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Fixed points of admissible almost contractive type mappings on b-metric spaces with an application to quadratic integral equations

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Abstract

Samet *et al.* in (Nonlinear Anal. 75:2154-2165, 2012) introduced the concepts of α-ψ-contractive type mappings and α-admissible mappings in metric spaces. The purpose of this paper is to present a new class of almost contractive mappings called almost generalized (α-ψ-φ-θ)-contractive mappings and to establish some fixed and common fixed point results for this class of mappings in complete ordered b-metric spaces. Our results improve and generalize several known results from the current literature and its extension. Moreover, an application to integral equations is given here to illustrate the usability of the obtained results.

MSC: 47H10; 54H25

Keywords: common fixed point; coincidence point; b-metric space; g-α-admissible mapping; α-regular; triangular α-admissible; integral equations; almost contractive mapping

1 Introduction

It is well known that the Banach contraction principle has been improved in different directions at different spaces by mathematicians over the years. In 1998, Czerwik [1, 2] introduced the concept of b-metric space. In the sequel, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in b-metric spaces (see, *e.g.*, [1, 3–20]). On the other hand, more recently, Samet *et al.* in [21] introduced the concepts of α-ψ-contractive type mappings and α-admissible mappings in metric spaces. Then, Karapınar and Samet [22] introduced the concept of generalized α-ψ-contractive type, which was inspired by the notion of α-ψ-contractive mappings. Furthermore, they [22] obtained various fixed point theorems for this generalized class of contractive mappings. Also, it should be noted that the study of common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity (see [23–27]). In this paper, first, we introduce the concept of almost generalized (α-ψ-φ-θ)-contractive mappings, and then we prove some common fixed point and coincidence fixed point theorems for this class of mappings in partially ordered complete b-metric spaces. Finally, as an application of our main results, we prove the existence of a unique solution to a class of nonlinear quadratic integral equations. The results of this paper improve and generalize the obtained results in papers [21, 22, 28].

Definition 1.1 [1] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric space iff for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x,y) = 0$ iff $x = y$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a b-metric space with the parameter s.

It is obvious that a b-metric space with base $s = 1$ is a metric space. There are examples of b-metric spaces which are not metric spaces (see, *e.g.*, Singh and Prasad [18]).

The notions of a Cauchy sequence and a convergent sequence in b-metric spaces are defined by Boriceanu [29]. As usual, a b-metric space is said to be complete if and only if each Cauchy sequence in this space is convergent. Note that a b-metric, in the general case, is not continuous [3].

Definition 1.2 [30] Let X be a nonempty set and $T, g : X \to X$ be given self-mappings on X. The pair $\{T, g\}$ is said to be weakly compatible if $Tgx = gTx$, whenever $Tx = gx$ for some x in X.

Samet *et al.* [21] defined the notion of α-admissible mappings as follows.

Definition 1.3 Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be α-admissible if

$$\alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Recently, Rosa and Vetro [31] introduced the following new notions of g-α-admissible mapping.

Definition 1.4 Let $T, g : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. The mapping T is g-α-admissible if, for all $x, y \in X$ such that $\alpha(gx, gy) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$. If g is the identity mapping, then T is called α-admissible.

Definition 1.5 [32] An α-admissible map T is said to be triangular α-admissible if

$$\alpha(x,z) \geq 1 \quad \text{and} \quad \alpha(z,y) \geq 1 \quad \Longrightarrow \quad \alpha(x,y) \geq 1.$$

2 Main results

In this section, we prove some common fixed point results for two self-mappings satisfying an almost generalized $(\alpha$-ψ-φ-$\theta)$-contractive mapping (for the notion of α-ψ-contractive type mappings, see Samet *et al.* [21]).

Let (X,d) be a b-metric space with a constant s and $T : X \to X$ and $g : X \to X$ be two mappings. Set

$$M(x,y) = \max\left\{d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{d(gx,Ty) + d(gy,Tx)}{2s}\right\}$$

and

$$N(x,y) = \min\{d(gx,Tx), d(gx,Ty), d(gy,Tx)\}.$$

Now, we introduce the novel notion of an almost generalized $(\alpha\text{-}\psi\text{-}\varphi\text{-}\theta)$-contractive mapping as follows.

Definition 2.1 Let T and g be two self-mappings on a b-metric space (X,d). We say that T is an almost generalized $(\alpha\text{-}\psi\text{-}\varphi\text{-}\theta)$-contractive mapping with respect to g if there exist $\alpha : X \times X \to \mathbb{R}$ and some $L \geq 0$ such that for all $x, y \in X$, we have

$$\alpha(x,y)\psi\big(s^3 d(Tx,Ty)\big) \leq \varphi\big(M(x,y)\big) + L\theta\big(N(x,y)\big), \tag{2.1}$$

where $\psi, \varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions with $\varphi(t) < \psi(t)$, $\theta(t) > 0$ for each $t > 0$, $\varphi(0) = \psi(0) = \theta(0) = 0$ and ψ is increasing.

Definition 2.2 Let (X,d) be a b-metric space, $g : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. X is α-regular with respect to g if for every sequence $\{x_n\} \subseteq X$ such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx \in gX$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that for all $k \in \mathbb{N}$, $\alpha(gx_{n(k)}, gx) \geq 1$. If g is the identity mapping, then T is called α-regular.

