). \tag{2.2}$$

By (2.2) and Minkowski's inequality, we get

$$\bigl\|S^*_{\phi}f\bigr\|_{L^2(\mathbb{R})} \le \bigl\|S^*_{\phi,0}f\bigr\|_{L^2(\mathbb{R})} + \sum_{k=1}^{\infty} \bigl\|S^*_{\phi,k}f\bigr\|_{L^2(\mathbb{R})}. \tag{2.3}$$

Now let us recall a result which will be used in our proof of Theorem 1.1.

Lemma 2.1 (see [18]) *Assume that the functions ω_1 and ω_2 belong to $L^2(\mathbb{R})$ and that the function m satisfies the following assumption: there is a number C independent of (t,ξ) such that*

$$\bigl|m(t,\xi)\bigr| \le C\omega_1(t), \qquad \left|\frac{\partial(m(t,\xi))}{\partial t}\right| \le C\bigl(\omega_1(t) + \omega_2(t)|\xi|^a\bigr), \quad a > 0.$$

Then there is a number C independent of f such that

$$\left(\int_{\mathbb{R}^n} \sup_{0<t<1} \left|\int_{\mathbb{R}^n} e^{ix\cdot\xi} m(t,\xi)\hat{f}(\xi)\,d\xi\right|^2 dx\right)^{1/2} \le C\|f\|_{L^2(\mathbb{R}^n)}, \quad \operatorname{supp}\hat{f} \subseteq \bigl\{\xi, |\xi| < 2\bigr\}.$$

We first prove that if $s > \frac{m_2}{4}$ for $m_2 > 0$ or $s > \frac{-m_2}{2}$ for $m_2 \le 0$, then

$$\bigl\|S^*_{\phi,0}f\bigr\|_{L^2(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}. \tag{2.4}$$

For $g \in \mathcal{S}(\mathbb{R})$ and $\operatorname{supp}\hat{g} \subseteq \{\xi, |\xi| < 2\}$, $s > \frac{m_2}{4}$ for $m_2 > 0$ or $s > \frac{-m_2}{2}$ for $m_2 \le 0$ and $0 < t < 1$, let

$$R_{0,t}g(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{ix\cdot\xi} e^{it\phi(|\xi|)} \bigl(1 + |\xi|^2\bigr)^{-s/2} \hat{g}(\xi)\,d\xi =: \int_{\mathbb{R}} e^{ix\cdot\xi} m(t,\xi)\hat{g}(\xi)\,d\xi,$$

where $m(t,\xi) = (2\pi)^{-1}e^{it\phi(|\xi|)}(1+|\xi|^2)^{-s/2}$. Define the maximal operator R_0^* by

$$R_0^* g(x) = \sup_{0<t<1} \left|R_{0,t} g(x)\right|, \quad x \in \mathbb{R}.$$

On the one hand, it is obvious that $|m(t,\xi)| \le \chi_{(0,1)}(t)$ for $\xi \in \mathbb{R}$ and $0 < t < 1$. On the other hand, by

$$\frac{\partial(m(t,\xi))}{\partial t} = \frac{i}{2\pi} e^{it\phi(|\xi|)} \phi\big(|\xi|\big)\big(1+|\xi|^2\big)^{-s/2},$$

it follows that

$$\left|\frac{\partial(m(t,\xi))}{\partial t}\right| \le \chi_{(0,1)}(t)\left|\phi\big(|\xi|\big)\right| \quad \text{for } \xi \in \mathbb{R} \text{ and } 0 < t < 1. \tag{2.5}$$

By the condition (K1), $|\phi(|\xi|)| \le C\max\{|\phi(1)|, 1\} \le C$ for $0 \le |\xi| < 1$. By (K2), we have, for $|\xi| \ge 1$,

$$\left|\phi\big(|\xi|\big)\right| \le \begin{cases} C|\xi|^{m_1}, & m_1 > 0, \\ C|\xi|^{m_1} \le C, & m_1 \le 0. \end{cases}$$

Hence, combining with (2.5) we get, for $\xi \in \mathbb{R}$,

$$\left|\frac{\partial(m(t,\xi))}{\partial t}\right| \le \begin{cases} C(\chi_{(0,1)}(t) + \chi_{(0,1)}(t)|\xi|^{m_1}), & m_1 > 0, \\ C(\chi_{(0,1)}(t) + \chi_{(0,1)}(t)|\xi|), & m_1 \le 0, \end{cases}$$

where C is independent of (t,ξ). It follows that $m(t,\xi)$ satisfies the assumptions of Lemma 2.1. Therefore, when $s > \frac{m_2}{4}$ for $m_2 > 0$ or $s > \frac{-m_2}{2}$ for $m_2 \le 0$, we obtain

$$\left\|R_0^* g\right\|_{L^2(\mathbb{R})} \le C\|g\|_{L^2(\mathbb{R})}. \tag{2.6}$$

We have

$$S_{t,\phi,0} f(x) = R_{0,t}\big(\mathcal{F}^{-1}\big(\varphi_0(\cdot)\big(1+|\cdot|^2\big)^{\frac{s}{2}}\hat{f}(\cdot)\big)\big)(x), \tag{2.7}$$

where $\mathcal{F}^{-1}$ denotes the Fourier inverse transform. Note that

$$\operatorname{supp} \varphi_0(\cdot)\big(1+|\cdot|^2\big)^{\frac{s}{2}}\hat{f}(\cdot) \subseteq \big\{\xi; |\xi| < 2\big\}.$$

Thus, by (2.7) and (2.6), we have

$$\begin{aligned} \left\|S_{\phi,0}^* f\right\|_{L^2(\mathbb{R})} &= \left\|R_0^*\big(\mathcal{F}^{-1}\big(\varphi_0(\cdot)\big(1+|\cdot|^2\big)^{\frac{s}{2}}\hat{f}(\cdot)\big)\big)\right\|_{L^2(\mathbb{R})} \\ &\le C\left\|\mathcal{F}^{-1}\big(\varphi_0(\cdot)\big(1+|\cdot|^2\big)^{\frac{s}{2}}\hat{f}(\cdot)\big)\right\|_{L^2(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}, \end{aligned}$$

which is just (2.4). Now we define the operator R_N by

$$R_N f(x) = N^{-s}\int_{\mathbb{R}} e^{ix\cdot\xi + it(x)\phi(|\xi|)}\varphi\left(\frac{\xi}{N}\right)\hat{f}(\xi)\,d\xi, \quad N \ge 2, \tag{2.8}$$

where $t(x)$ is a measurable function in $\mathbb{R}$ with $0 < t(x) < 1$.

