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Stability of functional equations in (n, β)-normed spaces

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Abstract
In this paper, we first introduce the notions of (n, β)-normed space and non-Archimedean (n, β)-normed space, then we study the Hyers-Ulam stability of the Cauchy functional equation and the Jensen functional equation in non-Archimedean (n, β)-normed spaces and that of the pexiderized Cauchy functional equation in (n, β)-normed spaces.

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1 Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940 concerning the stability of group homomorphisms. Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Rassias [3] proved a generalization of the Hyers theorem for additive mappings. The result of Rassias has provided a lot of influence during the past 36 years in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as the Hyers-Ulam-Rassias stability of functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem. A large list of references can be found in [4–11].

In [12, 13], Gähler introduced the theory of 2-norms and n-norms on a linear space. A systematic development of n-normed linear spaces is due to Kim and Cho [14], Malceski [15], Misiak [16] and Gunawan and Mashadi [17].

Recently, Park [18] investigated the approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in 2-Banach spaces. This is the first result for the stability problem of functional equations in 2-Banach spaces. In 2012, Xu and Rassias [19] examined the Hyers-Ulam stability of a general mixed additive and cubic functional equation in n-Banach spaces. In 2013, Xu [20] investigated approximate multi-Jensen, multi-Euler-Lagrange additive and quadratic mappings in n-Banach spaces.

In this paper, we first introduce the notions of (n,β)-normed space and non-Archimedean (n,β)-normed space, then we study the Hyers-Ulam stability of the Cauchy functional equation and the Jensen functional equation in non-Archimedean (n,β)-normed spaces in Section 2. Finally, in Section 3, we investigate the Hyers-Ulam stability of the pexiderized Cauchy functional equation in (n,β)-normed spaces.

Now, we give some concepts concerning the (n,β)-normed space.

Definition 1.1 Let X be a linear space over $\mathbb{R}$ with $\dim X \geq n$, $n \in \mathbb{N}$ and $0 < \beta \leq 1$, let $\|\cdot,\ldots,\cdot\|_\beta : X^n \to \mathbb{R}$ be a function satisfying the following properties:

(a) $\|x_1,\ldots,x_n\|_\beta = 0$ if and only if $x_1,\ldots,x_n$ are linearly dependent;

(b) $\|x_1,\ldots,x_n\|_\beta$ is invariant under permutations of $x_1,\ldots,x_n$;

(c) $\|\alpha x_1,\ldots,x_n\|_\beta = |\alpha|^\beta \|x_1,\ldots,x_n\|_\beta$;

(d) $\|x_1,\ldots,x_{n-1},y+z\|_\beta \leq \|x_1,\ldots,x_{n-1},y\|_\beta + \|x_1,\ldots,x_{n-1},z\|_\beta$

for all $x_1,\ldots,x_n \in X$ and $\alpha \in \mathbb{R}$.

Then the function $\|\cdot,\ldots,\cdot\|_\beta$ is called an (n,β)-norm on X and the pair $(X,\|\cdot,\ldots,\cdot\|_\beta)$ is called a linear (n,β)-normed space or an (n,β)-normed space.

We remark that the concept of a linear (n,β)-normed space is a generalization of a linear n-normed space ($\beta = 1$) and of a β-normed space ($n = 1$). Now we present two examples about n-normed space.

Example 1.2 [19] For $x_1,\ldots,x_n \in \mathbb{R}^n$, the Euclidean n-norm $\|x_1,\ldots,x_n\|_E$ is defined by

$$\|x_1,\ldots,x_n\|_E = \left|\det(x_{ij})\right| = \operatorname{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right), \tag{1.1}$$

where $x_i = (x_{i1},x_{i2},\ldots,x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\ldots,n$.

Example 1.3 [19] The standard n-norm on X, a real inner product space of dimension $\dim X \geq n$, is as follows:

$$\|x_1,x_2,\ldots,x_n\|_S = \begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_1,x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,x_1\rangle & \cdots & \langle x_n,x_n\rangle \end{vmatrix}^{1/2}, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on X. If $X = \mathbb{R}^n$, then this n-norm is exactly the same as the Euclidean n-norm $\|x_1,\ldots,x_n\|_E$ mentioned earlier. For $n = 1$, this n-norm is the usual norm $\|x_1\| = \langle x_1,x_1\rangle^{1/2}$.

Lemma 1.4 *Let $(X,\|\cdot,\ldots,\cdot\|_\beta)$ be a linear (n,β)-normed space, $n \geq 2$, $0 < \beta \leq 1$. If $x_1 \in X$ and $\|x_1,y_1,\ldots,y_{n-1}\|_\beta = 0$ for all $y_1,\ldots,y_{n-1} \in X$, then $x_1 = 0$.*

Proof Since $\dim X \geq n$, we can take $y_1,\ldots,y_n$ from X such that they are linearly independent. It follows from the assumption that $\|x_1,y_2,\ldots,y_n\|_\beta = 0$, then by the definition of linear (n,β)-normed space we have that $x_1, y_2, \ldots, y_n$ are linearly dependent. Thus there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ with $(\alpha_1, \alpha_2, \ldots, \alpha_n) \neq (0, \ldots, 0)$ such that

$$\alpha_1 x_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n = 0.$$

Then we have $\alpha_1 \neq 0$. (If $\alpha_1 = 0$, since $y_2, \ldots, y_n$ are linearly independent, then we have $\alpha_2 = 0, \ldots, \alpha_n = 0$; this is a contradiction.) So we have

$$x_1 = -\frac{\alpha_2}{\alpha_1} y_2 - \cdots - \frac{\alpha_n}{\alpha_1} y_n. \tag{1.3}$$

Hence $x_1 \in \operatorname{span}\{y_2, y_3, \ldots, y_n\}$. Similarly, let $A_i = \{y_1, y_2, \ldots, y_n\} \setminus \{y_i\}$, we can obtain that $x_1 \in \operatorname{span} A_i$, $i = 1, 2, \ldots, n$. In the n-dimensional space $\operatorname{span}\{y_1, y_2, \ldots, y_n\}$, it is easy to get that $\bigcap_{i=1}^n \operatorname{span} A_i = 0$, from which it follows that $x_1 = 0$. □

Remark 1.5 Let $(X, \|\cdot, \ldots, \cdot\|_\beta)$ be a linear (n,β)-normed space, $0 < \beta \leq 1$. One can show that conditions (b) and (d) in Definition 1.1 imply that

$$\bigl| \|x, z_1, \ldots, z_{n-1}\|_\beta - \|y, z_1, \ldots, z_{n-1}\|_\beta \bigr| \leq \|x - y, z_1, \ldots, z_{n-1}\|_\beta$$

for all $x, y \in X$ and $z_1, \ldots, z_{n-1} \in Y$.

