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Reverse Beckenbach-Dresher's inequality

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Abstract
In the paper, we establish an inverse of Beckenbach-Dresher's integral inequality, which provides new estimates on inequality of this type.

MSC: 26D15

Keywords: Beckenbach's inequality; Radon's inequality; Beckenbach-Dresher's inequality

1 Introduction

The well-known inequality due to Beckenbach can be stated as follows (see [1], also see [2], p.27).

Theorem A *If* $1 \le p \le 2$, *and* $x_i, y_i > 0$ *for* $i = 1, 2, \ldots, n$, *then*

$$\frac{\sum_{i=1}^{n}(x_i + y_i)^p}{\sum_{i=1}^{n}(x_i + y_i)^{p-1}} \le \frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} x_i^{p-1}} + \frac{\sum_{i=1}^{n} y_i^p}{\sum_{i=1}^{n} y_i^{p-1}}. \tag{1.1}$$

An integral analogue of Beckenbach's inequality easily follows.

Theorem B *Let* $1 \le p \le 2$. *If* f *and* g *are positive and continuous functions on* $[a, b]$, *then*

$$\frac{\int_a^b (f(x) + g(x))^p\,dx}{\int_a^b (f(x) + g(x))^{p-1}\,dx} \le \frac{\int_a^b f(x)^p\,dx}{\int_a^b f(x)^{p-1}\,dx} + \frac{\int_a^b g(x)^p\,dx}{\int_a^b g(x)^{p-1}\,dx}. \tag{1.2}$$

An extension of Beckenbach's inequality was obtained by Dresher [3] by an ingenious method using moment-space theory.

Theorem C *Let* f *and* g *be positive and continuous functions on* $[a, b]$. *If* $p \ge 1 \ge r \ge 0$, *then*

$$\left(\frac{\int_a^b (f(x) + g(x))^p\,dx}{\int_a^b (f(x) + g(x))^r\,dx}\right)^{1/(p-r)} \le \left(\frac{\int_a^b f^p(x)\,dx}{\int_a^b f^r(x)\,dx}\right)^{1/(p-r)} + \left(\frac{\int_a^b g^p(x)\,dx}{\int_a^b g^r(x)\,dx}\right)^{1/(p-r)}. \tag{1.3}$$

The inequality which we shall call Beckenbach-Dresher's inequality. In fact, this result was also established by Danskin [4], who employed a combination of Hölder's and Minkowski's inequalities.

Beckenbach-Dresher's inequality was studied extensively and numerous variants, generalizations, and extensions appeared in the literature (see [3–12] and the references cited therein). Research of reverse Beckenbach-Dresher's integral inequality is rare (see [13] and [14]). The aim of this paper is to discuss reverse Beckenbach-Dresher's integral inequality and establish the following reversed Beckenbach-Dresher integral inequality by deriving reverse Hölder's, Minkowski's and Radon's integral inequalities.

Theorem *Let f and g be continuous functions on $[a,b]$, $0 < m_1 \leq f(x) \leq M_1$ and $0 < m_2 \leq g(x) \leq M_2$. If $p \geq 1 \geq r \geq 0$, then*

$$\ell \cdot \left(\frac{\int_a^b (f(x)+g(x))^p\,dx}{\int_a^b (f(x)+g(x))^r\,dx}\right)^{1/(p-r)} \geq \left(\frac{\int_a^b f^p(x)\,dx}{\int_a^b f^r(x)\,dx}\right)^{1/(p-r)} + \left(\frac{\int_a^b g^p(x)\,dx}{\int_a^b g^r(x)\,dx}\right)^{1/(p-r)}, \tag{1.4}$$

where

$$\ell = \frac{L_{\alpha,\beta}(s,t,S,T)}{\Gamma_{\alpha,\beta}(m_1,m_2,M_1,M_2)}, \quad \frac{1}{\alpha}+\frac{1}{\beta}=1, \alpha>1, \tag{1.5}$$

$$L_{\alpha,\beta}(s,t,S,T) = \bigl(\Upsilon_{\alpha,\beta}\bigl(sT^{-\frac{m}{m+1}}, \bigl(stS^{-1}\bigr)^{\frac{m}{m+1}}, St^{-\frac{m}{m+1}}, \bigl(s^{-1}ST\bigr)^{\frac{m}{m+1}}\bigr)\bigr)^{m+1}, \quad m>0, \tag{1.6}$$