Lemma 2.2 *Suppose that ϕ satisfies the conditions in Theorem* 1.1. *If $s > \frac{m_2}{4}$ for $m_2 > 0$ or $s > \frac{-m_2}{2}$ for $m_2 \leq 0$, then there exist $\delta > 0$ and $C > 0$, such that, for all $N \geq 2$,*

$$\|R_N f\|_{L^2(\mathbb{R})} \leq CN^{-\delta}\|f\|_{L^2(\mathbb{R})}. \tag{2.9}$$

The proof of Lemma 2.2 will be given in the next subsection. Now we finish the proof of Theorem 1.1 by using Lemma 2.2. By linearizing the maximal operator, we have, for some real-valued function $t(x)$,

$$\begin{aligned} S^*_{\phi,k}f(x) &\leq \frac{1}{2\pi}\left|\int_{\mathbb{R}} e^{ix\cdot\xi+it(x)\phi(|\xi|)}\varphi\left(\frac{\xi}{2^k}\right)\hat{f}(\xi)\,d\xi\right| \\ &= \frac{1}{2\pi}\left|R_{2^k}\left(\mathcal{F}^{-1}\left(\chi_{\{2^{k-1}<|\xi|<2^{k+1}\}}2^{ks}\hat{f}\right)\right)(x)\right|. \end{aligned} \tag{2.10}$$

By (2.9) and (2.10), for $k \geq 1$, we have

$$\begin{aligned} \left\|S^*_{\phi,k}f\right\|_{L^2(\mathbb{R})} &\leq \left\|R_{2^k}\left(\mathcal{F}^{-1}\left(\chi_{\{2^{k-1}<|\xi|<2^{k+1}\}}2^{ks}\hat{f}\right)\right)\right\|_{L^2(\mathbb{R})} \\ &\leq C2^{-k\delta}\left\|\mathcal{F}^{-1}\left(\chi_{\{2^{k-1}<|\xi|<2^{k+1}\}}2^{ks}\hat{f}\right)\right\|_{L^2(\mathbb{R})}. \end{aligned}$$

From this we get

$$\left\|S^*_{\phi,k}f\right\|_{L^2(\mathbb{R})} \leq C2^{-k\delta}\|f\|_{H^s(\mathbb{R})}. \tag{2.11}$$

Summing up the estimates of (2.3), (2.4), and (2.11), we have

$$\begin{aligned} \left\|S^*_{\phi}f\right\|_{L^2(\mathbb{R})} &\leq \left\|S^*_{\phi,0}f\right\|_{L^2(\mathbb{R})} + \sum_{k=1}^{\infty}\left\|S^*_{\phi,k}f\right\|_{L^2(\mathbb{R})} \\ &\leq C\|f\|_{H^s(\mathbb{R})} + C\sum_{k=1}^{\infty}2^{-k\delta}\|f\|_{H^s(\mathbb{R})} \\ &\leq C\|f\|_{H^s(\mathbb{R})}. \end{aligned}$$

Therefore, to finish the proof of Theorem 1.1, it remains to show Lemma 2.2.

2.2 The proof of Lemma 2.2

Write

$$R_N f(x) = \int_{\mathbb{R}} e^{ix\cdot\xi}p_N(x,\xi)\hat{f}(\xi)\,d\xi, \quad N \geq 2,$$

where $f \in \mathcal{S}(\mathbb{R})$ and $p_N(x,\xi) = e^{it(x)\phi(|\xi|)}\varphi(\frac{\xi}{N})N^{-s}$. Take the function $\rho \in C_0^\infty(\mathbb{R})$ such that $\rho(x) = 1$ if $|x| < 1$, and $\rho(x) = 0$ if $|x| \geq 2$, and set $\psi = 1 - \rho$. Denote

$$p_{N,M}(x,\xi) = \rho\left(\frac{x}{M}\right)p_N(x,\xi), \quad M > 1$$

and

$$p_{N,M,\varepsilon}(x,\xi) = \psi\left(\frac{\xi}{\varepsilon}\right)p_{N,M}(x,\xi), \quad 0 < \varepsilon < 1.$$

For $N \geq 2$, $M > 1$, and $0 < \varepsilon < 1$, the corresponding operators $R_{N,M}$ and $R_{N,M,\varepsilon}$ are defined by

$$R_{N,M}f(x) = \int_{\mathbb{R}} e^{ix\cdot\xi} p_{N,M}(x,\xi)\hat{f}(\xi)\,d\xi$$

and

$$R_{N,M,\varepsilon}f(x) = \int_{\mathbb{R}} e^{ix\cdot\xi} p_{N,M,\varepsilon}(x,\xi)\hat{f}(\xi)\,d\xi.$$

Obviously, both of the operators $R_{N,M}$ and $R_{N,M,\varepsilon}$ are bounded on $L^2(\mathbb{R})$. On the other hand, it is easy to see that the adjoint operator $R'_{N,M,\varepsilon}$ of $R_{N,M,\varepsilon}$ is given by

$$R'_{N,M,\varepsilon}g(x) = \iint e^{i(x-y)\cdot\xi}\overline{p_{N,M,\varepsilon}(y,\xi)}g(y)\,dy\,d\xi$$

and it follows that

$$\lim_{\varepsilon\to 0} R'_{N,M,\varepsilon}g(x) = R'_{N,M}g(x), \quad g \in \mathcal{S}(\mathbb{R}), \tag{2.12}$$

where $R'_{N,M}$ denotes the adjoint operator of $R_{N,M}$. Since

$$\int \left|R'_{N,M,\varepsilon}g(x)\right|^2 dx = \lim_{L\to\infty}\int_{|x|<L}\left|R'_{N,M,\varepsilon}g(x)\right|^2 dx \tag{2.13}$$

and

$$\begin{aligned}\int_{|x|<L}\left|R'_{N,M,\varepsilon}g(x)\right|^2 dx &= \int_{|x|<L} R'_{N,M,\varepsilon}g(x)\overline{R'_{N,M,\varepsilon}g(x)}\,dx \\ &= \int_{|x|<L}\left(\iint e^{i(x-y)\cdot\xi}\overline{p_{N,M,\varepsilon}(y,\xi)}g(y)\,dy\,d\xi\right) \\ &\quad\times\left(\overline{\iint e^{i(x-z)\cdot\eta}\overline{p_{N,M,\varepsilon}(z,\eta)}g(z)\,dz\,d\eta}\right)dx.\end{aligned} \tag{2.14}$$