Definition 1.6 A sequence $\{x_m\}$ in a linear (n,β)-normed space X is called a convergent sequence if there is $x \in X$ such that

$$\lim_{m\to\infty} \|x_m - x, y_1, \ldots, y_{n-1}\|_\beta = 0$$

for all $y_1, \ldots, y_{n-1} \in X$. In this case, we call that $\{x_m\}$ converges to x or that x is the limit of $\{x_m\}$, write $x_m \to x$ as $m \to \infty$ or $\lim_{m\to\infty} x_m = x$.

Definition 1.7 A sequence $\{x_m\}$ in a linear (n,β)-normed space X is called a Cauchy sequence if

$$\lim_{m,k\to\infty} \|x_k - x_m, z_1, \ldots, z_{n-1}\|_\beta = 0$$

for all $z_1, \ldots, z_{n-1} \in X$.

We can easily get the following lemma by Remark 1.5.

Lemma 1.8 *For a convergent sequence $\{x_m\}$ in a linear (n,β)-normed space X,*

$$\lim_{m\to\infty} \|x_m, z_1, \ldots, z_{n-1}\|_\beta = \Bigl\| \lim_{m\to\infty} x_m, z_1, \ldots, z_{n-1} \Bigr\|_\beta$$

for all $z_1, \ldots, z_{n-1} \in X$.

Definition 1.9 A linear (n,β)-normed space in which every Cauchy sequence is convergent is called a complete (n,β)-normed space.

In 1897, Hensel [21] introduced a normed space which does not have the Archimedean property. It turns out that non-Archimedean spaces have many nice applications (see [22–24]).

Definition 1.10 A field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ is called a non-Archimedean field if the function $|\cdot|: K \to [0,\infty)$ satisfies the following conditions:

(1) $|r| = 0$ if and only if $r = 0$;
(2) $|rs| = |r||s|$;
(3) $|r+s| \leq \max\{|r|,|s|\}$ for all $r,s \in \mathbb{K}$;
(4) there exists a member $a_0 \in K$ such that $|a_0| \neq 0,1$.

Definition 1.11 [25] Let X be a vector space over a scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\|: X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(1′) $\|x\| = 0$ if and only if $x = 0$;
(2′) $\|rx\| = |r|\|x\|$;
(3′) $\|x+y\| \leq \max\{\|x\|,\|y\|\}$ for all $x,y \in X$ and $r \in K$.

The pair $(X,\|\cdot\|)$ is called a non-Archimedean space if $\|\cdot\|$ is a non-Archimedean norm on X.

Definition 1.12 Let X be a real vector space with $\dim X \geq n$ over a scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$, where n is a positive integer and β is a constant with $0 < \beta \leq 1$. A real-valued function $\|\cdot,\ldots,\cdot\|_\beta : X^n \to \mathbb{R}$ is called an (n,β)-norm on X if the following conditions hold:

(N1′) $\|x_1,\ldots,x_n\|_\beta = 0$ if and only if $x_1,\ldots,x_n$ are linearly dependent;
(N2′) $\|x_1,\ldots,x_n\|_\beta$ is invariant under permutations of $x_1,\ldots,x_n$;
(N3′) $\|\alpha x_1,x_2,\ldots,x_n\|_\beta = |\alpha|^\beta \|x_1,x_2,\ldots,x_n\|_\beta$;
(N4′) $\|x_0+x_1,x_2,\ldots,x_n\|_\beta \leq \max\{\|x_0,x_2,\ldots,x_n\|_\beta, \|x_1,x_2,\ldots,x_n\|_\beta\}$

for all $\alpha \in K$ and $x_0,x_1,\ldots,x_n \in X$.
Then $(X,\|\cdot,\ldots,\cdot\|_\beta)$ is called a non-Archimedean (n,β)-normed space.

It follows from the preceding definition that the non-Archimedean (n,β)-normed space is a non-Archimedean n-normed space if $\beta = 1$, and a non-Archimedean β-normed space if $n = 1$, respectively.

Remark 1.13 A sequence $\{x_m\}$ in a non-Archimedean (n,β)-normed space X is a Cauchy sequence if and only if $\{x_{m+1} - x_m\}$ converges to zero.

Proof It follows from (N4′) that

$$\begin{aligned}&\|x_m - x_k, y_1,\ldots,y_{n-1}\|_\beta \\ &\quad\leq \max\bigl\{\|x_{j+1} - x_j, y_1,\ldots,y_{n-1}\|_\beta : k \leq j \leq m-1\bigr\} \quad (m > k)\end{aligned}$$

for all $y_1, \ldots, y_{n-1} \in X$. So a sequence $\{x_m\}$ is a Cauchy sequence in X if and only if $\{x_{m+1} - x_m\}$ converges to zero. □

Throughout this paper, let $\mathbb{N}$ denote the set of positive integers and $j, k, m, n \in \mathbb{N}$, and let $n \geq 2$ be fixed.

2 Cauchy functional equations

In this section, we assume that $|2| \neq 1$. Under this condition we investigate the Hyers-Ulam stability of the Cauchy functional equation in which the target space Y is a complete non-Archimedean (n, β)-normed space. When the domain space X is a non-Archimedean β-normed space, we can formulate our result as follows.

Theorem 2.1 *Suppose that X is a non-Archimedean β_1-normed space and that Y is a complete non-Archimedean (n, β)-normed space, where $n \geq 2$, $0 < \beta, \beta_1 \leq 1$. Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $(p+q)\beta_1 > \beta$, and let $\psi : \underbrace{Y \times Y \times \cdots \times Y}_{n-1} \to [0, \infty)$ be a function. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| f(x+y) - f(x) - f(y), z_1, \ldots, z_{n-1} \right\|_\beta \leq \theta \|x\|_{\beta_1}^p \|y\|_{\beta_1}^q \psi(z_1, \ldots, z_{n-1}) \tag{2.1}$$

for all $x, y \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\| f(x) - A(x), z_1, \ldots, z_{n-1} \right\|_\beta \leq \theta \left|2^{-\beta}\right| \|x\|_{\beta_1}^{p+q} \psi(z_1, \ldots, z_{n-1}) \tag{2.2}$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$.

