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On the capacity of MIMO correlated Nakagami-*m* fading channels using copula

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Abstract

In this paper, a novel approach is proposed based on the probability density function (PDF) concept to achieve the capacity of a correlated ergodic multi-input multi-output (MIMO) channel with Nakagami-*m* fading. In our proposed method, channel parameters are unknown, and they are initially estimated by using the PDF of the received samples in the receiving antennas. The copula theory is employed to estimate the parameters of the channel in the proposed PDF-based approach. By appealing to copula, the notion of PDF estimation is simplified in the computation technique when we are faced with correlated signals. Since we are working on a correlated channel, the copula concept results in a powerful estimation approach for the PDF of the signals in the receivers. Accurate PDF estimation leads to having a precise calculation for channel parameters. Hence, the new approach guarantees that the capacity of a correlated ergodic channel is predicted reliably. In the previous works, either the capacity of simple uncorrelated Nakagami-*m* channels is presented or an asymptotic formulation is suggested for a correlated Nakagami-*m* channel. However, our proposed method introduces an analytic expression for the capacity of the MIMO correlated Nakagami-*m* fading channel relying on copula. All the results in both channel parameter estimation and channel capacity prediction are validated with some simulations.

Keywords: MIMO correlated Nakagami-*m* fading channel; PDF estimation; Copula theory; Channel capacity

1 Introduction

The transmission over multi-input multi-output (MIMO) channels offers significant increases in data throughput and link range without additional bandwidth or increased transmit power and results in higher capacity [1,2]. It is often supposed that the channel state information (CSI) is perfectly known at the receiver. However, in the actual environment, the channel has to be estimated. Precise estimation of the channel parameters critically helps in obtaining an appropriate design for the communication systems. Since there are more channel parameters in MIMO channels, a more powerful approach is required for the estimation.

There is another idealized assumption about channel coefficients that are considered to be independent and identically distributed (i.i.d) [3,4]. However, the mentioned assumption is not practical, on the other hand, in many practical situations, there exists a correlation among the antennas. This is due to poor scattering conditions or physical vicinity among the antennas [5]. Thus, the investigation about the behavior of MIMO systems in correlated fading environments is of interest [6].

Since the Rayleigh model is a reasonable assumption for the fading in many wireless communication systems, it is often supposed that the MIMO channel fading is Rayleigh distributed [6,7]. Nevertheless, the measurements [8] conclude that the Nakagami-*m* model presents a better fitting to the fading channel distribution. Achieving more similarity to the actual environment, the usual uniform probability density function (PDF) assumption for the phase of the Nakagami-*m* model is not adopted in this paper, and a more reliable PDF is considered [9].

In this paper, a 2×2 MIMO channel is considered in which the transmitting antennas are close and correlated, and the receiving antennas are far and independent [7,10]. Note that it is able to be generalized to arbitrary number of transmitters and receivers. The Nakagami-*m* model is also assumed for the fading environment. In addition to changing the amplitude of the transmitting signal due

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to the fading, it also results in a nonuniform phase shifting in the transmitting signal. A new PDF-based approach is applied to derive the Nakagami-m parameters related to two different paths from transmitters to two isolated receivers. This method is also capable of estimating the correlation parameter between the signals sent from two near transmitters.

Since we are faced with the correlated signals and our method is PDF-based, the powerful concept of copula effectively improves the proposed estimation method. The copula theory is suitable when two or several random variables are dependent. Thus, to calculate the total PDF of the received signal in the receivers, which includes some correlated parts, the copula theory helps us to attain a more precise PDF, and it results in having more reliable estimated parameters.

After estimation, the MIMO channel capacity is predicted relying on the estimated parameters. Since we have a correlated Nakagami-m channel, the copula theory is again employed to achieve the PDF of eigenvalues of the channel matrix, and by using the obtained PDF, the capacity is calculated.

The organization of the paper is in the following form: Section 2 includes some facts about the dependency problem and the role of copula theory in such problems. In Section 3, the MIMO channel that we are faced with is defined, and we discuss the correlation between the signals in the channel and the fading of the channel environment. The PDF of the received signal in each receiver is obtained based on the copula theory in Section 4, and the fading and correlation parameters are determined. The channel capacity is specified in Section 5 by using the PDF related to eigenvalues of the channel matrix. Some simulation results are presented in Section 6 to approve the proposed approach, and finally, some results are concluded in Section 7. For reader convenience, Table 1 provides a list of symbols and mathematical notations.