Lemma 2.1 *Let $T, g : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. Suppose that T is g-α-admissible and triangular α-admissible. Assume that there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$. Then*

$$\alpha(gx_m, gx_n) \geq 1 \quad \text{for all } m, n \in \mathbb{N} \text{ with } m < n,$$

where

$$gx_{n+1} = Tx_n.$$

Proof Since there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$ and T is g-α-admissible, we deduce that

$$\begin{aligned} &\alpha(gx_0, gx_1) = \alpha(gx_0, Tx_0) \geq 1 \quad \Longrightarrow \quad \alpha(gx_1, gx_2) = \alpha(Tx_0, Tx_1) \geq 1, \\ &\alpha(gx_1, gx_2) \geq 1 \quad \Longrightarrow \quad \alpha(gx_2, gx_3) = \alpha(Tx_1, Tx_2) \geq 1. \end{aligned}$$

By continuing this process, we get

$$\alpha(gx_n, gx_{n+1}) \geq 1, \quad n = 0, 1, 2, \ldots.$$

Suppose that $m < n$. Since $\alpha(gx_m, gx_{m+1}) \geq 1$, $\alpha(gx_{m+1}, gx_{m+2}) \geq 1$ and T is triangular α-admissible, we have $\alpha(gx_m, gx_{m+2}) \geq 1$. Again, since $\alpha(gx_m, gx_{m+2}) \geq 1$ and $\alpha(gx_{m+2}, gx_{m+3}) \geq 1$, we have $\alpha(gx_m, gx_{m+3}) \geq 1$. Continuing this process inductively, we obtain

$$\alpha(gx_m, gx_n) \geq 1. \qquad \square$$

Now, we establish some results for the existence of a common fixed point of mappings satisfying an almost generalized $(\alpha$-ψ-φ-$\theta)$-contractive condition in the setup of b-metric spaces. The main result in this paper is the following common fixed point theorem.

Theorem 2.2 *Let (X,d) be a complete b-metric space, $T,g:X\to X$ be such that $TX\subseteq gX$ and suppose that gX is closed. Assume that the mapping T is an almost generalized $(\alpha$-ψ-φ-$\theta)$-contractive mapping with respect to g and the following conditions hold*:

(i) *T is g-α-admissible and triangular α-admissible*;

(ii) *there exists $x_0\in X$ such that $\alpha(gx_0,Tx_0)\geq 1$*;

(iii) *X is α-regular with respect to g.*

Then T and g have a coincidence point.

Moreover, if the following conditions hold:

(a) *the pair $\{T,g\}$ is weakly compatible*;

(b) *either $\alpha(u,v)\geq 1$ or $\alpha(v,u)\geq 1$ whenever $Tu=gu$ and $Tv=gv$.*

Then T and g have a unique common fixed point.

Proof Let $x_0\in X$ be such that $\alpha(gx_0,Tx_0)\geq 1$ (using condition (ii)). Since $TX\subseteq gX$, we can choose a point $x_1\in X$ such that $Tx_0=gx_1$. Also, there exists $x_2\in X$ such that $Tx_1=gx_2$, this can be done through the reality $TX\subseteq gX$. Continuing this process having chosen $x_1,x_2,\ldots,x_n\in X$, we have $x_{n+1}\in X$ such that

$$gx_{n+1}=Tx_n,\quad n=0,1,2,\ldots. \tag{2.2}$$

By Lemma 2.1, we have

$$\alpha(gx_n,gx_{n+1})\geq 1,\quad n=0,1,2,\ldots. \tag{2.3}$$

If $Tx_{n_0}=Tx_{n_0+1}$ for some n_0, then by (2.2) we get

$$gx_{n_0+1}=Tx_{n_0}=Tx_{n_0+1},$$

that is, T and g have a coincidence point at $x=x_{n_0+1}$, and so the proof is completed. So, we suppose that for all $n\in\mathbb{N}$, $Tx_n\neq Tx_{n+1}$. Since the mapping T is an almost generalized $(\alpha$-ψ-φ-$\theta)$-contractive mapping with respect to g and using (2.3), we obtain

$$\begin{aligned}\psi\bigl(d(gx_n,gx_{n+1})\bigr)&\leq\psi\bigl(s^3d(gx_n,gx_{n+1})\bigr)=\psi\bigl(s^3d(Tx_{n-1},Tx_n)\bigr)\\&\leq\alpha(gx_{n-1},gx_n)\psi\bigl(s^3d(Tx_{n-1},Tx_n)\bigr)\\&\leq\varphi\bigl(M(x_{n-1},x_n)\bigr)+L\theta\bigl(N(x_{n-1},x_n)\bigr)\end{aligned} \tag{2.4}$$

for all $n\in\mathbb{N}$, where

$$\begin{aligned}N(x_{n-1},x_n)&=\min\bigl\{d(gx_{n-1},Tx_{n-1}),d(gx_{n-1},Tx_n),d(gx_n,Tx_{n-1})\bigr\}\\&=\min\bigl\{d(gx_{n-1},gx_n),d(gx_{n-1},gx_{n+1}),d(gx_n,gx_n)\bigr\}\\&=0\end{aligned}$$

and

$$\begin{aligned} &M(x_{n-1},x_n) \\ &\quad = \max\left\{d(gx_{n-1},gx_n), d(gx_{n-1},Tx_{n-1}), d(gx_n,Tx_n), \frac{d(gx_{n-1},Tx_n)+d(gx_n,Tx_{n-1})}{2s}\right\} \\ &\quad = \max\left\{d(gx_{n-1},gx_n), d(gx_{n-1},gx_n), d(gx_n,gx_{n+1}), \frac{d(gx_{n-1},gx_{n+1})+d(gx_n,gx_n)}{2s}\right\} \\ &\quad = \max\left\{d(gx_{n-1},gx_n), d(gx_n,gx_{n+1}), \frac{1}{2s}d(gx_{n-1},gx_{n+1})\right\}. \end{aligned}$$

Since

$$\frac{d(gx_{n-1},gx_{n+1})}{2s} \le \frac{d(gx_{n-1},gx_n)+d(gx_n,gx_{n+1})}{2} \le \max\bigl\{d(gx_{n-1},gx_n), d(gx_n,gx_{n+1})\bigr\},$$

then we get

$$\begin{aligned} &N(x_{n-1},x_n) = 0, \\ &M(x_{n-1},x_n) = \max\bigl\{d(gx_{n-1},gx_n), d(gx_n,gx_{n+1})\bigr\}. \end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we have

$$\psi\bigl(d(gx_n,gx_{n+1})\bigr) \le \varphi\bigl(\max\bigl\{d(gx_{n-1},gx_n), d(gx_n,gx_{n+1})\bigr\}\bigr). \tag{2.6}$$