Proof Putting $y = x$ in (2.1) and dividing both sides by $|2^\beta|$, we get

$$\left\| \frac{f(2x)}{2} - f(x), z_1, \ldots, z_{n-1} \right\|_\beta \leq \theta \left|2^{-\beta}\right| \|x\|_{\beta_1}^{p+q} \psi(z_1, \ldots, z_{n-1}) \tag{a}$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Replacing x by $2^m x$ in (a) and dividing both sides by $|2^{m\beta}|$, we get

$$\begin{aligned} &\left\| \frac{f(2^{m+1}x)}{2^{m+1}} - \frac{f(2^m x)}{2^m}, z_1, \ldots, z_{n-1} \right\|_\beta \\ &\quad \leq \theta \left| \frac{1}{2^{m\beta}} \right| \left| \frac{1}{2^\beta} \right| \left|2^{m(p+q)\beta_1}\right| \|x\|_{\beta_1}^{p+q} \psi(z_1, \ldots, z_{n-1}) \\ &\quad = \theta \left|2^{-\beta}\right| \left|2^{(p+q)\beta_1 - \beta}\right|^m \|x\|_{\beta_1}^{p+q} \psi(z_1, \ldots, z_{n-1}) \end{aligned}$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Since $(p+q)\beta_1 > \beta$ and $|2| \neq 1$, we have

$$\lim_{m \to \infty} \left\| 2^{-m-1} f\left(2^{m+1}x\right) - 2^{-m} f\left(2^m x\right), z_1, \ldots, z_{n-1} \right\|_\beta = 0$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Considering Remark 1.13, we get that $\{2^{-m} f(2^m x)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is a complete space, we can define the mapping

$A : X \to Y$ by

$$A(x) = \lim_{m\to\infty} 2^{-m} f\bigl(2^m x\bigr) \tag{b}$$

for all $x \in X$.

Next, we show that A is additive. It follows from (2.1), (b) and Lemma 1.8 that

$$\begin{aligned}
&\bigl\|A(x+y) - A(x) - A(y), z_1, \ldots, z_{n-1}\bigr\|_\beta \\
&\quad = \lim_{m\to\infty} \bigl|2^{-m\beta}\bigr| \bigl\|f\bigl(2^m x + 2^m y\bigr) - f\bigl(2^m x\bigr) - f\bigl(2^m y\bigr), z_1, \ldots, z_{n-1}\bigr\|_\beta \\
&\quad \le \lim_{m\to\infty} \theta \bigl|2^{-m\beta}\bigr| \bigl\|2^m x\bigr\|_{\beta_1}^p \bigl\|2^m y\bigr\|_{\beta_1}^q \psi(z_1, \ldots, z_{n-1}) \\
&\quad = \lim_{m\to\infty} \theta \bigl|2^{(p+q)\beta_1 - \beta}\bigr|^m \|x\|_{\beta_1}^p \|y\|_{\beta_1}^q \psi(z_1, \ldots, z_{n-1})
\end{aligned}$$

for all $x, y \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Since $(p+q)\beta_1 > \beta$ and $|2| \neq 1$, we get

$$\bigl\|A(x+y) - A(x) - A(y), z_1, \ldots, z_{n-1}\bigr\|_\beta = 0$$

for all $x, y \in X$ and $z_1, \ldots, z_{n-1} \in Y$. By Lemma 1.4, we get

$$A(x+y) - A(x) - A(y) = 0$$

for all $x, y \in X$. So the mapping A is additive.

Replacing x by $2x$ in (a) and dividing both sides by $|2^\beta|$, we get

$$\left\|\frac{f(2^2 x)}{2^2} - \frac{f(2x)}{2}, z_1, \ldots, z_{n-1}\right\|_\beta \le \theta \bigl|2^{-2\beta}\bigr| \|2x\|_{\beta_1}^{p+q} \psi(z_1, \ldots, z_{n-1}). \tag{c}$$

Thus by (a) and (c), we get

$$\begin{aligned}
&\left\|f(x) - \frac{f(2^2 x)}{2^2}, z_1, \ldots, z_{n-1}\right\|_\beta \\
&\quad \le \max\left\{\left\|\frac{f(2x)}{2} - f(x), z_1, \ldots, z_{n-1}\right\|_\beta, \left\|\frac{f(2^2 x)}{2^2} - \frac{f(2x)}{2}, z_1, \ldots, z_{n-1}\right\|_\beta\right\} \\
&\quad \le \max\bigl\{\theta \bigl|2^{-\beta}\bigr| \|x\|_{\beta_1}^{p+q} \psi(z_1, \ldots, z_{n-1}), \theta \bigl|2^{-2\beta}\bigr| \|2x\|_{\beta_1}^{p+q} \psi(z_1, \ldots, z_{n-1})\bigr\}
\end{aligned}$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Since $(p+q)\beta_1 > \beta$ and $|2| \neq 1$, we get

$$\bigl\|f(x) - 2^{-2} f(2x), z_1, \ldots, z_{n-1}\bigr\|_\beta \le \bigl|2^{-\beta}\bigr| \theta \|x\|_{\beta_1}^{p+q} \psi(z_1, \ldots, z_{n-1})$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$.

By induction on m, we can conclude that

$$\bigl\|f(x) - 2^{-m} f\bigl(2^m x\bigr), z_1, \ldots, z_{n-1}\bigr\|_\beta \le \bigl|2^{-\beta}\bigr| \theta \|x\|_{\beta_1}^{p+q} \psi(z_1, \ldots, z_{n-1}) \tag{d}$$

for all $m \in \mathbb{N}$, $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x with $2x$ in (d) and dividing both sides by $|2^\beta|$, we get

$$\left\|2^{-1}f(2x) - 2^{-m-1}f\left(2^{m+1}x\right), z_1, \dots, z_{n-1}\right\|_\beta \le \left|2^{-2\beta}\right|\theta\|2x\|_{\beta_1}^{p+q}\psi(z_1, \dots, z_{n-1}) \tag{e}$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in \mathbb{N}$. It follows from (a) and (e) that

$$\left\|f(x) - 2^{-m-1}f\left(2^{m+1}x\right), z_1, \dots, z_{n-1}\right\|_\beta \le \left|2^{-\beta}\right|\theta\|x\|_{\beta_1}^{p+q}\psi(z_1, \dots, z_{n-1})$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in \mathbb{N}$. This completes the proof of (d).

Taking the limit as $m \to \infty$ in (d), we can obtain (2.2).

Finally, we need to prove the uniqueness of A. Let A' be another additive mapping satisfying (2.2),

$$\begin{aligned}
&\left\|A(x) - A'(x), z_1, \dots, z_{n-1}\right\|_\beta \\
&\quad = \left|2^{-m\beta}\right|\left\|A\left(2^m x\right) - A'\left(2^m x\right), z_1, \dots, z_{n-1}\right\|_\beta \\
&\quad \le \left|2^{-m\beta}\right|\max\left\{\left\|A\left(2^m x\right) - f\left(2^m x\right), z_1, \dots, z_{n-1}\right\|_\beta, \left\|f\left(2^m x\right) - A'\left(2^m x\right), z_1, \dots, z_{n-1}\right\|_\beta\right\} \\
&\quad \le \left|2^{-m\beta}\right|\left|2^{-\beta}\right|\theta\left\|2^m x\right\|_{\beta_1}^{p+q}\psi(z_1, \dots, z_{n-1}) \\
&\quad = \theta\left|2^{(p+q)\beta_1-\beta}\right|^m\left|2^{-\beta}\right|\|x\|_{\beta_1}^{p+q}\psi(z_1, \dots, z_{n-1})
\end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Taking the limit as $m \to \infty$, we get

$$\left\|A(x) - A'(x), z_1, \dots, z_{n-1}\right\|_\beta = 0$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By Lemma 1.4, we get $A(x) = A'(x)$ for all $x \in X$. So A is the unique additive mapping satisfying (2.2). □

When the domain space X is a vector space, we get the following theorems with a generalized control function.