Table 1 Symbols and mathematical notations

Notation	Meaning
$(.)^T$	Transpose of matrix
$(.)^\dagger$	Complex conjugate transpose of matrix
$\|\mathbf{R}\|$	Determinant of matrix **R**
tr(.)	Trace operator
$\mathbb{E}$	Expectation value
C	Copula function
c	Copula density function
F	Cumulative distribution function
f	Probability density function
$\mathbf{I}$	Identity matrix
$\mathbf{H}$	Channel matrix
C_t	Channel capacity
ρ	Linear correlation parameter
α	Clayton copula parameter
ν	t copula parameter

2 Copula

One of the popular methods in modeling the dependencies is the copula approach. The copula was first employed by Sklar in mathematical and statistical problems [11]. Copula is a mathematical function that combines univariate PDFs to produce a joint PDF with a particular dependency structure. In this paper, the estimation of fading parameters is done by using the PDF of the received signal, given that the received signal is one of the MIMO system outputs including the sum of several correlated signals transmitted through the MIMO channel. Moreover, this signal is corrupted by an independent noise. Due to the correlated nature of the received signal, we are required to determine the PDF of a signal that is composed of several dependent components. Thus, the copula concept is a powerful tool that is suitable for our problem, and it facilitates the PDF estimation procedure. The fundamental theorem for copula was given by Sklar. Based on the Sklar theorem, for a given joint multivariate PDF and the relevant marginal PDFs, there exists a copula function that relates them. In a multivariate case, Sklar's theorem is as follows:

Let F be an n-dimensional cumulative distribution function (CDF) with margins $F_1, \dots, F_n$. Then there exists a function $C : [0,1]^n \to [0,1]$ such that:

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)). \tag{1}$$

Conversely, if C is a copula and $F_1, \dots, F_n$ are CDFs, then the function F defined by (1) is an n-dimensional CDF with margins $F_1, \dots, F_n$.

The proof of the theorem could be seen in [12]. Function C has some inherent properties, a description of which can be found thoroughly in [12]. Based on the copula properties, we can state that a copula is itself a CDF, defined on $[0,1]^n$, with uniform margins.

The construction of multivariate CDFs by employing the copula function provides a suitable flexibility, because we can select the margins and their dependence relationship separately [13]. For any copula function, there is a corresponding copula density function. To derive the copula density function, we firstly compute the joint PDF by taking the nth derivative of function C in (1) as:

$$f(x_1, \dots, x_n) = \frac{\partial^n C(F_1(x_1), \dots, F_n(x_n))}{\partial x_1 \cdots \partial x_n}. \tag{2}$$

By applying the chain rule to (2):

$$f(x_1,\dots,x_n) = \frac{\partial^n C(F_1(x_1),\dots,F_n(x_n))}{\partial F_1(x_1)\cdots\partial F_n(x_n)} \times \prod_{i=1}^{n} \frac{dF_i(x_i)}{dx_i} = c(F_1(x_1),\dots,F_n(x_n)) \prod_{i=1}^{n} f_i(x_i), \quad (3)$$

where $f_1(x_1),\dots,f_n(x_n)$ are the marginal PDFs and $c(.)$ is the copula density function. It is shown in (3) that a multivariate PDF is constructed by multiplying a copula density function and a set of marginal PDFs in which the copula density function can be selected independent of the margins.

The copulas are divided into two groups. The first one is the family of elliptical copulas. The most prominent elliptical copulas are normal and Student's *t*. We can specify different levels of dependency between the margins in an elliptical copula, and it is a suitable feature of this group. The second class of copulas is known as the Archimedean copulas. The ease with which they are constructed, the great variety of copulas that belong to this class, and modeling the dependence in arbitrarily high dimensions with only one parameter are the popular properties of this family [12].

In this paper, three kinds of copula, i.e., normal, Clayton, and *t* copula, are applied for the estimation. The mathematical relationships for the normal copula density function are presented in the following, and the relationships related to the other two copulas could be seen in Appendix A. The normal copula density function is given by:

$$c(x_1,\dots,x_n) = \frac{1}{|\mathbf{R}|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}u^T\left(\mathbf{R}^{-1} - I\right)u\right\}, \quad (4)$$

where $\mathbf{R}$ is the correlation matrix, u is the vector $u = \left[u_1,\dots,u_n\right]$ in which the ith element is $u_i = \Phi^{-1}(F_i(x_i))$ that Φ^{-1} is the inverse of the univariate standard normal CDF. $|.|$ and $(.)^T$ denote the determinant and transpose of the matrix, respectively.

It is called the normal copula because similar to normal distribution, it also enforces dependency by using pairwise correlations among the variables. However, in the normal copula, the marginal distributions are arbitrary. After discussing the copula concept and correlation modeling, a correlated channel is presented in the next section, and the parameters of the mentioned channel are estimated by using the copula function.