If for some $n \in \mathbb{N}$, $\max\{d(gx_{n-1},gx_n), d(gx_n,gx_{n+1})\} = d(gx_n,gx_{n+1})$, then by (2.6) and using the properties of the function φ, we get

$$\begin{aligned} \psi\bigl(d(gx_n,gx_{n+1})\bigr) &\le \varphi\bigl(\max\bigl\{d(gx_{n-1},gx_n), d(gx_n,gx_{n+1})\bigr\}\bigr) = \varphi\bigl(d(gx_n,gx_{n+1})\bigr) \\ &< \psi\bigl(d(gx_n,gx_{n+1})\bigr), \end{aligned}$$

which is a contradiction. So

$$\psi\bigl(d(gx_n,gx_{n+1})\bigr) \le \varphi\bigl(d(gx_{n-1},gx_n)\bigr) < \psi\bigl(d(gx_{n-1},gx_n)\bigr) \quad \text{for each } n \in \mathbb{N}. \tag{2.7}$$

From (2.7), we deduce that $\{\psi(d(gx_n,gx_{n+1}))\}$ is a nonnegative nonincreasing sequence. Since ψ is increasing, the sequence $\{d(gx_n,gx_{n+1})\}$ is nonincreasing, and consequently there exists $\delta \ge 0$ such that

$$\lim_{n\to\infty} d(gx_n,gx_{n+1}) = \delta.$$

We claim that $\delta = 0$. On the contrary, assume that

$$\lim_{n\to\infty} d(gx_n,gx_{n+1}) = \delta > 0. \tag{2.8}$$

Since ψ and φ are continuous, then from (2.7) and (2.8) we have

$$\psi(\delta) = \lim_{n\to\infty} \psi\bigl(d(gx_n,gx_{n+1})\bigr) = \lim_{n\to\infty} \varphi\bigl(d(gx_n,gx_{n+1})\bigr) = \varphi(\delta),$$

and so $\delta = 0$, that is a contradiction. Thus

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{2.9}$$

Now, we claim that

$$\lim_{n,m\to\infty} d(gx_n, gx_m) = 0. \tag{2.10}$$

Assume, on the contrary, that there exist $\epsilon > 0$ and subsequences $\{gx_{m(k)}\}$, $\{gx_{n(k)}\}$ of $\{gx_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(gx_{m(k)}, gx_{n(k)}) \geq \epsilon. \tag{2.11}$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.11) and $n(k) > m(k) \geq k$. Thus,

$$d(gx_{m(k)}, gx_{n(k)-1}) < \epsilon. \tag{2.12}$$

Using the triangle inequality in a b-metric space and (2.11) and (2.12), we obtain that

$$\begin{aligned}\epsilon &\leq d(gx_{n(k)}, gx_{m(k)}) \leq sd(gx_{n(k)}, gx_{n(k)-1}) + sd(gx_{n(k)-1}, gx_{m(k)}) \\ &< sd(gx_{n(k)}, gx_{n(k)-1}) + s\epsilon.\end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (2.9), we obtain

$$\epsilon \leq \limsup_{k\to\infty} d(gx_{m(k)}, gx_{n(k)}) \leq s\epsilon. \tag{2.13}$$

Also

$$\begin{aligned}\epsilon &\leq d(gx_{m(k)}, gx_{n(k)}) \leq sd(gx_{m(k)}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)}) \\ &\leq s^2 d(gx_{m(k)}, gx_{n(k)}) + s^2 d(gx_{n(k)}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)}) \\ &\leq s^2 d(gx_{m(k)}, gx_{n(k)}) + (s^2 + s) d(gx_{n(k)}, gx_{n(k)+1}).\end{aligned}$$

So, from (2.9) and (2.13), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(gx_{m(k)}, gx_{n(k)+1}) \leq s^2\epsilon. \tag{2.14}$$

Also

$$\begin{aligned}\epsilon &\leq d(gx_{n(k)}, gx_{m(k)}) \leq sd(gx_{n(k)}, gx_{m(k)+1}) + sd(gx_{m(k)+1}, gx_{m(k)}) \\ &\leq s^2 d(gx_{n(k)}, gx_{m(k)}) + s^2 d(gx_{m(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)}) \\ &\leq s^2 d(gx_{n(k)}, gx_{m(k)}) + (s^2 + s) d(gx_{m(k)}, gx_{m(k)+1}).\end{aligned}$$

So from (2.9) and (2.13), we get

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(gx_{n(k)}, gx_{m(k)+1}) \leq s^2\epsilon. \tag{2.15}$$

Also

$$d(gx_{m(k)+1}, gx_{n(k)}) \leq sd(gx_{m(k)+1}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)}),$$

so from (2.9) and (2.15), we have

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(gx_{n(k)+1}, gx_{m(k)+1}). \tag{2.16}$$

Taking (2.9), (2.13), (2.14) and (2.15) into account, we get

$$\begin{aligned}
&\limsup_{k\to\infty} M(x_{n(k)}, x_{m(k)}) \\
&\quad = \max\Bigg\{\limsup_{k\to\infty} d(gx_{n(k)}, gx_{m(k)}), \limsup_{k\to\infty} d(gx_{n(k)}, gx_{n(k)+1}), \limsup_{k\to\infty} d(gx_{m(k)}, gx_{m(k)+1}), \\
&\qquad \frac{\limsup_{k\to\infty} d(gx_{n(k)}, gx_{m(k)+1}) + \limsup_{k\to\infty} d(gx_{m(k)}, gx_{n(k)+1})}{2s}\Bigg\} \\
&\quad \leq \max\left\{s\epsilon, 0, 0, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.
\end{aligned}$$