Theorem 2.2 *Let X be a vector space and Y be a complete non-Archimedean (n, β)-normed space, where $n \ge 2$ and $0 < \beta \le 1$. Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty}\left|\frac{1}{2^{m\beta}}\right|\varphi\left(2^m x, 2^m y\right) = 0 \tag{2.3}$$

for all $x, y \in X$, and let $\psi : \underbrace{Y \times Y \times \cdots \times Y}_{n-1} \to [0, \infty)$ be a function. The limit

$$\lim_{m\to\infty}\max\left\{\left|2^{-j\beta}\right|\varphi\left(2^{j-1}x, 2^{j-1}x\right) : 1 \le j \le m\right\} \tag{2.4}$$

exists for all $x \in X$, and it is denoted by $\widetilde{\varphi}(x)$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$\left\|f(x + y) - f(x) - f(y), z_1, \dots, z_{n-1}\right\|_\beta \le \varphi(x, y)\psi(z_1, \dots, z_{n-1}) \tag{2.5}$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$. Then there exists an additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x), z_1, \dots, z_{n-1}\|_\beta \leq \widetilde{\varphi}(x)\psi(z_1, \dots, z_{n-1}) \tag{2.6}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k\to\infty} \lim_{m\to\infty} \max\{|2^{-j\beta}|\varphi(2^{j-1}x, 2^{j-1}x) : 1 + k \leq j \leq m + k\} = 0 \tag{2.7}$$

for all $x \in X$, then A is a unique additive mapping satisfying (2.6).

Proof Putting $y = x$ in (2.5) and dividing both sides by $|2^\beta|$, we get

$$\left\|\frac{f(2x)}{2} - f(x), z_1, \dots, z_{n-1}\right\|_\beta \leq |2^{-\beta}|\varphi(x, x)\psi(z_1, \dots, z_{n-1}) \tag{f}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $2^j x$ in (f) and dividing both sides by $|2^{j\beta}|$, we get

$$\left\|\frac{f(2^{j+1}x)}{2^{j+1}} - \frac{f(2^j x)}{2^j}, z_1, \dots, z_{n-1}\right\|_\beta \leq |2^{-j\beta}||2^{-\beta}|\varphi(2^j x, 2^j x)\psi(z_1, \dots, z_{n-1})$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $j \in \mathbb{N}$. Taking the limit as $j \to \infty$ and considering (2.3), we get

$$\lim_{j\to\infty} \left\|\frac{f(2^{j+1}x)}{2^{j+1}} - \frac{f(2^j x)}{2^j}, z_1, \dots, z_{n-1}\right\|_\beta = 0$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering Remark 1.13, we know that $\{2^{-m}f(2^m x)\}$ is a Cauchy sequence. Since Y is a complete space, we can define the mapping $A : X \to Y$ by

$$A(x) = \lim_{m\to\infty} 2^{-m}f(2^m x)$$

for all $x \in X$.

Next, we prove that A is additive:

$$\begin{aligned}
&\|A(x + y) - A(x) - A(y), z_1, \dots, z_{n-1}\|_\beta \\
&\quad \leq |2^{-m\beta}|\|A(2^m x + 2^m y) - A(2^m x) - A(2^m y), z_1, \dots, z_{n-1}\|_\beta \\
&\quad \leq |2^{-m\beta}|\varphi(2^m x, 2^m x)\psi(z_1, \dots, z_{n-1})
\end{aligned}$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$. Taking the limit as $m \to \infty$ and considering (2.3), we get

$$\|A(x + y) - A(x) - A(y), z_1, \dots, z_{n-1}\|_\beta = 0$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By Lemma 1.4, we know that A is additive.

Replacing x by $2x$ in (f) and dividing both sides by $|2^\beta|$, we get

$$\left\|\frac{f(2^2x)}{2^2}-\frac{f(2x)}{2},z_1,\dots,z_{n-1}\right\|_\beta\le\left|2^{-2\beta}\right|\varphi(2x,2x)\psi(z_1,\dots,z_{n-1})$$

for all $x\in X$ and $z_1,\dots,z_{n-1}\in Y$. Considering (f), we get

$$\left\|f(x)-\frac{f(2^2x)}{2^2},z_1,\dots,z_{n-1}\right\|_\beta\le\max\left\{\left|2^{-\beta}\right|\varphi(x,x),\left|2^{-2\beta}\right|\varphi(2x,2x)\right\}\psi(z_1,\dots,z_{n-1})$$

for all $x\in X$, $z_1,\dots,z_{n-1}\in Y$.

By induction on m, we get

$$\left\|f(x)-\frac{f(2^mx)}{2^m},z_1,\dots,z_{n-1}\right\|_\beta\le\max\left\{\frac{\varphi(2^{k-1}x,2^{k-1}x)}{|2^{k\beta}|}:1\le k\le m\right\}\psi(z_1,\dots,z_{n-1})\qquad\text{(g)}$$

for all $x\in X$ and $z_1,\dots,z_{n-1}\in Y$. Replacing x by $2x$ in (g) and dividing both sides by $|2^\beta|$, we get

$$\left\|\frac{f(2x)}{2}-\frac{f(2^{m+1}x)}{2^{m+1}},z_1,\dots,z_{n-1}\right\|_\beta\le\max\left\{\frac{\varphi(2^kx,2^kx)}{|2^{(k+1)\beta}|}:1\le k\le m\right\}\psi(z_1,\dots,z_{n-1})$$

for all $x\in X$, $z_1,\dots,z_{n-1}\in Y$ and $m\in\mathbb{N}$, which together with (f) implies

$$\begin{aligned}&\left\|f(x)-\frac{f(2^{m+1}x)}{2^{m+1}},z_1,\dots,z_{n-1}\right\|_\beta\\&\quad\le\max\left\{\frac{\varphi(x,x)}{|2^\beta|},\frac{\varphi(2^kx,2^kx)}{|2^{(k+1)\beta}|}:1\le k\le m\right\}\psi(z_1,\dots,z_{n-1})\\&\quad=\max\left\{\left|2^{-(k+1)\beta}\right|\varphi\left(2^kx,2^kx\right):0\le k\le m\right\}\psi(z_1,\dots,z_{n-1})\\&\quad=\max\left\{\left|2^{-k\beta}\right|\varphi\left(2^{k-1}x,2^{k-1}x\right):1\le k\le m+1\right\}\psi(z_1,\dots,z_{n-1})\end{aligned}$$

for all $x\in X$, $z_1,\dots,z_{n-1}\in Y$ and $m\in\mathbb{N}$. This completes the proof of (g).