$$s = \min\bigl\{m_1(b-a)^{1/p}, m_2(b-a)^{1/p}\bigr\}, \qquad S = \max\bigl\{M_1(b-a)^{1/p}, M_2(b-a)^{1/p}\bigr\},$$

$$t = \min\bigl\{m_1(b-a)^{1/r}, m_2(b-a)^{1/r}\bigr\}, \qquad T = \max\bigl\{M_1(b-a)^{1/r}, M_2(b-a)^{1/r}\bigr\},$$

$$\Upsilon_{\alpha,\beta}(m_1,m_2,M_1,M_2) = \max\left\{C_{\alpha,\beta}\left(\frac{M_1^\alpha}{m_1^\alpha(b-a)}, \frac{m_2^\beta}{M_2^\beta(b-a)}\right), C_{\alpha,\beta}\left(\frac{m_1^\alpha}{M_1^\alpha(b-a)}, \frac{M_2^\beta}{m_2^\beta(b-a)}\right)\right\}, \tag{1.7}$$

$$C_{\alpha,\beta}(\xi,\eta) = \frac{\xi/\alpha+\eta/\beta}{\xi^{1/\alpha}\eta^{1/\beta}}, \tag{1.8}$$

and

$$\begin{aligned}\Gamma_{\alpha,\beta}&(m_1,m_2,M_1,M_2)\\ &= \max\bigl\{\Upsilon_{\alpha,\beta}\bigl(m_1,(m_1+m_2)^{\alpha-1},M_1,(M_1+M_2)^{\alpha-1}\bigr),\\ &\quad \Upsilon_{\alpha,\beta}\bigl(m_2,(m_1+m_2)^{\alpha-1},M_2,(M_1+M_2)^{\alpha-1}\bigr)\bigr\}.\end{aligned} \tag{1.9}$$

2 Proof of theorem

Lemma 2.1 [15] *If $0 < m_1 \leq a \leq M_1$, $0 < m_2 \leq b \leq M_2$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $\alpha>1$, then*

$$\max\bigl\{C_{\alpha,\beta}(M_1,m_2), C_{\alpha,\beta}(m_1,M_2)\bigr\} \cdot \alpha\beta a^{1/\alpha}b^{1/\beta} \geq a\beta + b\alpha, \tag{2.1}$$

with equality if and only if either $(a,b)=(m_1,M_2)$ or $(a,b)=(M_1,m_2)$, where $C_{\alpha,\beta}(\xi,\eta)$ is as in (1.8).

Obviously, by using a way similar to the proof of (2.1), we may find that inequality (2.1) is reversed if $0<\alpha<1$ or $\alpha<0$. Here, we omit the details.

Lemma 2.2 *Let f and g be positive continuous functions on $[a,b]$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, $\alpha>1$ and f^α and g^β be integrable on $[a,b]$. If $0<m_1\le f(x)\le M_1$ and $0<m_2\le g(x)\le M_2$, then*

$$\left(\int_a^b f^\alpha(x)\,dx\right)^{1/\alpha}\left(\int_a^b g^\beta(x)\,dx\right)^{1/\beta}\le \Upsilon_{\alpha,\beta}(m_1,m_2,M_1,M_2)\cdot\int_a^b f(x)g(x)\,dx, \tag{2.2}$$

with equality if and only if f^α and g^β are proportional, where $\Upsilon_{\alpha,\beta}(m_1,m_2,M_1,M_2)$ is as in (1.7).

The inequality is reversed if $0<\alpha<1$ or $\alpha<0$.