So,

$$\limsup_{k\to\infty} M(x_{m(k)}, x_{n(k)}) \leq \epsilon s. \tag{2.17}$$

Similarly, we have

$$\limsup_{k\to\infty} N(x_{m(k)}, x_{n(k)}) = 0. \tag{2.18}$$

Now, using inequality (2.1) and Lemma 2.1, we have

$$\begin{aligned}
\psi(s\epsilon) &= \psi\left(s^3 \cdot \frac{\epsilon}{s^2}\right) \leq \psi\left(s^3 \limsup_{k\to\infty} d(gx_{m(k)+1}, gx_{n(k)+1})\right) \\
&= \limsup_{k\to\infty} \psi\left(s^3 d(gx_{m(k)+1}, gx_{n(k)+1})\right) = \limsup_{k\to\infty} \psi\left(s^3 d(Tx_{m(k)}, Tx_{n(k)})\right) \\
&\leq \limsup_{k\to\infty} \alpha(gx_{m(k)}, gx_{n(k)}) \psi\left(s^3 d(Tx_{m(k)}, Tx_{n(k)})\right) \\
&\leq \limsup_{k\to\infty} \left[\varphi\left(M(x_{m(k)}, x_{n(k)})\right) + L\theta\left(N(x_{m(k)}, x_{n(k)})\right)\right] \\
&= \varphi\left(\limsup_{k\to\infty} M(x_{m(k)}, x_{n(k)})\right) + L\theta\left(\limsup_{k\to\infty} N(x_{m(k)}, x_{n(k)})\right) \\
&\leq \varphi(s\epsilon) \\
&< \psi(s\epsilon),
\end{aligned}$$

which is a contradiction. So, we conclude that $\{gx_n\}$ is a Cauchy sequence in (X,d). By virtue of (2.2) we get $\{Tx_n\} = \{gx_{n+1}\} \subseteq gX$ and gX is closed, there exists $x \in X$ such that

$$\lim_{n\to\infty} gx_n = gx. \tag{2.19}$$

Now, we claim that x is a coincidence point of T and g. On the contrary, assume that $d(Tx,gx) > 0$. Since X is α-regular with respect to g and (2.19), we have

$$\alpha(gx_{n(k)+1},gx) \geq 1 \quad \text{for all } k \in \mathbb{N}. \tag{2.20}$$

Also by the use of triangle inequality in a b-metric space, we have

$$\begin{aligned} d(gx,Tx) &\leq sd(gx,gx_{n(k)+1}) + sd(gx_{n(k)+1},Tx) \\ &= sd(gx,gx_{n(k)+1}) + sd(Tx_{n(k)},Tx). \end{aligned}$$

In the above inequality, if k tends to infinity, then we have

$$d(gx,Tx) \leq \lim_{k\to\infty} sd(Tx_{n(k)},Tx). \tag{2.21}$$

By property of ψ, (2.20) and (2.21), we have

$$\begin{aligned} \psi\bigl(s^2 d(gx,Tx)\bigr) &\leq \lim_{k\to\infty} \psi\bigl(s^3 d(Tx_{n(k)},Tx)\bigr) \leq \lim_{k\to\infty} \alpha(gx_{n(k)+1},gx)\psi\bigl(s^3 d(Tx_{n(k)},Tx)\bigr) \\ &\leq \lim_{k\to\infty} \bigl[\varphi\bigl(M(x_{n(k)},x)\bigr) + L\theta\bigl(N(x_{n(k)},x)\bigr)\bigr] \\ &= \varphi\Bigl(\lim_{k\to\infty} M(x_{n(k)},x)\Bigr) + L\theta\Bigl(\lim_{k\to\infty} N(x_{n(k)},x)\Bigr) \\ &= \varphi\bigl(d(gx,Tx)\bigr) \\ &< \psi\bigl(d(gx,Tx)\bigr), \end{aligned}$$

which is a contradiction. Indeed,

$$\begin{aligned} &M(x_{n(k)},x) \\ &\quad = \max\biggl\{d(gx_{n(k)},gx), d(gx_{n(k)},Tx_{n(k)}), d(gx,Tx), \frac{d(gx_{n(k)},Tx) + d(gx,Tx_{n(k)})}{2s}\biggr\} \\ &\quad = \max\biggl\{d(gx_{n(k)},gx), d(gx_{n(k)},gx_{n(k)+1}), d(gx,Tx), \frac{d(gx_{n(k)},Tx) + d(gx,gx_{n(k)+1})}{2s}\biggr\} \end{aligned}$$

and

$$\begin{aligned} N(x_{n(k)},x) &= \min\bigl\{d(gx_{n(k)},Tx_{n(k)}), d(gx_{n(k)},Tx), d(gx,Tx_{n(k)})\bigr\} \\ &= \min\bigl\{d(gx_{n(k)},gx_{n(k)+1}), d(gx_{n(k)},Tx), d(gx,gx_{n(k)+1})\bigr\}. \end{aligned}$$

When n tends to infinity, we deduce

$$\lim_{k\to\infty} M(x_{n(k)},x) = \max\biggl\{d(gx,Tx), \frac{d(gx,Tx)}{2}\biggr\} = d(gx,Tx)$$

and

$$\lim_{k\to\infty} N(x_{n(k)}, x) = 0.$$

Hence, $d(gx, Tx) = 0$, that is, $gx = Tx$ and x is a coincidence point of T and g. We claim that if $Tu = gu$ and $Tv = gv$, then $gu = gv$. By hypotheses, $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$. Suppose that $\alpha(u, v) \geq 1$, then

$$\psi\left(s^3 d(gu, gv)\right) = \psi\left(s^3 d(Tu, Tv)\right) \leq \alpha(u, v)\psi\left(s^3 d(Tu, Tv)\right) \leq \varphi\left(M(u, v)\right) + L\theta\left(N(u, v)\right),$$

where

$$\begin{aligned} M(u, v) &= \max\left\{d(gu, gv), d(gu, Tu), d(gv, Tv), \frac{d(gu, Tv) + d(gv, Tu)}{2s}\right\} \\ &= \max\left\{d(gu, gv), d(gu, gu), d(gv, gv), \frac{d(gu, gv) + d(gv, gu)}{2s}\right\} \\ &= d(gu, gv) \end{aligned}$$

and

$$N(u, v) = \min\left\{d(gu, Tu), d(gu, Tv), d(gv, Tu)\right\} = \min\left\{d(gu, gu), d(gu, gv), d(gv, gu)\right\} = 0.$$