Taking the limit as $m\to\infty$ in (g), we can obtain (2.6).

Now we need to prove the uniqueness of A. Let A' be another additive mapping satisfying (2.6). Since

$$\begin{aligned}&\lim_{k\to\infty}\left|2^{-k\beta}\right|\widetilde{\varphi}\left(2^kx\right)\\&\quad=\lim_{k\to\infty}\left|2^{-k\beta}\right|\lim_{m\to\infty}\max\left\{\left|2^{-j\beta}\right|\varphi\left(2^{j+k-1}x,2^{j+k-1}x\right):1\le j\le m\right\}\\&\quad=\lim_{k\to\infty}\lim_{m\to\infty}\max\left\{\left|2^{-j\beta}\right|\varphi\left(2^{j-1}x,2^{j-1}x\right):1+k\le j\le m+k\right\}\end{aligned}$$

for all $x\in X$ and $z_1,\dots,z_{n-1}\in Y$, it follows from (2.7) that

$$\begin{aligned}&\left\|A(x)-A'(x),z_1,\dots,z_{n-1}\right\|_\beta\\&\quad=\lim_{k\to\infty}\left|2^{-k\beta}\right|\left\|A\left(2^kx\right)-A'\left(2^kx\right),z_1,\dots,z_{n-1}\right\|_\beta\end{aligned}$$

$$\leq \lim_{k\to\infty} |2^{-k\beta}| \max\{ \| A(2^k x) - f(2^k x), z_1, \ldots, z_{n-1} \|_\beta,$$
$$\| f(2^k x) - A'(2^k x), z_1, \ldots, z_{n-1} \|_\beta \}$$
$$\leq \lim_{k\to\infty} |2^{-k\beta}| \widetilde{\varphi}(2^k x) \psi(z_1, \ldots, z_{n-1})$$
$$= 0$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Considering Lemma 1.4, we prove that A is unique. □

Next, we study the Hyers-Ulam stability of Jensen functional equation in a non-Archimedean (n, β)-normed space.

Theorem 2.3 *Let X be a vector space and Y be a complete non-Archimedean (n, β)-normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} |2^{m\beta}| \varphi\left(\frac{x}{2^m}, \frac{y}{2^m} \right) = 0 \tag{2.8}$$

for all $x, y \in X$, and let $\psi : \underbrace{Y \times Y \times \cdots \times Y}_{n-1} \to [0, \infty)$ be a function. The limit

$$\lim_{m\to\infty} \max\left\{ |2^{j\beta}| \varphi\left(\frac{x}{2^j}, 0 \right) : 0 \leq j \leq m-1 \right\} \tag{2.9}$$

exists for all $x \in X$, which is denoted by $\widetilde{\varphi}(x)$. Suppose that a mapping $f : X \to Y$ and $f(0) = 0$ satisfies the inequality

$$\left\| 2f\left(\frac{x+y}{2} \right) - f(x) - f(y), z_1, \ldots, z_{n-1} \right\|_\beta \leq \varphi(x, y) \psi(z_1, \ldots, z_{n-1}) \tag{2.10}$$

for all $x, y \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Then there exists an additive mapping $A : X \to Y$ such that

$$\| f(x) - A(x), z_1, \ldots, z_{n-1} \|_\beta \leq \widetilde{\varphi}(x) \psi(z_1, \ldots, z_{n-1}) \tag{2.11}$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k\to\infty} \lim_{m\to\infty} \max\left\{ |2^{j\beta}| \varphi\left(\frac{x}{2^j}, 0 \right) : k \leq j \leq m+k-1 \right\} = 0 \tag{2.12}$$

for all $x \in X$, then A is a unique additive mapping satisfying (2.11).

Proof Putting $y = 0$ in (2.10), we get

$$\left\| 2f\left(\frac{x}{2} \right) - f(x), z_1, \ldots, z_{n-1} \right\|_\beta \leq \varphi(x, 0) \psi(z_1, \ldots, z_{n-1}) \tag{a1}$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Replacing x by $\frac{x}{2^m}$ in (a1) and multiplying both sides by $|2^{m\beta}|$, we get

$$\left\| 2^{m+1} f\left(\frac{x}{2^{m+1}} \right) - 2^m f\left(\frac{x}{2^m} \right), z_1, \ldots, z_{n-1} \right\|_\beta \leq |2^{m\beta}| \varphi\left(\frac{x}{2^m}, 0 \right) \psi(z_1, \ldots, z_{n-1})$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Taking the limit as $m \to \infty$ and considering (2.8), we get

$$\lim_{m\to\infty}\left\| 2^{m+1}f\left(\frac{x}{2^{m+1}}\right) - 2^m f\left(\frac{x}{2^m}\right), z_1, \dots, z_{n-1}\right\|_\beta = 0$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering Remark 1.13, we know that $\{2^m f(\frac{x}{2^m})\}$ is a Cauchy sequence. Since Y is a complete space, we can define the mapping $A : X \to Y$ by

$$A(x) = \lim_{m\to\infty} 2^m f\left(\frac{x}{2^m}\right) \tag{b1}$$

for all $x \in X$.

By induction on m, we get

$$\begin{aligned}&\left\| 2^m f\left(\frac{x}{2^m}\right) - f(x), z_1, \dots, z_{n-1}\right\|_\beta \\ &\quad\le \max\left\{ \left|2^{k\beta}\right| \varphi\left(\frac{x}{2^k}, 0\right) : 0 \le k \le m-1 \right\} \psi(z_1, \dots, z_{n-1})\end{aligned} \tag{c1}$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in \mathbb{N}$. Replacing x by $\frac{x}{2}$ in (c1) and multiplying both sides by $|2^\beta|$, we get

$$\begin{aligned}&\left\| 2^{m+1} f\left(\frac{x}{2^{m+1}}\right) - 2f\left(\frac{x}{2}\right), z_1, \dots, z_{n-1}\right\|_\beta \\ &\quad\le \max\left\{ \left|2^{(k+1)\beta}\right| \varphi\left(\frac{x}{2^{k+1}}, 0\right) : 0 \le k \le m-1 \right\} \psi(z_1, \dots, z_{n-1})\end{aligned}$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in \mathbb{N}$. Considering the above inequality and (a1), we have

$$\begin{aligned}&\left\| 2^{m+1} f\left(\frac{x}{2^{m+1}}\right) - f(x), z_1, \dots, z_{n-1}\right\|_\beta \\ &\quad\le \max\left\{ \varphi(x,0), \left|2^{(k+1)\beta}\right| \varphi\left(\frac{x}{2^{k+1}}, 0\right) : 0 \le k \le m-1 \right\} \psi(z_1, \dots, z_{n-1}) \\ &\quad= \max\left\{ \left|2^{k\beta}\right| \varphi\left(\frac{x}{2^k}, 0\right) : 0 \le k \le m \right\} \psi(z_1, \dots, z_{n-1})\end{aligned}$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in \mathbb{N}$. This completes the proof of (c1).