By (2.13), (2.14), and a similar calculation as [3], p.708, we have

$$\begin{aligned}&\int\left|R'_{N,M,\varepsilon}g(x)\right|^2 dx \\ &\quad= 2\pi\iint\left(\int e^{i(z-y)\xi}\overline{p_{N,M,\varepsilon}(y,\xi)}p_{N,M,\varepsilon}(z,\xi)\,d\xi\right)g(y)\overline{g(z)}\,dy\,dz \\ &\quad= 2\pi\iint\left(\int e^{i(z-y)\xi}\rho\left(\frac{y}{M}\right)\rho\left(\frac{z}{M}\right)\psi^2\left(\frac{\xi}{\varepsilon}\right)\overline{p_N(y,\xi)}p_N(z,\xi)\,d\xi\right) \\ &\qquad\times g(y)\overline{g(z)}\,dy\,dz.\end{aligned} \tag{2.15}$$

Therefore, invoking (2.12) and by Fatou's lemma, we obtain

$$\begin{aligned}&\int\left|R'_{N,M}g(x)\right|^2 dx \\ &\quad\leq \liminf_{\varepsilon\to 0}\int\left|R'_{N,M,\varepsilon}g(x)\right|^2 dx\end{aligned}$$

$$
\begin{aligned}
&= 2\pi \lim_{\varepsilon\to 0} \iint \left(\int e^{i(z-y)\xi} \rho\left(\frac{y}{M}\right)\rho\left(\frac{z}{M}\right)\psi^2\left(\frac{\xi}{\varepsilon}\right)\overline{p_N(y,\xi)}p_N(z,\xi)\,d\xi \right) \\
&\quad \times g(y)\overline{g(z)}\,dy\,dz \\
&\le C \iint \left| \int e^{i[(z-y)\xi+(t(z)-t(y))\phi(|\xi|)]}\varphi^2\left(\frac{\xi}{N}\right)d\xi N^{-2s}\right| \left|g(y)\right|\left|g(z)\right|\,dy\,dz. \qquad (2.16)
\end{aligned}
$$

It is easy to check that the constant C is independent of N and M. Now define

$$
I_N(x,\omega) = N^{-2s} \int e^{i[x\xi+\omega\phi(|\xi|)]}\varphi^2\left(\frac{\xi}{N}\right)d\xi \quad \text{for } x\in\mathbb{R}, -1<\omega<1, N\ge 2
$$

and

$$
J_N(x) = \sup_{|\omega|<1}\left|I_N(x,\omega)\right|, \quad x\in\mathbb{R}.
$$

We have the following conclusion.

Lemma 2.3 *Let J_N be defined as above, ϕ satisfies the conditions in Theorem* 1.1. *If $s > \frac{m_2}{4}$ for $m_2 > 0$ or $s > \frac{-m_2}{2}$ for $m_2 \le 0$, then there exist $\delta, C > 0$, such that, for all $N \ge 2$,*

$$
\|J_N\|_{L^1(\mathbb{R})} \le CN^{-2\delta}. \qquad (2.17)
$$

Below we first finish the proof of Lemma 2.2 by applying Lemma 2.3, whose proof will be given in the next subsection. By (2.16) and (2.17), invoking Hölder's inequality and Young's inequality, we have

$$
\begin{aligned}
\int \left|R'_{N,M}g(x)\right|^2 dx &\le C \iint \left|I_N\big(z-y, t(z)-t(y)\big)\right|\left|g(y)\right|\left|g(z)\right|\,dy\,dz \\
&\le C \iint J_N(z-y)\left|g(y)\right|\left|g(z)\right|\,dy\,dz \\
&= C \int \big(J_N * |g|\big)(z)\left|g(z)\right|\,dz \\
&\le C\left\|J_N * |g|\right\|_2 \|g\|_2 \\
&\le C\|J_N\|_1\|g\|_2^2 \le CN^{-2\delta}\|g\|_2^2.
\end{aligned}
$$

From this we get

$$
\left\|R'_{N,M}g\right\|_2 \le CN^{-\delta}\|g\|_2.
$$

Thus, $\|R_{N,M}g\|_2 \le CN^{-\delta}\|g\|_2$ by duality, where C is independent of N and M. Letting $M \to \infty$, we obtain

$$
\|R_N g\|_2 \le CN^{-\delta}\|g\|_2.
$$

It follows that (2.9) holds, and we complete the proof of Lemma 2.2 based on Lemma 2.3.

2.3 The proof of Lemma 2.3

Now we verify the estimate (2.17). We need the following results.

Lemma 2.4 (Van der Corput's lemma; see [21], p.309) *Let $\psi \in C_0^\infty(\mathbb{R})$ and $\phi \in C^\infty(\mathbb{R})$ satisfy $|\phi''(\xi)| > \lambda > 0$ on the support of ψ. Then*

$$\left|\int e^{i\phi(\xi)}\psi(\xi)\,d\xi\right| \le 10\lambda^{-\frac{1}{2}}\left\{\|\psi\|_\infty + \left\|\psi'\right\|_1\right\}.$$

Lemma 2.5 ([22]) *Let I denote an open integral in $\mathbb{R}$. For $g \in C_0^\infty(I)$ and the real-valued function $F \in C^\infty(I)$ with $F' \neq 0$, if $k \in \mathbb{N}$, then*

$$\int_I e^{iF(x)}g(x)\,dx = \int_I e^{iF(x)}h_k(x)\,dx,$$

where h_k is a linear combination of functions of the form

$$g^{(s)}\left(F'\right)^{-k-r}\prod_{q=1}^{r}F^{(j_q)}$$

with $0 \le s \le k$, $0 \le r \le k$, and $2 \le j_q \le k+1$.

We now return to the proof of Lemma 2.3. Recall that

$$I_N(x,\omega) = N^{-2s}\int e^{i[x\cdot\xi+\omega\phi(|\xi|)]}\varphi^2\left(\frac{\xi}{N}\right)d\xi, \quad x \in \mathbb{R}, -1 < \omega < 1, N \ge 2.$$

Performing a change of variable, we have

$$I_N(x,\omega) = N^{1-2s}\int e^{i(Nx\xi+\omega\phi(N|\xi|))}G(\xi)\,d\xi,$$

where $x \in \mathbb{R}$, $-1 < \omega < 1$, $N \ge 2$, and $G(\xi) = \varphi^2(\xi)$. It is obvious that, for all $x \in \mathbb{R}$, $-1 < \omega < 1$, and $N \ge 2$,

$$\left|I_N(x,\omega)\right| \le CN^{1-2s}. \tag{2.18}$$

Below we give more estimates of $|I_N(x,\omega)|$.

Step 1: *The other estimates of* $I_N(x,\omega)$.