So,

$$\psi\left(s^3 d(gu, gv)\right) \leq \varphi\left(d(gu, gv)\right) < \psi\left(d(gu, gv)\right),$$

which is a contradiction. Thus we deduce that $gu = gv$. Similarly, if $\alpha(v, u) \geq 1$, we can prove that $gu = gv$. Now, we show that T and g have a common fixed point. Indeed, if $w = Tu = gu$, owing to the weak compatibility of T and g, we get $Tw = T(gu) = g(Tu) = gw$. Thus w is a coincidence point of T and g, then $gu = gw = w = Tw$. Therefore, w is a common fixed point of T and g. The uniqueness of the common fixed point of T and g is a consequence of conditions (2.1) and (b), and so we omit the details. □

Example 2.1 Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |x(t)|\,dt < \infty$. Define $D : X \times X \to [0, \infty)$ by

$$D(x, y) = \left(\int_0^1 \left|x(t) - y(t)\right|\,dt\right)^2.$$

Then D is a b-metric on X, with $s = 2$.

The operator $T : X \to X$ is defined by

$$Tx(t) = \frac{\sqrt{2}}{4}\ln\left(\left|x(t)\right| + 1\right), \tag{2.22}$$

and the operator $g : X \to X$ is defined by

$$gx(t) = e^{\frac{4}{\sqrt{2}}|x(t)|} - 1. \tag{2.23}$$

Now, we prove that T and g have a unique common fixed point. For all $x, y \in X$, we have

$$\begin{aligned}
\sqrt{2^3 D(Tx,Ty)} &= \sqrt{2^3\left(\int_0^1 \left|Tx(t)-Ty(t)\right| dt\right)^2} \\
&\le 2\sqrt{2}\int_0^1 \left|\frac{\sqrt{2}}{4}\ln\left(\left|x(t)\right|+1\right)-\frac{\sqrt{2}}{4}\ln\left(\left|y(t)\right|+1\right)\right| dt \\
&\le \int_0^1 \left|\left(\ln\left(\left|x(t)\right|+1\right)-\ln\left(\left|y(t)\right|+1\right)\right)\right| dt \\
&\le \int_0^1 \ln\left(\frac{|x(t)|+1}{|y(t)|+1}\right) dt \\
&\le \int_0^1 \ln\left(1+\frac{|x(t)-y(t)|}{|y(t)|+1}\right) dt \\
&\le \ln\left(1+\int_0^1 \left|x(t)-y(t)\right| dt\right) \\
&\le \ln\left(1+\int_0^1 \left|e^{\frac{4}{\sqrt{2}}|x(t)|}-e^{\frac{4}{\sqrt{2}}|y(t)|}\right| dt\right) \\
&\le \ln\left(1+\sqrt{\left(\int_0^1 \left|e^{\frac{4}{\sqrt{2}}|x(t)|}-e^{\frac{4}{\sqrt{2}}|y(t)|}\right| dt\right)^2}\right) \\
&\le \ln\left(1+\sqrt{D(gx,gy)}\right).
\end{aligned}$$

Now, if we define $\varphi(t) = \ln(1+\sqrt{t})$, $\psi(t) = \sqrt{t}$, $\alpha(x,y) = 1$ and $x_0 = 0$. Thus, by using Theorem 2.2, we obtain that T and g have a unique common fixed point.

From Theorem 2.2, if we choose $g = I_X$ the identity mapping on X, we deduce the following corollary.

Corollary 2.3 *Let (X,d) be a complete b-metric space and $T : X \to X$ be a self-mapping on X. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and some $L \ge 0$ such that for all $x, y \in X$,*

$$\alpha(x,y)\psi\left(s^3 d(Tx,Ty)\right) \le \varphi\left(M(x,y)\right) + L\theta\left(N(x,y)\right),$$

where $\psi, \varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions with $\varphi(t) < \psi(t)$, $\theta(t) > 0$ for each $t > 0$, $\varphi(0) = \psi(0) = \theta(0) = 0$, ψ is increasing,

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}$$

and

$$N(x,y) = \min\left\{d(x,Tx), d(x,Ty), d(y,Tx)\right\}.$$

Suppose also that the following conditions hold:

(i) *T is α-admissible and triangular α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$*;

(iii) *X is α-regular;*
(iv) *either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$.*
Then T has a unique fixed point.