Taking the limit as $m \to \infty$ in (c1), we can obtain (2.11).

Next, we prove that A is additive. Considering (2.8), (2.10) and (b1), we have

$$\begin{aligned}&\left\| 2A\left(\frac{x+y}{2}\right) - A(x) - A(y), z_1, \dots, z_{n-1}\right\|_\beta \\ &\quad= \lim_{m\to\infty}\left|2^{m\beta}\right| \left\| 2f\left(\frac{x+y}{2^{m+1}}\right) - f\left(\frac{x}{2^m}\right) - f\left(\frac{y}{2^m}\right), z_1, \dots, z_{n-1}\right\|_\beta \\ &\quad\le \lim_{m\to\infty}\left|2^{m\beta}\right| \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right)\psi(z_1, \dots, z_{n-1}) \\ &\quad= 0\end{aligned}$$

for all $x, y \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Considering Lemma 1.4, we have $2A(\frac{x+y}{2}) - A(x) - A(y) = 0$ for all $x, y \in X$. Since $f(0) = 0$, $A(0) = 0$, we know that A is additive.

Now we need to prove the uniqueness of A. Let A' be another additive mapping satisfying (2.11). Since

$$\begin{aligned}
&\lim_{k\to\infty} |2^{k\beta}| \widetilde{\varphi}\left(\frac{x}{2^k}\right) \\
&\quad = \lim_{k\to\infty} |2^{k\beta}| \lim_{m\to\infty} \max\left\{ |2^{(j+k)\beta}| \varphi\left(\frac{x}{2^{j+k}}, 0\right) : 0 \le j \le m-1 \right\} \\
&\quad = \lim_{k\to\infty} \lim_{m\to\infty} \max\left\{ |2^{j\beta}| \varphi\left(\frac{x}{2^{j}}, 0\right) : k \le j \le m+k-1 \right\}
\end{aligned}$$

for all $x \in X$, it follows from (2.12) that

$$\begin{aligned}
&\left\| A(x) - A'(x), z_1, \ldots, z_{n-1} \right\|_\beta \\
&\quad = \lim_{k\to\infty} |2^{k\beta}| \left\| A\left(\frac{x}{2^k}\right) - A'\left(\frac{x}{2^k}\right), z_1, \ldots, z_{n-1} \right\|_\beta \\
&\quad \le \lim_{k\to\infty} |2^{k\beta}| \max\left\{ \left\| A\left(\frac{x}{2^k}\right) - f\left(\frac{x}{2^k}\right), z_1, \ldots, z_{n-1} \right\|_\beta, \right. \\
&\qquad \left. \left\| f\left(\frac{x}{2^k}\right) - A'\left(\frac{x}{2^k}\right), z_1, \ldots, z_{n-1} \right\|_\beta \right\} \\
&\quad \le \lim_{k\to\infty} |2^{k\beta}| \widetilde{\varphi}\left(\frac{x}{2^k}\right) \psi(z_1, \ldots, z_{n-1}) \\
&\quad = 0
\end{aligned}$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$. Considering Lemma 1.4, we prove that A is unique. □

3 Pexiderized Cauchy functional equations

In this section, we investigate the Hyers-Ulam stability of the pexiderized Cauchy functional equation in (n, β)-normed spaces.

Theorem 3.1 *Let X be a vector space and Y be a complete (n, β)-normed space with $0 < \beta \le 1$. Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying*

$$\Phi(x) = \sum_{i=1}^{\infty} 2^{-i\beta} \left(\varphi\left(2^{i-1}x, 0\right) + \varphi\left(0, 2^{i-1}x\right) + \varphi\left(2^{i-1}x, 2^{i-1}x\right) \right) < \infty \tag{3.1}$$

and

$$\lim_{m\to\infty} 2^{-m\beta} \varphi\left(2^m x, 2^m y\right) = 0 \tag{3.2}$$

for all $x, y \in X$. $\psi : \underbrace{Y \times Y \times \cdots \times Y}_{n-1} \to [0, \infty)$ is a function. If mappings $f, g, h : X \to Y$ satisfy the inequality

$$\left\| f(x+y) - g(x) - h(y), z_1, \ldots, z_{n-1} \right\|_\beta \le \varphi(x, y) \psi(z_1, \ldots, z_{n-1}) \tag{3.3}$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then there exists a unique additive mapping $A : X \to Y$ satisfying

$$\begin{aligned} &\|f(x) - A(x), z_1, \dots, z_{n-1}\|_\beta \\ &\quad \le \Phi(x)\psi(z_1, \dots, z_{n-1}) + \|h(0), z_1, \dots, z_{n-1}\|_\beta + \|g(0), z_1, \dots, z_{n-1}\|_\beta, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\|g(x) - A(x), z_1, \dots, z_{n-1}\|_\beta \\ &\quad \le \Phi(x)\psi(z_1, \dots, z_{n-1}) + \|g(0), z_1, \dots, z_{n-1}\|_\beta + 2\|h(0), z_1, \dots, z_{n-1}\|_\beta \\ &\qquad + \varphi(x, 0)\psi(z_1, \dots, z_{n-1}), \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\|h(x) - A(x), z_1, \dots, z_{n-1}\|_\beta \\ &\quad \le \Phi(x)\psi(z_1, \dots, z_{n-1}) + \|h(0), z_1, \dots, z_{n-1}\|_\beta + 2\|g(0), z_1, \dots, z_{n-1}\|_\beta \\ &\qquad + \varphi(0, x)\psi(z_1, \dots, z_{n-1}) \end{aligned} \tag{3.6}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$.

Proof Putting $y = x$ in inequality (3.3), we get

$$\|f(2x) - g(x) - h(x), z_1, \dots, z_{n-1}\|_\beta \le \varphi(x, x)\psi(z_1, \dots, z_{n-1}) \tag{3.7}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Putting $y = 0$ in inequality (3.3), we get

$$\|f(x) - g(x) - h(0), z_1, \dots, z_{n-1}\|_\beta \le \varphi(x, 0)\psi(z_1, \dots, z_{n-1}) \tag{3.8}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. It then follows from (3.8) that

$$\|f(x) - g(x), z_1, \dots, z_{n-1}\|_\beta \le \varphi(x, 0)\psi(z_1, \dots, z_{n-1}) + \|h(0), z_1, \dots, z_{n-1}\|_\beta \tag{3.9}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Putting $x = 0$ in inequality (3.3), we get

$$\|f(y) - g(0) - h(y), z_1, \dots, z_{n-1}\|_\beta \le \varphi(0, y)\psi(z_1, \dots, z_{n-1})$$

for all $y \in X$ and $z_1, \dots, z_{n-1} \in Y$. Thus, we obtain

$$\|f(x) - h(x), z_1, \dots, z_{n-1}\|_\beta \le \varphi(0, x)\psi(z_1, \dots, z_{n-1}) + \|g(0), z_1, \dots, z_{n-1}\|_\beta \tag{3.10}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$.