By the condition (K3), there exist $m_2 \in \mathbb{R}$ and $C_1 > 0$ such that $|\phi'(r)| \le C_1 r^{m_2-1}$ for $r \ge 1$. Denote

$$C_2 = \max_{\frac{1}{2}\le|\xi|\le 2}\left\{|\xi|^{m_2-1}\right\} \quad \text{and} \quad C_3 = \max\{C_1C_2, 1\}.$$

Now we give the following estimates of $I_N(x,\omega)$ for $x \in \mathbb{R}$, $-1 < \omega < 1$, and $N \ge 2$:

$$\left|I_N(x,\omega)\right| \le \begin{cases} C(N|x|)^{-2}N^{1-2s}, & |\omega| < \frac{N|x|}{2C_3N^{m_2}}, \\ C(N|x|)^{-\frac{1}{2}}N^{1-2s}, & |\omega| \ge \frac{N|x|}{2C_3N^{m_2}}. \end{cases} \tag{2.19}$$

Let $F(\xi) = Nx\xi + \omega\phi(N|\xi|)$. We have

$$F'(\xi) = Nx + N\operatorname{sgn}(\xi)\omega\phi'\bigl(N|\xi|\bigr),$$
$$F''(\xi) = N^2\omega\phi''\bigl(N|\xi|\bigr)$$

and

$$F^{(3)}(\xi) = N^3\operatorname{sgn}(\xi)\omega\phi^{(3)}\bigl(N|\xi|\bigr).$$

Noting $N|\xi| > 1$ by $N \geq 2$ and $\frac{1}{2} < |\xi| < 2$, by (K3) we get

$$\bigl|N\operatorname{sgn}(\xi)\omega\phi'\bigl(N|\xi|\bigr)\bigr| \leq C_1 N|\omega|\bigl(N|\xi|\bigr)^{m_2-1} \leq C_1C_2N^{m_2}|\omega| \leq C_3N^{m_2}|\omega|.$$

When $|\omega| < \frac{N|x|}{2C_3N^{m_2}}$ (equivalently, $C_3N^{m_2}|\omega| < \frac{1}{2}N|x|$), we have

$$\bigl|N\operatorname{sgn}(\xi)\omega\phi'\bigl(N|\xi|\bigr)\bigr| < \frac{1}{2}N|x|.$$

Therefore,

$$\bigl|F'(\xi)\bigr| \geq N|x| - \bigl|N\operatorname{sgn}(\xi)\omega\phi'\bigl(N|\xi|\bigr)\bigr| > \frac{1}{2}N|x|. \tag{2.20}$$

Since ϕ satisfies (K4) and (K5) with $m_4 \leq m_3 = m_2$, we have

$$\bigl|F^{(j)}(\xi)\bigr| \leq CN^{m_2}|\omega| \quad \text{for } j = 2, 3. \tag{2.21}$$

By the fact $\frac{N^{m_2}|\omega|}{N|x|} \leq \frac{1}{2C_3}$ and Lemma 2.5 for $k = 2$ and (2.20), (2.21), we get

$$\begin{aligned}
&\left|\int e^{iF(\xi)}G(\xi)\,d\xi\right| \\
&\quad\leq C\int_{\frac{1}{2}<|\xi|<2}\frac{1}{|F'(\xi)|^2}\left(1+\frac{|F''(\xi)|}{|F'(\xi)|}++\left(\frac{|F''(\xi)|}{|F'(\xi)|}\right)^2+\frac{|F^{(3)}(\xi)|}{|F'(\xi)|}\right)d\xi \\
&\quad\leq C\bigl(N|x|\bigr)^{-2}\sum_{r=0}^{2}\left(\frac{N^{m_2}|\omega|}{N|x|}\right)^r \\
&\quad\leq C\bigl(N|x|\bigr)^{-2},
\end{aligned}$$

from which follows the first estimate in (2.19). On the other hand, since ϕ satisfies (K4) with $m_3 = m_2$, we get, for $\frac{1}{2} < |\xi| < 2$,

$$\bigl|F''(\xi)\bigr| \geq CN^2|\omega|\bigl(N|\xi|\bigr)^{m_2-2} > CN^{m_2}|\omega| > 0.$$

Note that $\|G\|_\infty \leq C$ and $\|G'\|_1 \leq C$ on the support of φ. By Lemma 2.4 and noting that $|\omega| \geq \frac{N|x|}{2C_3N^{m_2}}$ (equivalently, $C_3N^{m_2}|\omega| \geq \frac{1}{2}N|x|$), we have

$$\bigl|I_N(x,\omega)\bigr| \leq C\bigl(N^{m_2}|\omega|\bigr)^{-\frac{1}{2}}\bigl(\|G\|_\infty + \bigl\|G'\bigr\|_1\bigr)N^{1-2s} \leq C\bigl(N|x|\bigr)^{-\frac{1}{2}}N^{1-2s}.$$

This is just the second estimate in (2.19).

Step 2: *Proof of Lemma* 2.3 *for* $s > \frac{m_2}{4}$ ($m_2 > 0$).

We now prove (2.17) for the case $s > \frac{m_2}{4}$ ($m_2 > 0$). Since $m_2 > 0$, $N \geq 2$, and $2C_3 > 1$, we write

$$\begin{aligned}\int \bigl|J_N(x)\bigr|\,dx &= \int_{0<|x|\leq\frac{1}{N}} \bigl|J_N(x)\bigr|\,dx + \int_{\frac{1}{N}<|x|\leq 2C_3N^{m_2-1}} \bigl|J_N(x)\bigr|\,dx \\ &\quad+ \int_{|x|>2C_3N^{m_2-1}} \bigl|J_N(x)\bigr|\,dx \\ &=: E_1 + E_2 + E_3.\end{aligned}$$

The estimate of E_1 is simple. Since $|I_N(x,\omega)| \leq CN^{1-2s}$ by (2.18), by the definition of J_N, we see that

$$E_1 \leq C\int_{0<|x|\leq\frac{1}{N}} N^{1-2s}\,dx \leq CN^{-2s}. \tag{2.22}$$

As for E_2, we first prove that if $\frac{1}{N} < |x| \leq 2C_3N^{m_2-1}$, then

$$J_N(x) \leq C\bigl(N|x|\bigr)^{-\frac{1}{2}}N^{1-2s}. \tag{2.23}$$

By the definition of J_N, to prove (2.23) it suffices to show that, if $\frac{1}{N} < |x| \leq 2C_3N^{m_2-1}$ and $|\omega| < 1$, then

$$\bigl|I_N(x,\omega)\bigr| \leq C\bigl(N|x|\bigr)^{-\frac{1}{2}}N^{1-2s}. \tag{2.24}$$

In fact, if $|\omega| < \frac{N|x|}{2C_3N^{m_2}}$, by the first estimate in (2.19) and $N|x| > 1$, then

$$\bigl|I_N(x,\omega)\bigr| \leq C\bigl(N|x|\bigr)^{-2}N^{1-2s} \leq C\bigl(N|x|\bigr)^{-\frac{1}{2}}N^{1-2s}.$$