Example 2.2 Let X be the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |x(t)|\,dt < \infty$. Define $D : X \times X \to [0,\infty)$ by

$$D(x,y) = \left(\int_0^1 \bigl|x(t) - y(t)\bigr|\,dt\right)^2.$$

Then D is a b-metric on X, with $s = 2$.
The operator $T : X \to X$ defined by

$$Tx(t) = \frac{\sqrt{2}}{4} \ln\bigl(\bigl|x(t)\bigr| + 1\bigr). \tag{2.24}$$

Now, we prove that T has a unique fixed point. For all $x, y \in X$, we have

$$\begin{aligned}
\sqrt{2^3 D(Tx,Ty)} &= \sqrt{2^3\left(\int_0^1 \bigl|Tx(t) - Ty(t)\bigr|\,dt\right)^2} \\
&\leq 2\sqrt{2}\int_0^1 \left|\frac{\sqrt{2}}{4}\ln\bigl(\bigl|x(t)\bigr| + 1\bigr) - \frac{\sqrt{2}}{4}\ln\bigl(\bigl|y(t)\bigr| + 1\bigr)\right| dt \\
&\leq \int_0^1 \bigl|\bigl(\ln\bigl(\bigl|x(t)\bigr| + 1\bigr) - \ln\bigl(\bigl|y(t)\bigr| + 1\bigr)\bigr)\bigr|\,dt \\
&\leq \int_0^1 \ln\left(\frac{|x(t)| + 1}{|y(t)| + 1}\right) dt \\
&\leq \int_0^1 \ln\left(1 + \frac{|x(t) - y(t)|}{|y(t)| + 1}\right) dt \\
&\leq \ln\left(1 + \int_0^1 \bigl|x(t) - y(t)\bigr|\,dt\right) \\
&\leq \ln\left(1 + \sqrt{\left(\int_0^1 \bigl|x(t) - y(t)\bigr|\,dt\right)^2}\right) \\
&\leq \ln\bigl(1 + \sqrt{D(x,y)}\bigr).
\end{aligned}$$

Now, if we define $\varphi(t) = \ln(1 + \sqrt{t})$, $\psi(t) = \sqrt{t}$, $\alpha(x,y) = 1$ and $x_0 = 0$. Thus, by Corollary 2.3 we obtain that T has a unique fixed point.

From Theorem 2.2, if the function $\alpha : X \times X \to \mathbb{R}$ is such that $\alpha(x,y) = 1$ for all $x, y \in X$, we deduce the following corollary.

Corollary 2.4 *Let (X,d) be a complete b-metric space, $T, g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed and there exists $L \geq 0$ such that for all $x, y \in X$,*

$$\psi\bigl(s^3 d(Tx,Ty)\bigr) \leq \varphi\bigl(M(x,y)\bigr) + L\theta\bigl(N(x,y)\bigr),$$

where $\psi, \varphi, \theta : [0, \infty) \to [0, \infty)$ *are continuous functions with* $\varphi(t) < \psi(t)$, $\theta(t) > 0$ *for each* $t > 0$, $\varphi(0) = \psi(0) = \theta(0) = 0$, ψ *is increasing,*

$$M(x,y) = \max\left\{d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{d(gx,Ty) + d(gy,Tx)}{2s}\right\}$$

and

$$N(x,y) = \min\{d(gx,Tx), d(gx,Ty), d(gy,Tx)\}.$$

Then T and g have a coincidence point. Moreover, if T and g are weakly compatible, then T and g have a unique common fixed point.

From Theorem 2.2, if $\psi(t) = \psi_1(t)$ and $\varphi(t) = \psi_1(t) - \varphi_1(t)$ for each $t \in \mathbb{R}_+$, where $\psi_1, \varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions such that $\psi_1(t) > \varphi_1(t) > 0$ for $t > 0$, $\psi_1(0) = \varphi_1(0) = 0$ and ψ_1 is increasing, we deduce the following corollary.

Corollary 2.5 *Let* (X,d) *be a complete b-metric space,* $T, g : X \to X$ *be such that* $TX \subseteq gX$. *Assume that gX is closed and there exist* $\alpha : X \times X \to \mathbb{R}$ *and* $L \geq 0$ *such that for all* $x, y \in X$,

$$\alpha(x,y)\psi_1\bigl(s^3 d(Tx,Ty)\bigr) \leq \psi_1\bigl(M(x,y)\bigr) - \varphi_1\bigl(M(x,y)\bigr) + L\theta\bigl(N(x,y)\bigr),$$

where $\psi_1, \varphi_1, \theta : \mathbb{R}_+ \to \mathbb{R}_+$ *are continuous functions such that* $\psi_1(t) > \varphi_1(t) > 0$ *for* $t > 0$, $\psi_1(t) = \varphi_1(t) = \theta(t) = 0$ *if and only if* $t = 0$ *and* ψ_1 *is increasing,*

$$M(x,y) = \max\left\{d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{d(gx,Ty) + d(gy,Tx)}{2s}\right\}$$

and

$$N(x,y) = \min\{d(gx,Tx), d(gx,Ty), d(gy,Tx)\}.$$

Assume also that the following conditions hold:

(i) *T is g-*α*-admissible and triangular* α*-admissible*;
(ii) *there exists* $x_0 \in X$ *such that* $\alpha(gx_0, Tx_0) \geq 1$;
(iii) *X is* α*-regular with respect to g.*

Then T and g have a coincidence point.

Moreover, the following conditions hold:

(a) *the pair* $\{T, g\}$ *is weakly compatible*;
(b) *either* $\alpha(u,v) \geq 1$ *or* $\alpha(v,u) \geq 1$ *whenever* $Tu = gu$ *and* $Tv = gv$.

Then T and g have a unique common fixed point.

From Corollary 2.5, if we choose $L = 0$ and $g = I_X$ the identity mapping on X, we deduce the following corollary.

Corollary 2.6 *Let* (X,d) *be a complete b-metric space,* $T : X \to X$ *be a self-mapping on X and* $\alpha : X \times X \to \mathbb{R}$. *Assume that the following condition holds*:

$$\alpha(x,y)\psi_1\bigl(s^3 d(Tx,Ty)\bigr) \leq \psi_1\bigl(M(x,y)\bigr) - \varphi_1\bigl(M(x,y)\bigr)$$

for all $x, y \in X$, where $\psi_1, \varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions such that $\psi_1(t) > \varphi_1(t) > 0$ for $t > 0$, $\psi_1(0) = \varphi_1(0) = 0$ and ψ_1 is increasing and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s}\right\}.$$

Assume also that the following conditions hold:

(i) *T is α-admissible and triangular α-admissible*;

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;

(iii) *X is α-regular*;

(iv) *either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$.*

Then T has a unique fixed point.