Proof If we set successively

$$\bar{a}=\frac{f^\alpha(x)}{X},\qquad X=\int_a^b f^\alpha(x)\,dx,$$
$$\bar{b}=\frac{g^\beta(x)}{Y},\qquad Y=\int_a^b g^\beta(x)\,dx.$$

Notice that

$$\frac{m_1^\alpha}{M_1^\alpha(b-a)}\le\bar{a}\le\frac{M_1^\alpha}{m_1^\alpha(b-a)},$$

and

$$\frac{m_2^\beta}{M_2^\beta(b-a)}\le\bar{b}\le\frac{M_2^\beta}{m_2^\beta(b-a)}.$$

By using Lemma 2.1, we have

$$\max\left\{C_{\alpha,\beta}\left(\frac{M_1^\alpha}{m_1^\alpha(b-a)},\frac{m_2^\beta}{M_2^\beta(b-a)}\right),C_{\alpha,\beta}\left(\frac{m_1^\alpha}{M_1^\alpha(b-a)},\frac{M_2^\beta}{m_2^\beta(b-a)}\right)\right\}\cdot\frac{f(x)g(x)}{X^{1/\alpha}Y^{1/\beta}}$$
$$\ge\frac{1}{\alpha}\frac{f^\alpha(x)}{X}+\frac{1}{\beta}\frac{g^\beta(x)}{Y},$$

with equality if and only if either

$$(\bar{a},\bar{b})=\left(\frac{m_1^\alpha}{M_1^\alpha(b-a)},\frac{M_2^\beta}{m_2^\beta(b-a)}\right)$$

or

$$(\bar{a},\bar{b})=\left(\frac{M_1^\alpha}{m_1^\alpha(b-a)},\frac{m_2^\beta}{M_2^\beta(b-a)}\right).$$

Therefore

$$\Upsilon_{\alpha,\beta}(m_1,m_2,M_1,M_2)\cdot\frac{\int_a^b f(x)g(x)\,dx}{X^{1/\alpha}Y^{1/\beta}}\ge\frac{1}{\alpha}\frac{\int_a^b f^\alpha(x)\,dx}{X}+\frac{1}{\beta}\frac{\int_a^b g^\beta(x)\,dx}{Y}=1. \tag{2.3}$$

From (2.3), inequality (2.2) easily follows.

In the following, we discuss the equality condition of (2.2). In view of the equality conditions of Lemma 2.1, the equality in (2.3) holds if and only if

$$\left(\frac{f^\alpha(x)}{\int_a^b f^\alpha(x)\,dx}, \frac{g^\beta(x)}{\int_a^b g^\beta(x)\,dx}\right) = \left(\frac{m_1^\alpha}{M_1^\alpha(b-a)}, \frac{M_2^\beta}{m_2^\beta(b-a)}\right),$$

or

$$\left(\frac{f^\alpha(x)}{\int_a^b f^\alpha(x)\,dx}, \frac{g^\beta(x)}{\int_a^b g^\beta(x)\,dx}\right) = \left(\frac{M_1^\alpha}{m_1^\alpha(b-a)}, \frac{m_2^\beta}{M_2^\beta(b-a)}\right).$$

Hence $f^\alpha(x) = \mu g^\beta(x)$, where

$$\mu = \frac{m_1^\alpha m_2^\beta}{M_2^\beta M_1^\alpha}\frac{\|f\|_\alpha^\alpha}{\|g\|_\beta^\beta},$$

or

$$\mu = \frac{M_1^\alpha M_2^\beta}{m_2^\beta m_1^\alpha}\frac{\|f\|_\alpha^\alpha}{\|g\|_\beta^\beta}$$

is a constant. It follows that the equality in (2.2) holds if and only if f^α and g^β are proportional.

This proof is completed. □

Lemma 2.3 *Let f and g be non-negative continuous functions on $[a,b]$. If $0 < m_1 \le f(x) \le M_1$, $0 < m_2 \le g(x) \le M_2$ and $\alpha > 1$, then*

$$\left(\int_a^b \big(f(x)+g(x)\big)^\alpha\,dx\right)^{1/\alpha}$$
$$\ge \Gamma_{\alpha,\beta}(m_1,m_2,M_1,M_2)\left(\left(\int_a^b f^\alpha(x)\,dx\right)^{1/\alpha} + \left(\int_a^b g^\alpha(x)\,dx\right)^{1/\alpha}\right), \tag{2.4}$$

with equality if and only if f and g are proportional, where $\Gamma_{\alpha,\beta}(m_1,m_2,M_1,M_2)$ is as in (1.9).

The inequality is reversed if $0 < \alpha < 1$ or $\alpha < 0$.