If $|\omega| \geq \frac{N|x|}{2C_3N^{m_2}}$, by the second estimate in (2.19), we obtain

$$\bigl|I_N(x,\omega)\bigr| \leq C\bigl(N|x|\bigr)^{-\frac{1}{2}}N^{1-2s}.$$

Thus (2.24) holds and so (2.23). Hence, by (2.23), we get

$$E_2 \leq C\int_{|x|\leq 2C_3N^{m_2-1}} \bigl(N|x|\bigr)^{-\frac{1}{2}}N^{1-2s}\,dx \leq CN^{\frac{m_2}{2}-2s}. \tag{2.25}$$

Finally, we consider E_3. We first show that if $|x| > 2C_3N^{m_2-1}$, then

$$\bigl|J_N(x)\bigr| \leq C\bigl(N|x|\bigr)^{-2}N^{1-2s}. \tag{2.26}$$

In fact, if $|x| > 2C_3N^{m_2-1}$ and $|\omega| < 1$, then $|x| > 2C_3N^{m_2-1}|\omega|$. Equivalently, $|\omega| < \frac{N|x|}{2C_3N^{m_2}}$. Thus, by the first inequality in (2.19), we obtain

$$\bigl|I_N(x,\omega)\bigr| \leq C\bigl(N|x|\bigr)^{-2}N^{1-2s},$$

and (2.26) follows from this. By (2.26), we obtain

$$E_3 \le C \int_{|x|>2C_3N^{m_2-1}} \big(N|x|\big)^{-2} N^{1-2s}\, dx \le CN^{-m_2-2s}. \tag{2.27}$$

Since $m_2 > 0$, by (2.22), (2.25), and (2.27), we have

$$\big|J_N(x)\big| \le CN^{\frac{m_2}{2}-2s} =: CN^{-2\delta},$$

where $2\delta = 2s - \frac{m_2}{2} > 0$ since $s > \frac{m_2}{4}$ and $m_2 > 0$.

Step 3: *Proof of Lemma* 2.3 *for* $s > \frac{-m_2}{2}$ ($m_2 \le 0$).

First we consider the case where $2C_3N^{m_2-1} > \frac{1}{N}$. Write

$$\begin{aligned}\int \big|J_N(x)\big|\, dx &= \int_{0<|x|\le\frac{1}{N}} \big|J_N(x)\big|\, dx + \int_{\frac{1}{N}<|x|\le 2C_3N^{m_2-1}} \big|J_N(x)\big|\, dx \\ &\quad + \int_{|x|>2C_3N^{m_2-1}} \big|J_N(x)\big|\, dx \\ &=: E_1 + E_2 + E_3.\end{aligned}$$

Since $m_2 \le 0$, by (2.22), (2.25), and (2.27), we have

$$\big|J_N(x)\big| \le CN^{-m_2-2s} =: CN^{-2\delta},$$

where $2\delta = 2s + m_2 > 0$ since $s > \frac{-m_2}{2}$ and $m_2 \le 0$. On the other hand, if $2C_3N^{m_2-1} \le \frac{1}{N}$, we have

$$\begin{aligned}\int \big|J_N(x)\big|\, dx &\le \int_{0<|x|\le\frac{1}{N}} \big|J_N(x)\big|\, dx + \int_{|x|>2C_3N^{m_2-1}} \big|J_N(x)\big|\, dx \\ &=: E_1 + E_3.\end{aligned}$$

Since $m_2 \le 0$, by (2.22) and (2.27), we have

$$\big|J_N(x)\big| \le CN^{-m_2-2s} =: CN^{-2\delta},$$

where $2\delta = 2s + m_2 > 0$ by $s > \frac{-m_2}{2}$ and $m_2 \le 0$. Thus, we complete the proof of Lemma 2.3.

3 The proof of Theorem 1.2

Assume $n \ge 2$. Let f be radial and belong to $\mathcal{S}(\mathbb{R}^n)$; we need to show that

$$\big\|S_\phi^* f\big\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)} \tag{3.1}$$

holds for $s > \frac{m_2}{4}$ if $m_2 > 0$ or $s > \frac{-m_2}{2}$ if $m_2 < 0$.

Let $t(x)$ is a measurable radial function with $0 < t(x) < 1$. Denote

$$Tf(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it(x)\phi(|\xi|)} \hat{f}(\xi)\, d\xi.$$

Recall the Bessel function $J_m(r)$ is defined by

$$J_m(r) = \frac{(\frac{r}{2})^m}{\Gamma(m+\frac{1}{2})\pi^{\frac{1}{2}}} \int_{-1}^{1} e^{irt}\left(1-t^2\right)^{m-\frac{1}{2}}\,dt, \quad m > -\frac{1}{2}.$$

Since f is radial,

$$\hat{f}(\xi) = (2\pi)^{\frac{n}{2}} |\xi|^{1-\frac{n}{2}} \int_0^\infty f(s) J_{\frac{n}{2}-1}\left(s|\xi|\right) s^{\frac{n}{2}}\,ds.$$

Therefore,

$$Tf(u) = (2\pi)^{\frac{n}{2}-n} u^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(ru) e^{it(u)\phi(r)} \hat{f}(r) r^{\frac{n}{2}}\,dr, \quad u > 0. \tag{3.2}$$

Here $Tf(u) = Tf(x)$ if $u = |x|$ and $\hat{f}(r) = \hat{f}(\xi)$ if $r = |\xi|$. By linearizing the maximal operator and using polar coordinates, to prove (3.1) it suffices to prove that

$$\left(\int_0^\infty \left|Tf(u)\right|^2 u^{n-1}\,du\right)^{1/2} \le \left(\int_0^\infty \left|\hat{f}(r)\right|^2 \left(1+r^2\right)^s r^{n-1}\,dr\right)^{1/2}. \tag{3.3}$$

Denote

$$g(r) = \hat{f}(r)\left(1+r^2\right)^{\frac{s}{2}} r^{\frac{n-1}{2}}, \quad r > 0. \tag{3.4}$$

By (3.2) and (3.4), it follows that

$$\begin{aligned} Tf(u)u^{\frac{n-1}{2}} &= (2\pi)^{-\frac{n}{2}} u^{\frac{1}{2}} \int_0^\infty J_{\frac{n}{2}-1}(ru) e^{it(u)\phi(r)} \hat{f}(r) r^{\frac{n}{2}}\,dr \\ &= (2\pi)^{-\frac{n}{2}} u^{\frac{1}{2}} \int_0^\infty J_{\frac{n}{2}-1}(ru) e^{it(u)\phi(r)} g(r)\left(1+r^2\right)^{-\frac{s}{2}} r^{\frac{1}{2}}\,dr. \end{aligned}$$

Let

$$Pg(u) = u^{\frac{1}{2}} \int_0^\infty J_{\frac{n}{2}-1}(ru) e^{it(u)\phi(r)} g(r)\left(1+r^2\right)^{-\frac{s}{2}} r^{\frac{1}{2}}\,dr.$$

Thus, we have

$$Tf(u)u^{\frac{n-1}{2}} = (2\pi)^{-\frac{n}{2}} Pg(u). \tag{3.5}$$

By (3.5), to prove (3.3) it suffices to prove that

$$\left(\int_0^\infty \left|Pg(u)\right|^2\,du\right)^{1/2} \le C\left(\int_0^\infty \left|g(r)\right|^2\,dr\right)^{1/2} \tag{3.6}$$

holds for $s > \frac{m_2}{4}$ ($m_2 > 0$) or $s > \frac{-m_2}{2}$ ($m_2 \le 0$). Let us recall a well-known estimate of J_m.