From Corollary 2.5, if the function $\alpha : X \times X \to \mathbb{R}$ is such that $\alpha(x,y) = 1$ for all $x, y \in X$, we deduce the following corollary.

Corollary 2.7 *Let (X,d) be a complete b-metric space, $T, g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed and there exists $L \geq 0$ such that for all $x, y \in X$,*

$$\psi_1\big(s^3 d(Tx,Ty)\big) \leq \psi_1\big(M(x,y)\big) - \varphi_1\big(M(x,y)\big) + L\theta\big(N(x,y)\big),$$

where $\psi_1, \varphi_1, \theta : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions such that $\psi_1(t) > \varphi_1(t) > 0$ for $t > 0$ and $\psi_1(t) = \varphi_1(t) = \theta(t) = 0$ if and only if $t = 0$ and ψ_1 is increasing,

$$M(x,y) = \max\left\{d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{d(gx,Ty) + d(gy,Tx)}{2s}\right\}$$

and

$$N(x,y) = \min\big\{d(gx,Tx), d(gx,Ty), d(gy,Tx)\big\}.$$

Then T and g have a coincidence point. Moreover, if T and g are weakly compatible, then T and g have a unique common fixed point.

From Corollary 2.7, if $\psi_1(t) = t$, $g = I$ and $L = 0$, we deduce the following corollary.

Corollary 2.8 *Let (X,d) be a complete b-metric space, $T : X \to X$ be a self-mapping on X. Assume that the following condition holds*:

$$s^3 d(Tx,Ty) \leq M(x,y) - \varphi_1\big(M(x,y)\big) \tag{2.25}$$

for all $x, y \in X$, where $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\varphi_1(t) < t$ for $t > 0$, $\varphi_1(0) = 0$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s}\right\}.$$

Then T has a unique fixed point.

From Theorem 2.2, if $\psi(t) = t$ and $\varphi(t) = \beta(t)t$ which $\beta \in \mathcal{F}$ ($\mathcal{F}$ defined in [28]), we deduce the following corollary.

Corollary 2.9 *Let (X,d) be a complete b-metric space, $T,g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed and there exist $\alpha : X \times X \to \mathbb{R}$ and $L \geq 0$ such that for all $x,y \in X$,*

$$\alpha(x,y)s^3 d(Tx,Ty) \leq \beta\bigl(M(x,y)\bigr)M(x,y) + L\theta\bigl(N(x,y)\bigr),$$

where $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\theta(t) = 0$ if and only if $t = 0$, $\beta \in S$ and

$$M(x,y) = \max\left\{d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{d(gx,Ty) + d(gy,Tx)}{2s}\right\}$$

and

$$N(x,y) = \min\bigl\{d(gx,Tx), d(gx,Ty), d(gy,Tx)\bigr\}.$$

Assume also that the following conditions hold:

(i) *T is g-α-admissible and triangular α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$*;
(iii) *X is α-regular with respect to g.*

Then T and g have a coincidence point.

Moreover, if the following conditions hold:

(a) *the pair $\{T,g\}$ is weakly compatible*;
(b) *either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.*

Then T and g have a unique common fixed point.

From Corollary 2.9, if the function $\alpha : X \times X \to \mathbb{R}$ is such that $\alpha(x,y) = 1$ for all $x,y \in X$, we deduce the following corollary.

Corollary 2.10 *Let (X,d) be a complete b-metric space, $T,g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed and that the following conditions hold*:

$$s^3 d(Tx,Ty) \leq \beta\bigl(M(x,y)\bigr)M(x,y)$$

for all $x,y \in X$, $\beta \in S$ and

$$M(x,y) = \max\left\{d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{d(gx,Ty) + d(gy,Tx)}{2s}\right\}.$$

Then T and g have a coincidence point. Moreover, if T and g are weakly compatible, then T and g have a unique common fixed point.

Remark 2.1 Since a b-metric space is a metric space when $s = 1$, so our results can be viewed as the generalization and the extension of several comparable results.

3 Application to integral equations

Here, in this section, we wish to study the existence of a unique solution for a nonlinear quadratic integral equation, as an application of our fixed point theorem. Consider the nonlinear quadratic integral equation

$$x(t) = h(t) + \lambda \int_0^1 k(t,s) f\bigl(s, x(s)\bigr)\, ds, \quad t \in I = [0,1], \lambda \geq 0. \tag{3.1}$$

Let Γ denote the class of those functions $\gamma : [0,+\infty) \to [0,+\infty)$ which satisfy the following conditions:

(i) γ is nondecreasing and $(\gamma(t))^p \leq \gamma(t^p)$ for all $p \geq 1$.

(ii) There exists $\varphi : [0,+\infty) \to [0,+\infty)$ which is a continuous function and $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$ such that $\gamma(t) = t - \varphi(t)$ for all $t \in [0,+\infty)$.

For example, $\gamma_1(t) = kt$, where $0 \leq k < 1$ and $\gamma_2(t) = \frac{t}{t+1}$ are in Γ.

We will analyze Eq. (3.1) under the following assumptions:

(A_1) $h : I \to \mathbb{R}$ is a continuous function.

(A_2) $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(t,x) \geq 0$ and there exist constant $0 \leq L < 1$ and $\gamma \in \Gamma$ such that for all $x, y \in \mathbb{R}$,

$$\bigl|f(t,x) - f(t,y)\bigr| \leq L\gamma\bigl(|x-y|\bigr).$$

(A_3) $k : I \times I \to \mathbb{R}$ is continuous at $t \in I$ for every $s \in I$ and measurable at $s \in I$ for all $t \in I$ such that $k(t,x) \geq 0$ and $\int_0^1 k(t,s)\, ds \leq K$.

(A_4) $\lambda^p K^p L^p \leq \frac{1}{2^{3p-3}}$.