3 MIMO system model

A wireless MIMO channel model with N_t transmitting and N_r receiving antennas is described by:

$$\mathbf{q} = \mathbf{Hs} + \mathbf{n}, \quad (5)$$

where $\mathbf{H}$ is the $N_r \times N_t$ channel matrix with random entries $h_{k\ell}$ denoting the gain of the radio channel between the ℓth transmitting antenna and the kth receiving antenna. The vectors $\mathbf{s} \in \mathcal{C}^{N_t}$ and $\mathbf{q} \in \mathcal{C}^{N_r}$ are the transmitted and received signal vectors, respectively. The vector $\mathbf{n}$ is a complex N_r-dimensional noise vector whose elements are complex white Gaussian noise samples with zero-mean and variance σ_n^2, and $\mathbb{E}[\mathbf{n}\,\mathbf{n}^\dagger] = \sigma_n^2\mathbf{I}$, where † denotes the complex conjugate transpose, $\mathbf{I}$ is the identity matrix, and $\mathbb{E}$ denotes expectation. The entries of the channel matrix $\mathbf{H}$ are supposed to be signals with the following general form:

$$Z = R\exp(j\Theta), \quad (6)$$

where the envelope R and phase Θ are independent. Assume:

$$X \triangleq R\cos(\Theta) \quad \text{and} \quad Y \triangleq R\sin(\Theta). \quad (7)$$

Thus, X and Y are the in-phase and quadrature components of the signal Z. For integer m, also:

$$X^2 = \sum_{i=1}^{m} X_i^2, \quad \text{and} \quad Y^2 = \sum_{i=1}^{m} Y_i^2, \quad (8)$$

where X_i and Y_i are i.i.d. zero-mean Gaussian samples with variance $\Omega/2m$. Therefore, the PDF of R is Nakagami-m distribution:

$$f_R(r) = \frac{2m^m\, r^{2m-1}}{\Omega^m\Gamma(m)} \exp\left(-\frac{mr^2}{\Omega}\right), \quad (9)$$

and phase Θ has the following PDF [9]:

$$f_\Theta(\theta) = \frac{\Gamma(m)|\sin(2\theta)|^{m-1}}{2^m\Gamma^2(m/2)}. \quad (10)$$

$\Omega = \mathbb{E}[R^2]$ and $m = \frac{\Omega^2}{\mathbb{E}[(R^2-\Omega)^2]}$ are the scale and shape parameters, respectively, and $\Gamma(\cdot)$ is the gamma function. Based on the density functions in (9) and (10), the PDF of the in-phase and quadrature components of the signal Z are the same and given by [9]:

$$f_X(x) = \frac{m^{m/2}|x|^{m-1}}{\Omega^{m/2}\Gamma(m/2)} \exp\left(-\frac{mx^2}{\Omega}\right), \quad -\infty < x < \infty, \quad (11)$$

and

$$f_Y(y) = \frac{m^{m/2}|y|^{m-1}}{\Omega^{m/2}\Gamma(m/2)} \exp\left(-\frac{my^2}{\Omega}\right), \quad -\infty < y < \infty. \quad (12)$$

While the PDFs are derived for integer values of m, there are no mathematical constraints for (11) and (12) to be used for any $m > 0.5$. For uncorrelated MIMO channels, the entries of $\mathbf{H}$ are independent. However, there is generally no such ideal case in practice. Hence, a study of the correlation among these entries is of interest [14].

When the receiving antennas are correlated, the columns of **H** are independent random vectors, but there exists the correlation among the elements of each column. On the other hand, if the transmitting antennas are correlated, the rows of the channel matrix are independent and the elements of each row are correlated. In this paper, as depicted in Figure 1, the transmitting antennas are assumed to be adjacent and correlated, but, the receiving antennas are far enough from each other that they could be considered independent [7,10].

4 Channel parameter estimation

In this section, the PDF of the received signals in receiving antennas is employed in order to estimate the MIMO channel parameters. To transmit information over a MIMO channel, there are different methods to modulate the information. Quadrature amplitude modulation, phase shift keying, frequency shift keying, and continuous phase modulation are some prominent modulation methods. All these methods use a sinusoidal function as the carrier signal. Hence, we base signaling assumption on a sinusoidal transmission entering a multipath environment infested by noise. Thus, the procedure is extensible to all of the above types of modulation schemes. For simplicity, it is supposed that there are two transmitters and two receivers, i.e., $N_t = 2$ and $N_r = 2$. Assume that the transmitting antennas transmit a signal with the following form:

$$s_\ell(t) = A\cos(\omega_c t + \theta_\ell), \quad \ell = 1, 2, \tag{13}$$

where $A > 0$, ω_c and θ_ℓ are the amplitude, carrier frequency and the phase of the transmitted signal, respectively, and ℓ is the number of the transmitter. It is concluded that the fading effect turns the signal in (13) to the following signal [15]:

$$s'_\ell(t) = R(t)\, A\cos(\omega_c t + \Theta(t)). \tag{14}$$

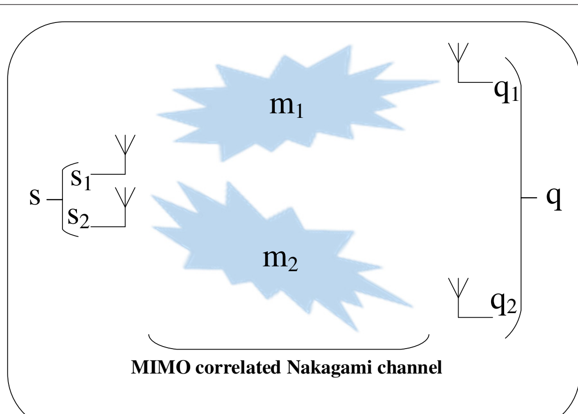

Figure 1 A 2 × 2 MIMO correlated Nakagami-*m* fading channel.

Since the Nakagami fading model is supposed for our MIMO environment, the random processes $R(t)$ and $\Theta(t)$ in (14) have the obtained PDFs in (9) and (10), respectively. Thus, the received signals in the two receivers are given by:

$$q_k(t) = \sum_{\ell=1}^{2} R_{k\ell}(t)\, A\cos(\omega_c t + \Theta_{k\ell}(t)) + n_k(t), \quad k = 1, 2, \tag{15}$$

where $n_k(t)$ is an independent zero-mean normally distributed random process, and ℓ and k are the numbers of the transmitter and the receiver, respectively. The envelope $R_{k\ell}(t)$ and phase $\Theta_{k\ell}(t)$ of Nakagami fading model include the shape parameter $m_{k\ell}$ and scale parameter $\Omega_{k\ell}$ that should be estimated.

The second-order moment of the Nakagami-m fading envelope is equal to the scale parameter. Thus, it is obtained simply. However, the estimation of the shape parameter is not straightforward and should be noticed more. We focus on estimating it, and we call it the fading parameter.

As shown in Figure 1, the adjacency of transmitting antennas leads to actually having similar fading parameters between the transmitters and a particular receiver:

$$m_{k1} = m_{k2} \triangleq m_k, \quad \textit{for} \quad k = 1, 2. \tag{16}$$

Therefore, the estimation of the channel parameters is reduced to obtaining the parameters m_k for $k = 1, 2$.

Since our parameter estimation is PDF-based, the PDF of the received signal $q_k(t)$ in (15) is required. For simplicity, the noise-free part of the received signal in (15) is defined by $q'_k(t)$, and therefore, the received signals could be given by:

$$q_k(t) = q'_k(t) + n_k(t), \quad k = 1, 2. \tag{17}$$

Assume that $f_{q'_k}\left(q'_k\right)$ and $f_{n_k}(n_k)$ are the PDF of random processes $q'_k(t)$ and $n_k(t)$ at time t, respectively. The independence of signals $q'_k(t)$ and noise $n_k(t)$ leads to having the PDF of the received signal $q_k(t)$ from the convolution of the PDFs $f_{q'_k}\left(q'_k\right)$ and $f_{n_k}(n_k)$. The PDF of the normal distributed noise, i.e., $f_{n_k}(n_k)$, is known. Thus, the problem is the calculation of the PDF $f_{q'_k}\left(q'_k\right)$.

Define:

$$q'_{k\ell}(t) \triangleq R_{k\ell}(t)\, A\cos(\omega_c t + \Theta_{k\ell}(t)), \quad \ell = 1, 2. \tag{18}$$

Using (15), (17), and (18), we have:

$$q'_k(t) = q'_{k1}(t) + q'_{k2}(t). \tag{19}$$

Both signals $q'_{k1}(t)$ and $q'_{k2}(t)$ have similar stochastic behavior; thus, they possess identical PDFs. Now, the PDF of the signal $q'_{k1}(t)$ is obtained. Define:

$$Q_1(t) \triangleq R_{k1}(t),$$
$$Q_2(t) \triangleq A\cos(\omega_c t + \Theta_{k1}(t)). \tag{20}$$

It is concluded in Appendix B that $f_{Q_2}(Q_2)$ is given by:

$$f_{Q_2}(Q_2) = \frac{2^{-m}\Gamma(m)}{A^{2m-2}\Gamma^2(m/2)\sqrt{A^2-Q_2^2}} \times \left(|\xi+\zeta|^{m-1} + |\xi-\zeta|^{m-1}\right), \tag{21}$$

where:

$$\xi \triangleq 2Q_2\sqrt{A^2-Q_2^2}\cos(2\omega_c t),$$
$$\zeta \triangleq \left(2Q_2^2 - A^2\right)\sin(2\omega_c t). \tag{22}$$