Lemma 3.1 ([19], p.158) *$J_m(r) = \sqrt{\frac{2}{\pi r}}\cos(r - \frac{\pi m}{2} - \frac{\pi}{4}) + O(r^{-\frac{3}{2}})$ as $r \to \infty$. In particular, $J_m(r) = O(r^{-\frac{1}{2}})$ as $r \to \infty$.*

By Lemma 3.1, we may get

$$t^{\frac{1}{2}}J_{\frac{n}{2}-1}(t)=b_1e^{it}+b_2e^{-it}+O\left(\min\left(1,\frac{1}{t}\right)\right),\quad t>0, \tag{3.7}$$

where b_1 and b_2 are the constants depending on n. In fact, by Lemma 3.1, as $t\to\infty$, we have

$$J_{\frac{n}{2}-1}(t)=\sqrt{\frac{2}{\pi t}}\cos\left(t-\frac{\pi(n-1)}{4}\right)+O\big(t^{-\frac{3}{2}}\big).$$

It follows that, as $t\to\infty$, we have

$$\begin{aligned}t^{\frac{1}{2}}J_{\frac{n}{2}-1}(t)&=\sqrt{\frac{2}{\pi}}\cos\left(\frac{\pi(n-1)}{4}\right)\cos t+\sqrt{\frac{2}{\pi}}\sin\left(\frac{\pi(n-1)}{4}\right)\sin t+O\big(t^{-1}\big)\\&=(b_1+b_2)\cos t+i(b_1-b_2)\sin t+O\big(t^{-1}\big)\\&=b_1e^{it}+b_2e^{-it}+O\big(t^{-1}\big),\end{aligned}$$

where

$$b_1=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\cos\left(\frac{\pi(n-1)}{4}\right)+i\sin\left(\frac{\pi(n-1)}{4}\right)\right)$$

and

$$b_2=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\cos\left(\frac{\pi(n-1)}{4}\right)-i\sin\left(\frac{\pi(n-1)}{4}\right)\right).$$

It follows that, when $t>1$, we have

$$\left|t^{\frac{1}{2}}J_{\frac{n}{2}-1}(t)-\big(b_1e^{it}+b_2e^{-it}\big)\right|\le Ct^{-1}. \tag{3.8}$$

On the other hand, by the definition of the Bessel function

$$J_m(t)=\frac{(\frac{t}{2})^m}{\Gamma(m+\frac{1}{2})\pi^{\frac{1}{2}}}\int_{-1}^{1}e^{its}\big(1-s^2\big)^{m-\frac{1}{2}}\,ds,\quad m>-\frac{1}{2},$$

we have $|J_m(t)|\le Ct^m$ for $m>-\frac{1}{2}$ and $t>0$. Thus, $|J_m(t)|\le Ct^{-\frac{1}{2}}$ when $m>-\frac{1}{2}$ and $0<t<1$. Since $n\ge 2$, so $|J_{\frac{n}{2}-1}(t)|\le Ct^{-\frac{1}{2}}$ for $0<t<1$. Therefore, when $0<t<1$, we have

$$\begin{aligned}\left|t^{\frac{1}{2}}J_{\frac{n}{2}-1}(t)-\big(b_1e^{it}+b_2e^{-it}\big)\right|&\le\left|t^{\frac{1}{2}}J_{\frac{n}{2}-1}(t)\right|+\left|b_1e^{it}\right|+\left|b_2e^{-it}\right|\\&\le Ct^{\frac{1}{2}}t^{-\frac{1}{2}}+|b_1|+|b_2|\le C.\end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that (3.7) holds. Invoking (3.7), we have

$$\begin{aligned}Pg(u)&=b_1\int_0^\infty e^{iru}e^{it(u)\phi(r)}g(r)\big(1+r^2\big)^{-\frac{s}{2}}\,dr\\&\quad+b_2\int_0^\infty e^{-iru}e^{it(u)\phi(r)}g(r)\big(1+r^2\big)^{-\frac{s}{2}}\,dr+E(u)+F(u)\\&=:b_1D_1(u)+b_2D_2(u)+E(u)+F(u),\end{aligned} \tag{3.10}$$

where

$$\left|E(u)\right| \leq C \int_0^{\frac{1}{u}} \left|g(r)\right| dr$$

and

$$\left|F(u)\right| \leq C \frac{1}{u} \int_{\frac{1}{u}}^{\infty} \frac{1}{r} \left|g(r)\right| dr.$$

From [17], pp.59-61, we have

$$\left(\int_0^{\infty} \left|E(u)\right|^2 du \right)^{1/2} \leq C \|g\|_{L^2(0,\infty)} \tag{3.11}$$

and

$$\left(\int_0^{\infty} \left|F(u)\right|^2 du \right)^{1/2} \leq C \|g\|_{L^2(0,\infty)}. \tag{3.12}$$

Thus, to prove (3.6), it remains to estimate D_1 and D_2. Denote $\hat{h}(r) = g(r)(1+r^2)^{-\frac{s}{2}} \chi_{(0,\infty)}$, and we get

$$D_1(u) = \int_0^{\infty} e^{iru} e^{it(u)\phi(r)} g(r) \left(1+r^2\right)^{-\frac{s}{2}} dr = \int_{\mathbb{R}} e^{iru} e^{it(u)\phi(r)} \hat{h}(r)\, dr$$

and

$$D_2(u) = \int_0^{\infty} e^{-iru} e^{it(u)\phi(r)} g(r) \left(1+r^2\right)^{-\frac{s}{2}} dr = \int_{\mathbb{R}} e^{-iru} e^{it(u)\phi(r)} \hat{h}(r)\, dr.$$

Therefore, we have

$$\left|D_i(u)\right| \leq S_\phi^* h(u) \quad \text{for } i = 1, 2. \tag{3.13}$$