Proof From the hypotheses, we have

$$\big\|f(x)+g(x)\big\|_\alpha^\alpha = \big\|f(x)\big[f(x)+g(x)\big]^{\alpha-1}\big\|_1 + \big\|g(x)\big[f(x)+g(x)\big]^{\alpha-1}\big\|_1. \tag{2.5}$$

By using Lemma 2.2, we obtain

$$\big\|f(x)\big[f(x)+g(x)\big]^{\alpha-1}\big\|_1 \ge \big[\Upsilon_{\alpha,\beta}\big(m_1,(m_1+m_2)^{\alpha-1},M_1,(M_1+M_2)^{\alpha-1}\big)\big]^{-1}$$
$$\times \big\|f(x)\big\|_\alpha \cdot \big\|f(x)+g(x)\big\|_\alpha^{\alpha/\beta}, \tag{2.6}$$

with equality if and only if $f^\alpha(x)$ and $(f(x)+g(x))^\alpha$ are proportional. It follows that the equality holds if and only if $f(x)$ and $g(x)$ are proportional.

$$\begin{aligned}\big\|g(x)\big[f(x)+g(x)\big]^{\alpha-1}\big\|_1 &\geq \big[\Upsilon_{\alpha,\beta}\big(m_2,(m_1+m_2)^{\alpha-1},M_2,(M_1+M_2)^{\alpha-1}\big)\big]^{-1}\\ &\quad\times \big\|g(x)\big\|_\alpha \cdot \big\|f(x)+g(x)\big\|_\alpha^{\alpha/\beta},\end{aligned} \tag{2.7}$$

with equality if and only if $g^\alpha(x)$ and $(f(x)+g(x))^\alpha$ are proportional. It follows that the equality holds if and only if $f(x)$ and $g(x)$ are proportional. Hence

$$\big\|f(x)+g(x)\big\|_\alpha^\alpha \geq \Gamma_{\alpha,\beta}(m_1,m_2,M_1,M_2)\cdot \big\|f(x)+g(x)\big\|_\alpha^{\alpha/\beta}\big(\big\|f(x)\big\|_\alpha+\big\|g(x)\big\|_\alpha\big), \tag{2.8}$$

where $\Gamma_{\alpha,\beta}(m_1,m_2,M_1,M_2)=\max\{M,N\}$,

$$M=\Upsilon_{\alpha,\beta}\big(m_1,(m_1+m_2)^{\alpha-1},M_1,(M_1+M_2)^{\alpha-1}\big),$$

and

$$N=\Upsilon_{\alpha,\beta}\big(m_2,(m_1+m_2)^{\alpha-1},M_2,(M_1+M_2)^{\alpha-1}\big).$$

Dividing both sides of (2.8) by $\|f(x)+g(x)\|_\alpha^{\alpha/\beta}$, we have

$$\big\|f(x)+g(x)\big\|_\alpha \geq \Gamma_{\alpha,\beta}(m_1,m_2,M_1,M_2)\cdot\big(\big\|f(x)\big\|_\alpha+\big\|g(x)\big\|_\alpha\big). \tag{2.9}$$

Moreover, in view of the equality conditions of (2.6) and (2.7), it follows that the equality in (2.4) holds if and only if $f(x)$ and $g(x)$ are proportional.

This proof is completed. □

Lemma 2.4 *Let f and g be continuous functions on $[a,b]$, $0<m_1\leq f(x)\leq M_1$ and $0<m_2\leq g(x)\leq M_2$. If $m>0$, then*

$$\int_a^b \frac{f^{m+1}(x)}{g^m(x)}\,dx \leq L_{\alpha,\beta}(m_1,m_2,M_1,M_2)\frac{(\int_a^b f(x)\,dx)^{m+1}}{(\int_a^b g(x)\,dx)^m}, \tag{2.10}$$

where $L_{\alpha,\beta}(m_1,m_2,M_1,M_2)$ is as in (1.6).