Also, consider the space $X = C(I)$ of continuous functions defined on $I = [0,1]$ with the standard metric given by

$$\rho(x,y) = \sup_{t \in I} \bigl|x(t) - y(t)\bigr| \quad \text{for } x, y \in C(I).$$

Now, for $p \geq 1$, we define

$$d(x,y) = \bigl(\rho(x,y)\bigr)^p = \Bigl(\sup_{t \in I} \bigl|x(t) - y(t)\bigr|\Bigr)^p = \sup_{t \in I} \bigl|x(t) - y(t)\bigr|^p \quad \text{for } x, y \in C(I).$$

It is easy to see that (X,d) is a complete b-metric space with $s = 2^{p-1}$ [3].

We formulate the main result of this section.

Theorem 3.1 *Under assumptions* (A_1)-(A_4), *Eq.* (3.1) *has a unique solution in* $C(I)$.

Proof We consider the operator $T : X \to X$ defined by

$$T(x)(t) = h(t) + \lambda \int_0^1 k(t,s) f\bigl(s, x(s)\bigr)\, ds \quad \text{for } t \in I.$$

By virtue of our assumptions, T is well defined (this means that if $x \in X$ then $Tx \in X$). Also, for $x, y \in X$, we have

$$\begin{aligned} \bigl|T(x)(t) - T(y)(t)\bigr| &= \left|h(t) + \lambda \int_0^1 k(t,s) f\bigl(s, x(s)\bigr)\, ds - h(t) - \lambda \int_0^1 k(t,s) f\bigl(s, y(s)\bigr)\, ds\right| \\ &\leq \lambda \int_0^1 k(t,s) \bigl|f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr|\, ds \\ &\leq \lambda \int_0^1 k(t,s) L\gamma\bigl(\bigl|x(s) - y(s)\bigr|\bigr)\, ds. \end{aligned}$$

Since the function γ is nondecreasing, we have

$$\gamma\left(\left|x(s)-y(s)\right|\right)\leq\gamma\left(\sup_{t\in I}\left|x(s)-y(s)\right|\right)=\gamma\left(\rho(x,y)\right),$$

hence

$$\left|T(x)(t)-T(y)(t)\right|\leq\lambda KL\gamma\left(\rho(x,y)\right).$$

Then we can obtain

$$\begin{aligned}d(Tx,Ty)&=\sup_{t\in I}\left|T(x)(t)-T(y)(t)\right|^{p}\\&\leq\left\{\lambda KL\gamma\left(\rho(x,y)\right)\right\}^{p}\\&\leq\lambda^{p}K^{p}L^{p}\gamma\left(d(x,y)\right)\\&\leq\lambda^{p}K^{p}L^{p}\gamma\left(M(x,y)\right)\\&\leq\lambda^{p}K^{p}L^{p}\left[M(x,y)-\varphi\left(M(x,y)\right)\right]\\&\leq\frac{1}{2^{3p-3}}\left[M(x,y)-\varphi\left(M(x,y)\right)\right].\end{aligned}$$

This proves that the operator T satisfies the contractive condition (2.25) appearing in Corollary 2.8. So Eq. (3.1) has a unique solution in $C(I)$ and the proof is complete. □

Example 3.1 Consider the following functional integral equation:

$$x(t)=\frac{t^{2}}{1+t^{4}}+\frac{1}{27}\int_{0}^{1}\frac{s\cos t}{18(1+t)}\frac{|x(s)|}{1+|x(s)|}\,ds \tag{3.2}$$

for $t\in[0,1]$. Observe that this equation is a special case of Eq. (3.1) with

$$\begin{aligned}h(t)&=\frac{t^{2}}{1+t^{4}},\\k(t,s)&=\frac{s}{1+t},\\f(t,x)&=\frac{\cos t}{18}\frac{|x|}{1+|x|}.\end{aligned}$$

Indeed, by using $\gamma(t)=\frac{1}{3}t$, we see that $\gamma\in\Phi$ and $(\gamma(t))^{p}=(\frac{1}{3}t)^{p}=\frac{1}{3^{p}}t^{p}\leq\frac{1}{3}t^{p}=\gamma(t^{p})$ for all $p\geq1$. Further, for arbitrarily fixed $x,y\in\mathbb{R}$ such that $x\geq y$ and for $t\in[0,1]$, we obtain

$$\begin{aligned}\left|f(t,x)-f(t,y)\right|&=\left|\frac{\cos t}{18}\frac{|x|}{1+|x|}-\frac{\cos t}{18}\frac{|y|}{1+|y|}\right|\\&\leq\frac{1}{18}|x-y|=\frac{1}{6}\gamma\left(|x-y|\right).\end{aligned}$$

Thus, the function f satisfies assumption (A_2) with $L=\frac{1}{6}$. It is also easily seen that h is a continuous function. Further, notice that the function k is continuous in $t\in I$ for every

$s \in I$ and measurable in $s \in I$ for all $t \in I$ and $k(t,s) \geq 0$. Moreover, we have

$$\int_0^1 k(t,s)\,ds = \int_0^1 \frac{s}{1+t}\,ds = \frac{1}{2[1+t]} \\ \leq \frac{1}{2} = K.$$

This shows that assumption (A_3) holds. Taking $L = \frac{1}{6}$, $K = \frac{1}{2}$ and $\lambda = \frac{1}{27}$, then inequality $L^p \lambda^p K^p \leq \frac{1}{2^{3p-3}}$ appearing in assumption (A_4) has the following form:

$$\frac{1}{6^p} \times \frac{1}{27^p} \times \frac{1}{2^p} \leq \frac{1}{2^{3p-3}}.$$

It is easily seen that each number $p \geq 1$ satisfies the above inequality. Consequently, all the conditions of Theorem 3.1 are satisfied. Hence the integral equation (3.2) has a unique solution in $C(I)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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