Let us define

$$\begin{aligned} &u(x, z_1, \dots, z_{n-1}) \\ &\quad = \|g(0), z_1, \dots, z_{n-1}\|_\beta + \|h(0), z_1, \dots, z_{n-1}\|_\beta + \varphi(x, x)\psi(z_1, \dots, z_{n-1}) \\ &\qquad + \varphi(x, 0)\psi(z_1, \dots, z_{n-1}) + \varphi(0, x)\psi(z_1, \dots, z_{n-1}). \end{aligned}$$

Using (3.7), (3.9) and (3.10), we have

$$\begin{aligned}
&\|f(2x) - 2f(x), z_1, \dots, z_{n-1}\|_\beta \\
&\quad \le \|f(2x) - g(x) - h(x), z_1, \dots, z_{n-1}\|_\beta + \|g(x) - f(x), z_1, \dots, z_{n-1}\|_\beta \\
&\qquad + \|h(x) - f(x), z_1, \dots, z_{n-1}\|_\beta \\
&\quad \le \|g(0), z_1, \dots, z_{n-1}\|_\beta + \|h(0), z_1, \dots, z_{n-1}\|_\beta + \varphi(x,0)\psi(z_1, \dots, z_{n-1}) \\
&\qquad + \varphi(0,x)\psi(z_1, \dots, z_{n-1}) + \varphi(x,x)\psi(z_1, \dots, z_{n-1}) \\
&\quad = u(x, z_1, \dots, z_{n-1})
\end{aligned} \tag{3.11}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x with $2x$ in (3.11), we get

$$\|f(2^2 x) - 2f(2x), z_1, \dots, z_{n-1}\|_\beta \le u(2x, z_1, \dots, z_{n-1}) \tag{3.12}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. It then follows from (3.11) and (3.12) that

$$\begin{aligned}
&\|f(2^2 x) - 2^2 f(x), z_1, \dots, z_{n-1}\|_\beta \\
&\quad \le \|f(2^2 x) - 2f(2x), z_1, \dots, z_{n-1}\|_\beta + 2^\beta \|f(2x) - 2f(x), z_1, \dots, z_{n-1}\|_\beta \\
&\quad \le u(2x, z_1, \dots, z_{n-1}) + 2^\beta u(x, z_1, \dots, z_{n-1})
\end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$.

Applying an induction argument on m, we will prove that

$$\|f(2^m x) - 2^m f(x), z_1, \dots, z_{n-1}\|_\beta \le \sum_{i=1}^{m} 2^{(i-1)\beta} u(2^{m-i} x, z_1, \dots, z_{n-1}) \tag{3.13}$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in N$. In view of (3.11), inequality (3.13) is true for $m = 1$. Assume that (3.13) is true for some $m > 1$. Substituting $2x$ for x in (3.13), we obtain

$$\|f(2^{m+1} x) - 2^m f(2x), z_1, \dots, z_{n-1}\|_\beta \le \sum_{i=1}^{m} 2^{(i-1)\beta} u(2^{m+1-i} x, z_1, \dots, z_{n-1})$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Hence, it follows from (3.11) that

$$\begin{aligned}
&\|f(2^{m+1} x) - 2^{m+1} f(x), z_1, \dots, z_{n-1}\|_\beta \\
&\quad \le \|f(2^{m+1} x) - 2^m f(x), z_1, \dots, z_{n-1}\|_\beta + 2^{n\beta} \|f(2x) - 2f(x), z_1, \dots, z_{n-1}\|_\beta \\
&\quad \le \sum_{i=1}^{m} 2^{(i-1)\beta} u(2^{m+1-i} x, z_1, \dots, z_{n-1}) + 2^{m\beta} u(x, z_1, \dots, z_{n-1}) \\
&\quad = \sum_{i=1}^{m+1} 2^{(i-1)\beta} u(2^{m+1-i} x, z_1, \dots, z_{n-1})
\end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$, which proves inequality (3.13). By (3.13), we have

$$\left\|2^{-m}f\left(2^m x\right) - f(x), z_1, \dots, z_{n-1}\right\|_\beta \le \sum_{i=1}^{m} 2^{(i-1-m)\beta} u\left(2^{m-i}x, z_1, \dots, z_{n-1}\right) \tag{3.14}$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in \mathbb{N}$. Moreover, if $m, k \in \mathbb{N}$ with $m < k$, then it follows from (3.11) that

$$\begin{aligned}
&\left\|2^{-k}f\left(2^k x\right) - 2^{-m}f\left(2^m x\right), z_1, \dots, z_{n-1}\right\|_\beta \\
&\quad\le \sum_{i=m}^{k-1} \left\|2^{-i}f\left(2^i x\right) - 2^{-(i+1)}f\left(2^{i+1}x\right), z_1, \dots, z_{n-1}\right\|_\beta \\
&\quad\le \sum_{i=m}^{k-1} 2^{-(i+1)\beta} \left\|2f\left(2^i x\right) - f\left(2^{i+1}x\right), z_1, \dots, z_{n-1}\right\|_\beta \\
&\quad= \sum_{i=m}^{k-1} 2^{-(i+1)\beta} u\left(2^i x, z_1, \dots, z_{n-1}\right) \\
&\quad= \sum_{i=m}^{k-1} 2^{-(i+1)\beta} \left[\varphi\left(2^i x, 0\right)\psi(z_1, \dots, z_{n-1}) + \varphi\left(0, 2^i x\right)\psi(z_1, \dots, z_{n-1})\right. \\
&\qquad \left. + \varphi\left(2^i x, 2^i x\right)\psi(z_1, \dots, z_{n-1}) + \left\|h(0), z_1, \dots, z_{n-1}\right\|_\beta + \left\|g(0), z_1, \dots, z_{n-1}\right\|_\beta\right] \\
&\quad\le \sum_{i=m}^{k-1} 2^{-(i+1)\beta} \left[\varphi\left(2^i x, 0\right) + \varphi\left(0, 2^i x\right) + \varphi\left(2^i x, 2^i x\right)\right]\psi(z_1, \dots, z_{n-1}) \\
&\qquad + 2^{-m}\left(\left\|h(0), z_1, \dots, z_{n-1}\right\|_\beta + \left\|g(0), z_1, \dots, z_{n-1}\right\|_\beta\right)
\end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Taking the limit as $m, k \to \infty$ and considering (3.1), we get