Proof Let $\alpha=m+1$, $\beta=(m+1)/m$ and replacing $f(x)$ and $g(x)$ by $u(x)$ and $v(x)$ in (2.2), respectively, we have

$$\begin{aligned}&\left(\int_a^b u(x)^{m+1}\,dx\right)^{1/(m+1)}\left(\int_a^b v(x)^{(m+1)/m}\,dx\right)^{m/(m+1)}\\ &\quad\leq \Upsilon_{\alpha,\beta}(m_1,m_2,M_1,M_2)\cdot\int_a^b u(x)v(x)\,dx.\end{aligned} \tag{2.11}$$

Taking for

$$u(x)=\left(\frac{f(x)}{g(x)}\right)^{1/(m+1)},\qquad v(x)=f^{m/(m+1)}(x)g^{1/(m+1)}(x)$$

in (2.11), and in view of

$$\left(\frac{m_1}{M_2}\right)^{\frac{1}{m+1}}\leq u(x)\leq\left(\frac{M_1}{m_2}\right)^{\frac{1}{m+1}}$$

and

$$m_1^{\frac{m}{m+1}} m_2^{\frac{1}{m+1}} \le \nu(x) \le M_1^{\frac{m}{m+1}} M_2^{\frac{1}{m+1}},$$

we obtain

$$\Upsilon_{\alpha,\beta}\big((m_1M_2^{-1})^{\frac{1}{m+1}}, m_1^{\frac{m}{m+1}} m_2^{\frac{1}{m+1}}, (M_1m_2^{-1})^{\frac{1}{m+1}}, M_1^{\frac{m}{m+1}} M_2^{\frac{1}{m+1}}\big) \int_a^b f(x)\,dx$$
$$\ge \left(\int_a^b \frac{f(x)}{g(x)}\,dx\right)^{1/(m+1)} \left(\int_a^b f(x)g^{1/m}(x)\,dx\right)^{m/(m+1)}.$$

Hence

$$\int_a^b \frac{f(x)}{g(x)}\,dx \le \frac{[\Upsilon_{\alpha,\beta}((m_1M_2^{-1})^{\frac{1}{m+1}}, m_1^{\frac{m}{m+1}} m_2^{\frac{1}{m+1}}, (M_1m_2^{-1})^{\frac{1}{m+1}}, M_1^{\frac{m}{m+1}} M_2^{\frac{1}{m+1}}) \int_a^b f(x)\,dx]^{m+1}}{(\int_a^b f(x)g^{1/m}(x)\,dx)^m}. \tag{2.12}$$

On the other hand, in (2.12), replacing $f(x)$ and $g(x)$ by $u(x)$ and $\nu(x)$, respectively, and letting $u(x) = f(x)$ and $\nu(x) = (\frac{g(x)}{f(x)})^m$, and in view of

$$m_1 \le u(x) \le M_1$$

and

$$\left(\frac{m_2}{M_1}\right)^m \le \nu(x) \le \left(\frac{M_2}{m_1}\right)^m,$$

we have

$$\int_a^b \frac{f^{m+1}(x)}{g^m(x)}\,dx \le \frac{[\Upsilon_{\alpha,\beta}(m_1M_2^{-\frac{m}{m+1}}, (m_1m_2M_1^{-1})^{\frac{m}{m+1}}, M_1m_2^{-\frac{m}{m+1}}, (m_1^{-1}M_1M_2)^{\frac{m}{m+1}}) \int_a^b f(x)\,dx]^{m+1}}{(\int_a^b g(x)\,dx)^m}$$
$$= \frac{L_{\alpha,\beta}(m_1, m_2, M_1, M_2)(\int_a^b f(x)\,dx)^{m+1}}{(\int_a^b g(x)\,dx)^m}.$$

This proof is completed. □

Let $f(x)$ and $g(x)$ reduce to positive real sequences a_i and b_i ($i = 1, \ldots, n$), respectively, and with appropriate changes in the proof of (2.10), we have the following.

Lemma 2.5 *Let a_i and b_i be positive real sequences and $0 < m_1 \le a_i \le M_1$, $0 < m_2 \le b_i \le M_2$, $i = 1, \ldots, n$. If $m > 0$, then*

$$\sum_{i=1}^n \frac{a_i^{m+1}}{b_i^m} \le L_{\alpha,\beta}(m_1, m_2, M_1, M_2) \frac{(\sum_{i=1}^n a_i)^{m+1}}{(\sum_{i=1}^n b_i)^m}, \tag{2.13}$$

where $L_{\alpha,\beta}(m_1, m_2, M_1, M_2)$ is as in Lemma 2.4.