The envelope $R_{k1}(t)$ and phase $\Theta_{k1}(t)$ are independent processes [9]; therefore, $Q_1(t)$ and $Q_2(t)$ are also independent. Since the signal $q'_{k1}(t)$ is the product of two independent signals $Q_1(t)$ and $Q_2(t)$, we have:

$$f_{q'_{k1}}\left(q'_{k1}\right) = \int_{-\infty}^{\infty} \frac{1}{|\eta|} f_{Q_1Q_2}(\eta, q'_{k1}/\eta)\, d\eta. \tag{23}$$

Using (21) and (23), the PDF $f_{q'_{k1}}\left(q'_{k1}\right)$ is calculated in the following form:

$$f_{q'_{k1}}\left(q'_{k1}\right) = \frac{2^{1-m}}{A^{2m-2}\,\Gamma^2\left({}^{m}/_{2}\right)}\left(\frac{m}{\Omega}\right)^m \int\limits_{q'_{k1/A}}^{\infty} \frac{\eta}{\sqrt{A^2\eta^2 - q'^2_{k1}}} \times \exp\left(-\frac{m\eta^2}{\Omega}\right)\left(|\chi+\psi|^{m-1} + |\chi-\psi|^{m-1}\right) d\eta \tag{24}$$

where:

$$\chi \triangleq 2q'_{k1}\sqrt{A^2\eta^2 - q'^2_{k1}}\cos(2\omega_c t),$$
$$\psi \triangleq \left(2q'^2_{k1} - A^2\eta^2\right)\sin(2\omega_c t). \tag{25}$$

As a result, the PDF of signal $q'_{k1}(t)$ is obtained. Similar statistical behavior for the two signals $q'_{k1}(t)$ and $q'_{k2}(t)$ results in an identical PDF for the second signal.

Now, we calculate the PDF of signal $q'_k(t)$ from (19). If the signals $q'_{k1}(t)$ and $q'_{k2}(t)$ are assumed to be independent, the PDF of $q'_k(t)$, i.e., $f_{q'}\left(q'\right)$, would be obtained by using the convolution of the PDFs $f_{q'_{k1}}\left(q'_{k1}\right)$ and $f_{q'_{k2}}\left(q'_{k2}\right)$. However, since the transmitting antennas are assumed to be adjacent and correlated, it is more realistic to suppose that there is a dependency between the signals $q'_{k1}(t)$ and $q'_{k2}(t)$, and the convolution could not be employed. The copula theory is capable to help us in this calculation. Based on (19), the PDF $f_{q'_k}\left(q'_k\right)$ is given by:

$$f_{q'_k}\left(q'_k\right) = \int_{-\infty}^{\infty} f_{q'_{k1}q'_{k2}}\left(q' - q'_{k2}, q'_{k2}\right) dq'_{k2}. \tag{26}$$

Thus, we only require to estimate the joint PDF $f_{q'_{k1}q'_{k2}}$, because when this joint PDF is derived, the integral in (26) is simply obtained. If the PDFs of the signals $q'_{k1}(t)$ and $q'_{k2}(t)$ are considered as the marginal density functions in the copula theory, the joint PDF $f_{q'_{k1}q'_{k2}}$ is simply obtained from (3):

$$f_{q'_{k1}q'_{k2}}\left(q'_{k1}, q'_{k2}\right) = f_{q'_{k1}}\left(q'_{k1}\right) f_{q'_{k2}}\left(q'_{k2}\right) c\left(F_{q'_{k1}}\left(q'_{k1}\right), F_{q'_{k2}}\left(q'_{k2}\right); \rho_k\right). \tag{27}$$

$F_{q'_{k1}}\left(q'_{k1}\right)$ and $F_{q'_{k2}}\left(q'_{k2}\right)$ are the marginal CDFs of the signals $q'_{k1}(t)$ and $q'_{k2}(t)$, respectively, and ρ_k is the linear correlation parameter between these two signals. The linear correlation is a measure of dependency in this paper and is also called Pearson's correlation. Note that since the signals in two transmitters are produced independently, the linear correlation between $q'_{k1}(t)$ and $q'_{k2}(t)$ are the same as the linear correlation related to the channel. Thus, the estimation of this parameter leads to specifying the channel correlation parameter.

As previously mentioned, three kinds of copula, i.e., normal, Clayton, and *t* copula, are applied for the estimation. Note that the linear correlation parameter ρ_k in (27) is not exactly the copula parameter, and we should obtain the copula parameter from ρ_k based on the related copula.

For the normal copula, the entries of the correlation matrix R in (4) are normal copula parameters, and fortunately, these parameters are almost the same as linear correlation parameters that present pairwise correlations among the variables.

In *t* copula, there are two parameters, one of which is the degrees of freedom and is considered equal to 2 in our simulations. The other one is exactly the same as the normal copula parameter and therefore is identical to the linear correlation parameter.