$$\lim_{m,k\to\infty} \left\|2^{-k}f\left(2^k x\right) - 2^{-m}f\left(2^m x\right), z_1, \dots, z_{n-1}\right\|_\beta = 0$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. According to Definition 1.7, we know that $\{2^{-m}f(2^m x)\}$ is a Cauchy sequence for every $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Since Y is a complete (n, β)-normed space, we can define a function $A : X \to Y$ by

$$A(x) = \lim_{m\to\infty} 2^{-m}f\left(2^m x\right).$$

Replacing x, y by $2^m x$, $2^m y$ in (3.3) and dividing both sides by $2^{m\beta}$, we get

$$\begin{aligned}
&2^{-m\beta}\left\|f\left(2^m x + 2^m y\right) - g\left(2^m x\right) - h\left(2^m y\right), z_1, \dots, z_{n-1}\right\|_\beta \\
&\quad\le 2^{-m\beta}\varphi\left(2^m x, 2^m y\right)\psi(z_1, \dots, z_{n-1})
\end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. It follows from (3.9) that

$$\begin{aligned}
&\left\|2^{-m}f\left(2^m x\right) - 2^{-m}g\left(2^m x\right), z_1, \dots, z_{n-1}\right\|_\beta \\
&\quad\le 2^{-m\beta}\left[\left\|h(0), z_1, \dots, z_{n-1}\right\|_\beta + \varphi\left(2^m x, 0\right)\psi(z_1, \dots, z_{n-1})\right] \qquad (3.15)
\end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering (3.1), we get

$$\begin{aligned}
&2^{-m\beta}\varphi\big(2^m x, 0\big)\psi(z_1, \dots, z_{n-1}) \\
&\quad \le 2^\beta \sum_{i=m}^{\infty} 2^{-(i+1)\beta}\big[\varphi\big(2^i x, 0\big)\psi(z_1, \dots, z_{n-1}) + \varphi\big(0, 2^i x\big)\psi(z_1, \dots, z_{n-1}) \\
&\qquad + \varphi\big(2^i x, 2^i x\big)\psi(z_1, \dots, z_{n-1})\big] \\
&\quad \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

It follows from (3.15) that

$$\lim_{m\to\infty} 2^{-m} g\big(2^m x\big) = \lim_{m\to\infty} 2^{-m} f\big(2^m x\big) = A(x) \tag{3.16}$$

for all $x \in X$. Also, by (3.10), we have

$$\begin{aligned}
&\big\|2^{-m} h\big(2^m x\big) - 2^{-m} f\big(2^m x\big), z_1, \dots, z_{n-1}\big\|_\beta \\
&\quad \le 2^{-m\beta}\big[\big\|g(0), z_1, \dots, z_{n-1}\big\|_\beta + \varphi\big(0, 2^m x\big)\psi(z_1, \dots, z_{n-1})\big]
\end{aligned} \tag{3.17}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Similarly, it follows from (3.17) that

$$\lim_{m\to\infty} 2^{-m} h\big(2^m x\big) = \lim_{m\to\infty} 2^{-m} f\big(2^m x\big) = A(x) \tag{3.18}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Thus, by (3.2), (3.16), (3.18) and Lemma 1.8, we get

$$\begin{aligned}
&\big\|A(x+y) - A(x) - A(y), z_1, \dots, z_{n-1}\big\|_\beta \\
&\quad = \lim_{m\to\infty} \big\|2^{-m} f\big(2^m x + 2^m y\big) - 2^{-m} g\big(2^m x\big) - 2^{-m} h\big(2^m y\big), z_1, \dots, z_{n-1}\big\|_\beta \\
&\quad \le \lim_{m\to\infty} 2^{-m\beta}\varphi\big(2^m x, 2^m y\big)\psi(z_1, \dots, z_{n-1}) \\
&\quad = 0
\end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Hence $A(x+y) - A(x) - A(y) = 0$.

Taking the limit as $m \to \infty$ in (3.14), we get

$$\begin{aligned}
&\big\|A(x) - f(x), z_1, \dots, z_{n-1}\big\|_\beta \\
&\quad \le \lim_{m\to\infty} \sum_{i=1}^{m} 2^{(i-1-m)\beta} u\big(2^{m-i} x, z_1, \dots, z_{n-1}\big) \\
&\quad = \lim_{m\to\infty} \big(1 - 2^{-m\beta}\big)\big(\big\|g(0), z_1, \dots, z_{n-1}\big\|_\beta + \big\|h(0), z_1, \dots, z_{n-1}\big\|_\beta\big) \\
&\qquad + \lim_{m\to\infty} \sum_{i=1}^{m} 2^{(i-m-1)\beta}\big(\varphi\big(2^{m-i} x, 0\big)\psi(z_1, \dots, z_{n-1}) + \varphi\big(0, 2^{m-i} x\big)\psi(z_1, \dots, z_{n-1}) \\
&\qquad + \varphi\big(2^{m-i} x, 2^{m-i} x\big)\psi(z_1, \dots, z_{n-1})\big) \\
&\quad = \big\|h(0), z_1, \dots, z_{n-1}\big\|_\beta + \big\|g(0), z_1, \dots, z_{n-1}\big\|_\beta + \Phi(x)\psi(z_1, \dots, z_{n-1})
\end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$, which proves (3.4).

It remains to prove the uniqueness of A. Assume that $A' : X \to Y$ is another additive mapping which satisfies (3.4). Then we have

$$\begin{aligned}
&\|A(x) - A'(x), z_1, \ldots, z_{n-1}\|_\beta \\
&\quad \le 2^{-m\beta} \|A(2^m x) - f(2^m x), z_1, \ldots, z_{n-1}\|_\beta + 2^{-m\beta} \|f(2^m x) - A'(2^m x), z_1, \ldots, z_{n-1}\|_\beta \\
&\quad \le 2^{-m\beta+1} \bigl(\|g(0), z_1, \ldots, z_{n-1}\|_\beta + \|h(0), z_1, \ldots, z_{n-1}\|_\beta + \Phi(2^m x)\psi(z_1, \ldots, z_{n-1})\bigr) \\
&\quad = 2^{-m\beta+1} \bigl(\|g(0), z_1, \ldots, z_{n-1}\|_\beta + \|h(0), z_1, \ldots, z_{n-1}\|_\beta\bigr) \\
&\qquad + 2 \sum_{i=m+1}^{\infty} 2^{-i\beta} \bigl(\varphi(2^{i-1}x, 0) + \varphi(0, 2^{i-1}x) + \varphi(2^{i-1}x, 2^{i-1}x)\bigr)\psi(z_1, \ldots, z_{n-1}) \\
&\quad \to 0 \quad \text{as } m \to \infty
\end{aligned}$$

for all $x \in X$ and $z_1, \ldots, z_{n-1} \in Y$, which together with Lemma 1.4 implies that $A(x) = A'(x)$ for all $x \in X$. Using (3.4) and (3.9), we can get (3.5), and also using (3.4) and (3.10), we can get (3.6). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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