This is just an inverse of the following well-known Radon's inequality [16], *p*.61

$$\sum_{i=1}^{n}\frac{a_i^{m+1}}{b_i^m}\geq\frac{(\sum_{i=1}^{n}a_i)^{m+1}}{(\sum_{i=1}^{n}b_i)^m},$$

where $m>0$, $a_i\geq 0$ *and* $b_i>0$, $i=1,2,\ldots,n$.

Proof of Theorem Let

$$\alpha_1=\left(\int_a^b f^p(x)\,dx\right)^{1/p},\qquad \beta_1=\left(\int_a^b f^r(x)\,dx\right)^{1/r},$$
$$\alpha_2=\left(\int_a^b g^p(x)\,dx\right)^{1/p},\qquad \beta_2=\left(\int_a^b g^r(x)\,dx\right)^{1/r},$$

then

$$0<m_1(b-a)^{1/p}\leq\alpha_1\leq M_1(b-a)^{1/p},$$
$$0<m_2(b-a)^{1/p}\leq\alpha_2\leq M_2(b-a)^{1/p},$$
$$0<m_1(b-a)^{1/r}\leq\beta_1\leq M_1(b-a)^{1/r},$$

and

$$0<m_2(b-a)^{1/r}\leq\beta_2\leq M_2(b-a)^{1/r}.$$

Let

$$s=\min\bigl\{m_1(b-a)^{1/p},m_2(b-a)^{1/p}\bigr\},\qquad S=\max\bigl\{M_1(b-a)^{1/p},M_2(b-a)^{1/p}\bigr\}$$

and

$$t=\min\bigl\{m_1(b-a)^{1/r},m_2(b-a)^{1/r}\bigr\},\qquad T=\max\bigl\{M_1(b-a)^{1/r},M_2(b-a)^{1/r}\bigr\}.$$

From reverse Radon's inequality (2.13) in Lemma 2.5, we have, for $m>0$,

$$\frac{\alpha_1^{m+1}}{\beta_1^m}+\frac{\alpha_2^{m+1}}{\beta_2^m}\leq L_{\alpha,\beta}(s,t,S,T)\frac{(\alpha_1+\alpha_2)^{m+1}}{(\beta_1+\beta_2)^m}. \tag{2.14}$$

If $m=\frac{r}{p-r}$, then

$$\begin{aligned}&\left(\frac{\int f^p(x)\,dx}{\int f^r(x)\,dx}\right)^{1/(p-r)}+\left(\frac{\int g^p(x)\,dx}{\int g^r(x)\,dx}\right)^{1/(p-r)}\\&\quad\leq L_{\alpha,\beta}(s,t,S,T)\frac{[(\int f^p(x)\,dx)^{1/p}+(\int g^p(x)\,dx)^{1/p}]^{p/(p-r)}}{[(\int f^r(x)\,dx)^{1/r}+(\int g^r(x)\,dx)^{1/r}]^{r/(p-r)}}.\end{aligned} \tag{2.15}$$

We have assumed $p>r>0$, since $m=\frac{r}{p-r}>0$.

On the other hand, by using the Minkowski inequality (2.4) and its reverse form, with $p \geq 1$ and $0 < r \leq 1$, respectively,

$$\Gamma_{\alpha,\beta}(m_1, m_2, M_1, M_2)^p \left[\left(\int f^p(x)\,dx \right)^{1/p} + \left(\int g^p(x)\,dx \right)^{1/p} \right]^p \leq \int \bigl(f(x) + g(x)\bigr)^p\,dx, \tag{2.16}$$

with equality if and only if f and g are proportional, and

$$\Gamma_{\alpha,\beta}(m_1, m_2, M_1, M_2)^r \left[\left(\int f^r(x)\,dx \right)^{1/r} + \left(\int g^r(x)\,dx \right)^{1/r} \right]^r \geq \int \bigl(f(x) + g(x)\bigr)^r(x)\,dx, \tag{2.17}$$

with equality if and only if f and g are proportional.

From (2.15), (2.16) and (2.17), (1.4) follows. This proof is completed. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

C-JZ and W-SC jointly contributed to the main results. All authors read and approved the final manuscript.

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