Since ϕ satisfies the conditions in Theorem 1.1, by the results of Theorem 1.1, when $s > \frac{m_2}{4}$ ($m_2 > 0$) or $s > \frac{-m_2}{2}$ ($m_2 \leq 0$), we have

$$\left\|S_\phi^* h\right\|_{L^2(\mathbb{R})} \leq C \|h\|_{H^s(\mathbb{R})}. \tag{3.14}$$

Since $u > 0$ and by (3.13) and (3.14), for $i = 1, 2$, we have

$$\begin{aligned} \|D_i\|_{L^2(0,\infty)} &\leq \|D_i\|_{L^2(\mathbb{R})} \leq C \left\|S_\phi^* h\right\|_{L^2(\mathbb{R})} \leq C \|h\|_{H^s(\mathbb{R})} \\ &= C \left(\int_0^{\infty} \left|g(r)\right|^2 \left(1+r^2\right)^{-s} \left(1+r^2\right)^{s} dr \right)^{1/2} \\ &= C \|g\|_{L^2(0,\infty)}. \end{aligned} \tag{3.15}$$

Thus, (3.6) follows from (3.10), (3.11), (3.12), and (3.15). We hence complete the proof of Theorem 1.2.

4 The proof of Theorem 1.3

In this case $k = 0$, Theorem 1.3 follows from Theorem 1.2. Hence we only give the proof of Theorem 1.3 for $k \geq 1$. We first recall a well-known result.

Lemma 4.1 ([19], p.158) *Suppose $n \geq 2$ and $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ has the form $f(x) = f_0(|x|)P(x)$, where $P(x)$ is a solid spherical harmonic of degree k, then $\hat{f}$ has the form $\hat{f}(x) = F_0(|x|)P(x)$, where*

$$F_0(r) = (2\pi)^{\frac{n}{2}} i^{-k} r^{-\frac{n}{2}-k+1} \int_0^\infty f_0(s) J_{\frac{n}{2}+k-1}(rs) s^{\frac{n}{2}+k}\,ds,$$

where J_m denotes the Bessel function.

Let us return to the proof of Theorem 1.3. First we show that, for $f \in \mathcal{H}_k$ $(k \geq 1)$,

$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\sum_{j=1}^{a_k} \int_0^\infty \left|F_j(r)\right|^2 \left(1 + r^2\right)^s r^{n+2k-1}\,dr \right)^{1/2}. \tag{4.1}$$

In fact, $f(x) = \sum_{j=1}^{a_k} f_j(|x|)P_j(x)$ where f_j are radial functions in $\mathcal{S}(\mathbb{R}^n)$ and $\{P_j\}_1^{a_k}$ is an orthonormal basis in $\mathscr{A}_k$. By Lemma 4.1 we get

$$\hat{f}(x) = \sum_{j=1}^{a_k} F_j\left(|x|\right)P_j(x), \tag{4.2}$$

where

$$F_j(r) = (2\pi)^{\frac{n}{2}} i^{-k} r^{1-\frac{n}{2}-k} \int_0^\infty f_j(s) J_{\frac{n}{2}+k-1}(rs) s^{\frac{n}{2}+k}\,ds, \qquad r > 0.$$

By (4.2) and noting that $\{P_1, P_1, \ldots, P_{a_k}\}$ is an orthonormal basis in $\mathscr{A}_k$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi \\
&\quad = \int_0^\infty \left(\int_{S^{n-1}} \left|\hat{f}\left(r\xi'\right)\right|^2 d\sigma\left(\xi'\right) \right) \left(1 + r^2\right)^s r^{n-1}\,dr \\
&\quad = \int_0^\infty \left(\int_{S^{n-1}} \left(\sum_{j=1}^{a_k} F_j(r)P_j\left(r\xi'\right) \right) \left(\sum_{i=1}^{a_k} \overline{F_i(r)} \overline{P_i\left(r\xi'\right)} \right) d\sigma\left(\xi'\right) \right) \left(1 + r^2\right)^s r^{n-1}\,dr \\
&\quad = \int_0^\infty \left(\sum_{j=1}^{a_k} \left|F_j(r)\right|^2 \right) r^{2k} \left(1 + r^2\right)^s r^{n-1}\,dr \\
&\quad = \sum_{j=1}^{a_k} \int_0^\infty \left|F_j(r)\right|^2 \left(1 + r^2\right)^s r^{n+2k-1}\,dr,
\end{aligned}$$

which is just (4.1). On the other hand, by (4.2), we have

$$S_{t,\phi}f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it\phi(|\xi|)} \hat{f}(\xi)\,d\xi = \sum_{j=1}^{a_k} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(e^{it\phi(|\xi|)} F_j\left(|\xi|\right)P_j(\xi)\right) d\xi.$$

Applying Lemma 4.1, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{ix\cdot\xi}\big(e^{it\phi(|\xi|)}F_j\big(|\xi|\big)P_j(\xi)\big)\,d\xi \\
&\quad= \big(e^{it\phi(|\cdot|)}F_j\big(|\cdot|\big)P_j(-\cdot)\big)^\wedge(x) \\
&\quad= (2\pi)^{\frac{n}{2}} i^{-k} s^{1-\frac{n}{2}-k}\left(\int_0^\infty J_{\frac{n}{2}+k-1}(rs)e^{it\phi(r)}F_j(r)r^{\frac{n}{2}+k}\,dr\right)P_j(-x),
\end{aligned}$$

where $s = |x| > 0$. Therefore, we have

$$\begin{aligned}
S_{t,\phi}f(x) &= \sum_{j=1}^{a_k}(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\big(e^{it\phi(|\xi|)}F_j\big(|\xi|\big)P_j(\xi)\big)\,d\xi \\
&= \sum_{j=1}^{a_k}(2\pi)^{-\frac{n}{2}} i^{-k}|x|^{1-\frac{n}{2}-k} \\
&\quad\times\left(\int_0^\infty J_{\frac{n}{2}+k-1}\big(r|x|\big)e^{it\phi(r)}F_j(r)r^{\frac{n}{2}+k}\,dr\right)P_j(-x).
\end{aligned} \tag{4.3}$$

Denote by $\mathscr{F}_n$ the Fourier transform in $\mathbb{R}^n$. Then $F_j = i^{-k}\mathscr{F}_{n+2k}f_j$. Note that for a radial function $h \in \mathcal{S}(\mathbb{R}^{n+2k})$, its Fourier transform is

$$\mathscr{F}_{n+2k}h(x) = (2\pi)^{\frac{n}{2}}|x|^{1-\frac{n}{2}-k}\int_0^\infty h(r)J_{\frac{n}{2}+k-1}\big(r|x|\big)r^{\frac{n}{2}+k}\,dr.$$