The Clayton copula has a parameter α which is different from the linear correlation parameter, and the relationship between them for the bivariate case is given by:

$$\alpha = \frac{\sin^{-1}(\rho_k)}{\pi - 2\sin^{-1}(\rho_k)}. \tag{28}$$

For generalizing (28) to the multivariate case, one can calculate α for each pair separately and consider the average of all obtained α values as the main Clayton copula parameter.

Until now, the PDF $f_{q'_k}\left(q'_k\right)$ is estimated, and thus, the PDF $f_{q_k}\left(q_k\right)$ is obtained analytically by using (17). Thus, the PDF of the received signals in both receivers, $f_{q_1}\left(q_1\right)$ and $f_{q_2}\left(q_2\right)$, are at hand. Using the obtained analytic PDF of the received signal in the kth receiver, the parameters

m_k and ρ_k in the route between the transmitters and the kth receiver could be estimated as follows.

To achieve the parameters m_k and ρ_k, the nonlinear minimum mean square error (NMMSE) estimator is employed. In addition to analytic PDF, the NMMSE estimator also requires the statistical PDF of the received signal in the kth receiver, which is calculated based on the samples of the received signal in the following form:

$$\hat{f}_{q_k}\left(q_k\right) = \frac{1}{N_k h}\sum_{i=0}^{N_k-1}\Psi\left(\frac{q_k - q_{k_i}}{h}\right). \tag{29}$$

(29) is the kernel estimator which is noticed as an approach to estimate the PDF of an arbitrary signal statistically. Ψ is the kernel function that must integrate to 1, and h is the window width or bandwidth of the kernel. N_k is the number of the received samples in the kth receiver, and q_{k_i} is the value of the ith sample. Utilizing both analytical and statistical obtained PDFs, the NMMSE estimator presents channel parameters:

$$\left(\hat{m}_k, \hat{\rho}_k\right) = \arg\min_{m_k,\rho_k}\left|f_{q_k}\left(q_k\right) - \hat{f}_{q_k}\left(q_k\right)\right|^2. \tag{30}$$

In the next section, a novel method is expressed to calculate the capacity of the proposed MIMO channel based on the parameters estimated in (30).

5 Capacity analysis

Since our method in the previous section obtains the channel parameters, i.e., m_k, ρ_k, hence, it is plausible to assume that the CSI for $\mathbf{H}$ channel is perfectly known at the receiver in the absence of channel knowledge at the transmitter. On the other hand, assume that a total transmit power P is uniformly distributed among the N_t transmitting antennas. The instantaneous capacity for the ergodic channel $\mathbf{H}$ is given by [6]:

$$C_I = \log_2 \det\left(\mathbf{I}_{N_r} + \frac{P}{\sigma_n^2 N_t}\mathbf{H}\mathbf{H}^\dagger\right), \tag{31}$$

where $\frac{P}{\sigma_n^2}$ is the average signal-to-noise ratio (SNR) at each receiver branch. Since $\mathbf{H}$ is randomly varying, C_I is also randomly varying. Thus, the information theoretic capacity C_t should be calculated as:

$$C_t = \mathbb{E}\left[\log_2 \det\left(\mathbf{I}_{N_r} + \frac{P}{\sigma_n^2 N_t}\mathbf{H}\mathbf{H}^\dagger\right)\right]. \tag{32}$$

Based on (9) and (10), if the entries of $\mathbf{H}$ are independent, the joint distribution of $\mathbf{H}$ entries is computed by multiplying the PDFs of the entries:

$$f\left(\mathbf{H}\right) = C_1 \exp\left(-\frac{m\,\mathrm{tr}\left(\mathbf{H}\mathbf{H}^\dagger\right)}{\Omega}\right)\prod_{i,j=1}^{2}\left|\left|h_{ij}\right|^2 \sin\left(2\theta_{ij}\right)\right|^{(m-1)}, \tag{33}$$

where $C_1 = \left(\frac{m^m}{2^{m-1}\Omega^m\Gamma^2\left(\frac{m}{2}\right)}\right)^4$, tr(.) denotes the trace operator, and θ_{ij} is the phase of h_{ij}. It is supposed in (33) that all entries have the same m and Ω, but it is simply generalized to the situation in which we have different values for the parameters.