Now we define the operator $S^{n+2k}_{t,\phi}$ on the set of all radial function in $\mathcal{S}(\mathbb{R}^{n+2k})$ by

$$S^{n+2k}_{t,\phi}h(x) := (2\pi)^{-n-2k}\int_{\mathbb{R}^{n+2k}} e^{ix\cdot\xi}e^{it\phi(|\xi|)}\mathscr{F}_{n+2k}h\big(|\xi|\big)\,d\xi.$$

Obviously, $S^{n+2k}_{t,\phi}h$ is still a radial function. Then

$$\begin{aligned}
S^{n+2k}_{t,\phi}f_j\big(|x|\big) &= i^k(2\pi)^{-n-2k}\int_{\mathbb{R}^{n+2k}} e^{ix\cdot\xi}\big(e^{it\phi(|\xi|)}F_j(\xi)\big)\,d\xi \\
&= i^k(2\pi)^{-\frac{n}{2}-2k}|x|^{1-\frac{n}{2}-k} \\
&\quad\times\int_0^\infty J_{\frac{n}{2}+k-1}\big(r|x|\big)e^{it\phi(r)}F_j(r)r^{\frac{n}{2}+k}\,dr.
\end{aligned} \tag{4.4}$$

By (4.3) and (4.4), we have

$$S_{t,\phi}f(x) = i^{-2k}(2\pi)^{2k}\sum_j S^{n+2k}_{t,\phi}f_j\big(|x|\big)\cdot P_j(-x), \quad x\in\mathbb{R}^n, \tag{4.5}$$

where we may see $S^{n+2k}_{t,\phi}f_j(|x|)$ as a function on $\mathbb{R}^n$, since $S^{n+2k}_{t,\phi}f_j$ is a radial function. Denote

$$S^{n+2k,*}_{\phi}f_j\big(|y|\big) = \sup_{0<t<1}\big|S^{n+2k}_{t,\phi}f_j\big(|y|\big)\big|, \quad y\in\mathbb{R}^{n+2k} \text{ or } y\in\mathbb{R}^n. \tag{4.6}$$

Then by (4.5) and (4.6), we obtain

$$S^*_\phi f(x) \le C_{n,k}\sum_j\big(S^{n+2k,*}_\phi f_j\big(|x|\big)\big)|x|^k. \tag{4.7}$$

Using the notation $v = |x|$ and $r = |\xi|$, by (4.7), we have

$$\|S_\phi^* f\|_{L^2(\mathbb{R}^n)}^2 \le C\sum_{j=1}^{a_k}\int_{\mathbb{R}^n}\left|S_\phi^{n+2k,*}f_j(v)\right|^2 v^{2k}\,dx. \tag{4.8}$$

Using the representation of polar coordinates and noting (4.6), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|S_\phi^{n+2k,*}f_j(v)\right|^2 v^{2k}\,dx\\ &\quad= \omega_{n-1}\int_0^\infty\left|S_\phi^{n+2k,*}f_j(v)\right|^2 v^{n+2k-1}\,dv\\ &\quad= \frac{\omega_{n-1}}{\omega_{n+2k-1}}\int_{\mathbb{R}^{n+2k}}\left|S_\phi^{n+2k,*}f_j(v)\right|^2\,dx,\end{aligned} \tag{4.9}$$

where ω_{n-1} and ω_{n+2k-1} denote the area of the unit sphere in $\mathbb{R}^n$ and $\mathbb{R}^{n+2k}$, respectively. Applying Theorem 1.2, when $s > \frac{m_2}{4}$ ($m_2 > 0$) or $s > \frac{-m_2}{2}$ ($m_2 \le 0$), we have

$$\int_{\mathbb{R}^{n+2k}}\left|S_\phi^{n+2k,*}f_j(v)\right|^2\,dx \le C\|f_j\|_{H^s(\mathbb{R}^{n+2k})}^2. \tag{4.10}$$

Note that $\mathscr{F}_{n+2k}f_j = i^k F_j$, and we get

$$\begin{aligned}\|f_j\|_{H^s(\mathbb{R}^{n+2k})}^2 &= \int_{\mathbb{R}^{n+2k}}\left|F_j(|\xi|)\right|^2\left(1+|\xi|^2\right)^s\,d\xi\\ &= \omega_{n+2k-1}\int_0^\infty\left|F_j(r)\right|^2\left(1+r^2\right)^s r^{n+2k-1}\,dr.\end{aligned} \tag{4.11}$$

Therefore, by (4.8), (4.9), (4.10), (4.11), and (4.1), we obtain

$$\|S_\phi^* f\|_{L^2(\mathbb{R}^n)}^2 \le C\sum_{j=1}^{a_k}\int_0^\infty\left|F_j(r)\right|^2\left(1+r^2\right)^s r^{n+2k-1}\,dr = C\|f\|_{H^s(\mathbb{R}^n)}^2. \tag{4.12}$$

Thus, we complete the proof of Theorem 1.3.

5 Some applications

We now give some examples to show that (1.10) includes some well-known equations.

Example 1 Let $\phi(r) = r^2$, then (1.10) is the *classical Schrödinger equation* (1.1).

Example 2 Let $\phi(r) = r^a$ ($a > 0$, $a \neq 1$), then (1.10) is the *fractional Schrödinger equation* (1.3). In this case, $\phi(r)$ satisfies (K1)-(K5) with $l_1 = m_1 = m_2 = m_3 = m_4 = a$.

Example 3 Let $\phi(r) = r^2 + r^4$, then (1.10) is the *fourth-order Schrödinger equation*:

$$\begin{cases} i\partial_t u + \Delta^2 u - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R},\\ u(x,0) = f(x).\end{cases} \tag{5.1}$$

In this case, $\phi(r)$ satisfies (K1) with $l_1 = 2 \ge 0$, (K2)-(K5) with $m_1 = m_2 = m_3 = m_4 = 4 > 0$.

Example 4 Recall the definition of the *beam equation*:

$$\begin{cases} \partial_{tt}u + \Delta^2 u + u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), \\ \partial_t u(x,0) = 0. \end{cases} \tag{5.2}$$

Note that the solution of (5.2) can be formally written as the real part of

$$u(x,t) = e^{it\sqrt{I+\Delta^2}} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it\sqrt{1+|\xi|^4}} \hat{f}(\xi)\, d\xi.$$

Thus, taking $\phi(r) = \sqrt{1+r^4}$, we see that $\phi(r)$ satisfies (K1) with $l_1 = 0 \geq 0$, (K2)-(K5), with $m_1 = m_2 = m_3 = m_4 = 2 > 0$, and the solution of (5.2) is the real part of

$$S_{t,\phi} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it\phi(|\xi|)} \hat{f}(\xi)\, d\xi.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YD participated in the design of the study and in the discussions of all results. YN participated in the discussions of all results and drafted the manuscript. All authors read and approved the final manuscript.

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