When the entries of channel matrix $\mathbf{H}$ are dependent, the copula theory helps us to extract the joint distribution of $\mathbf{H}$ by using (33). Only a new term, that is, a copula density function, is added to (33). Since our proposed MIMO correlated channel is based on Figure 1 in Section 3, matrix $\mathbf{H}$ is 2×2, and includes four entries h_{11}, h_{21}, h_{12}, and h_{22}, respectively, with a correlation matrix in the following form:

$$\mathbf{R} = \begin{bmatrix} 1 & \rho_1 & 0 & 0 \\ \rho_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho_2 \\ 0 & 0 & \rho_2 & 1 \end{bmatrix}. \tag{34}$$

For simplicity, let us assume $\rho_1 = \rho_2 = \rho$. Thus, the added copula density function should be constructed based on the correlation matrix. For instance, for the normal copula, the copula density function of (4) is employed, in which the correlation matrix is the same as the matrix $\mathbf{R}$ in (34), and is given by:

$$\begin{aligned} c\left(u_1,u_2,u_3,u_4\right) = \frac{1}{1-\rho^2}\exp\Bigg(\frac{-1}{2\left(1-\rho^2\right)} \times \Bigg[\rho^2\left(\sum_{i=1}^{4}\Phi^{-1}\left(u_i\right)\right)^2 \\ -2\rho\left(\Phi^{-1}(u_1)\,\Phi^{-1}\left(u_2\right) + \Phi^{-1}\left(u_3\right)\Phi^{-1}(u_4)\right)\Bigg]\Bigg), \end{aligned} \tag{35}$$

where:

$$\begin{aligned} u_1 = F_{h_{11}}\left(|h_{11}|\right), u_2 = F_{h_{21}}\left(|h_{21}|\right), \\ u_3 = F_{h_{12}}\left(|h_{12}|\right), u_4 = F_{h_{22}}\left(|h_{22}|\right), \end{aligned} \tag{36}$$

and F denotes CDF. When ρ_1 and ρ_2 are different, (35) is simply generalized. After extracting the PDF of matrix $\mathbf{H}$, the PDF of $\mathbf{H}\mathbf{H}^\dagger$ should be determined. To derive the matrix $\mathbf{H}\mathbf{H}^\dagger$, it is better that matrix $\mathbf{H}$ be decomposed into a product $\mathbf{H} = \mathbf{L}_H\mathbf{Q}_H$ by using LQ decomposition, where $\mathbf{L}_H$ is a complex lower triangular matrix with real positive diagonals in the following form:

$$\mathbf{L}_H = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix}, \tag{37}$$

where l_{11} and l_{22} are real and l_{21} is complex. $\mathbf{Q}_H$ is a complex orthogonal matrix that could be given by [16]:

$$\mathbf{Q}_H = \begin{pmatrix} e^{j\phi_1}\cos\left(\delta\right) & e^{j\phi_2}\sin\left(\delta\right) \\ -e^{j(\phi_3-\phi_2)}\sin\left(\delta\right) & e^{j(\phi_3-\phi_1)}\cos\left(\delta\right) \end{pmatrix}, \tag{38}$$

where $0 \le \phi_1, \phi_2, \phi_3 \le 2\pi$, and $0 \le \delta \le \pi/2$.

To derive the joint PDF $f\left(\mathbf{L}_H, \mathbf{Q}_H\right)$, the Jacobian of the transformation from $f\left(\mathbf{H}\right)$ to $f\left(\mathbf{L}_H, \mathbf{Q}_H\right)$ is required and is given by [17]:

$$J_1 = l_{11}^3 l_{22} \sin(\delta) \cos(\delta). \tag{39}$$

Thus:

$$\begin{aligned} f(\mathbf{L}_H, \mathbf{Q}_H) = {} & l_{11}^{4m-1} l_{22} \sin^{2m-1}(2\delta) \\ & \times \frac{C_1}{2^{2m-1}(1-\rho^2)} \exp\left(-\frac{m \operatorname{tr}\left(\mathbf{L}_H \mathbf{L}_H^\dagger\right)}{\Omega} - \frac{\text{Temp5}}{2(1-\rho^2)}\right) \\ & \times \left(l_{22}^2 \cos^2(\delta) + l_{21}^2 \sin^2(\delta) - \text{Temp6}\right)^{m-1} \\ & \times \left(l_{21}^2 \cos^2(\delta) + l_{22}^2 \sin^2(\delta) + \text{Temp6}\right)^{m-1} \\ & \times \prod_{i,j=1}^{2} \left|\sin(2\theta_{ij})\right|^{(m-1)}, \end{aligned} \tag{40}$$

where:

$$\begin{aligned} \text{Temp5} = {} & \rho^2 \left(\sum_{i=1}^{4} \Phi^{-1}(u_i)\right)^2 - 2\rho\left(\Phi^{-1}(u_1)\Phi^{-1}(u_2)\right. \\ & \left. + \Phi^{-1}(u_3)\Phi^{-1}(u_4)\right), \end{aligned} \tag{41}$$

and:

$$\begin{aligned} \text{Temp6} = {} & l_{22} \sin(2\delta) \times \left(\Im(l_{21}) \sin(\phi_1 + \phi_2 - \phi_3)\right. \\ & \left. - \Re(l_{21}) \cos(\phi_1 + \phi_2 - \phi_3)\right), \end{aligned} \tag{42}$$

where $\Re$ and $\Im$ denote the real part and the imaginary part, respectively. Note that the parameters u_is in (41) and θ_{ij}s in (40) should also be written in terms of l_{11}, l_{21}, l_{22}, ϕ_is, and δ. Therefore, the PDF $f(\mathbf{L}_H)$ is given as:

$$f(\mathbf{L}_H) = \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(\mathbf{L}_H, \mathbf{Q}_H)\, d\phi_1 d\phi_2 d\phi_3 d\delta. \tag{43}$$

Now, define:

$$\mathbf{W} \triangleq \mathbf{L}_H \mathbf{L}_H^\dagger. \tag{44}$$

The Jacobian of the transformation from $\mathbf{L}_H$ to $\mathbf{W}$ is $J_2 = 4 l_{11}^3 l_{22}$. Using J_2, the PDF of matrix $\mathbf{W}$ is simply obtained from $f(\mathbf{L}_H)$ in (43). The eigenvalue decomposition helps us to have:

$$\mathbf{W} = \Sigma \Lambda \Sigma^\dagger, \tag{45}$$

where the eigenvalue matrix Λ is defined in the following form:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \tag{46}$$

and the matrix Σ could be supposed in the following form:

$$\Sigma = \begin{bmatrix} \cos(\gamma) & -e^{j\mu} \sin(\gamma) \\ e^{-j\mu} \sin(\gamma) & \cos(\gamma) \end{bmatrix}, \tag{47}$$

where $0 \le \mu \le 2\pi$, and $0 \le \gamma \le \pi/2$.

The Jacobian of the transformation in (45) is calculated as:

$$J_3 = \frac{1}{2}(\lambda_1 - \lambda_2)^2 \sin(\gamma). \tag{48}$$

Using J_3, the PDF $f(\lambda_1, \lambda_2, \mu, \gamma)$ is obtained. By integrating over μ and γ, we obtain the PDF $f(\lambda_1, \lambda_2)$ as:

$$f(\lambda_1, \lambda_2) = \int_0^{\pi/2} \int_0^{2\pi} f(\lambda_1, \lambda_2, \mu, \gamma)\, d\mu d\gamma. \tag{49}$$

Now, we return to (32) for specifying the capacity. The capacity in (32) could be expressed in terms of the eigenvalues of positive definite matrix $\mathbf{W}$, i.e., λ_1 and λ_2:

$$\begin{aligned} C_t &= \mathbb{E}\left[\sum_{i=1}^{2} \log_2\left(1 + \frac{P}{\sigma_n^2 N_t}\lambda_i\right)\right] \\ &= 2\,\mathbb{E}\left[\log_2\left(1 + \frac{P}{\sigma_n^2 N_t}\lambda\right)\right], \end{aligned} \tag{50}$$

where λ is one of the eigenvalues λ_1 and λ_2 which is randomly selected uniformly. Thus, the capacity is given as:

$$C_t = 2\int_0^\infty \log_2\left(1 + \frac{P}{\sigma_n^2 N_t}\lambda\right) f(\lambda)\, d\lambda, \tag{51}$$

where:

$$f(\lambda) = \int_0^\infty f(\lambda_1, \lambda_2)\, d\lambda', \tag{52}$$

where λ' is one of the eigenvalues λ_1 and λ_2 which is not selected as λ. Since $f(\lambda_1, \lambda_2)$ is obtained in (49), the capacity is calculated.

Finally, the asymptotic case is also noticed. First, assume that the average SNR in (31), i.e., SNR = $\frac{P}{\sigma_n^2}$, tends to zero. Thus, (51) turns to:

$$C_t = 2\int_0^\infty \log_2(1+0) f(\lambda)\, d\lambda = 2\int_0^\infty 0 \times f(\lambda)\, d\lambda = 0. \tag{53}$$

On the other hand, when average SNR grows to infinity, (51) could be expressed as:

$$\begin{aligned} C_t &= 2\int_0^\infty \log_2\left(1 + \text{SNR}\,\frac{\lambda}{N_t}\right) f(\lambda)\, d\lambda \\ &= 2\int_0^\infty \log_2\left(\text{SNR}\,\frac{\lambda}{N_t}\right) f(\lambda)\, d\lambda \\ &= 2\int_0^\infty \left[\log_2(\text{SNR}) + \log_2(\lambda/N_t)\right] f(\lambda)\, d\lambda. \end{aligned} \tag{54}$$

In actual environment, there is a large M that the value of PDF $f(\lambda)$ is almost near to zero for the λs larger than that M. Thus, the upper bound of the integral in (54) could be decreased from infinity to M:

$$C_t = 2\int_0^M \left[\log_2(\text{SNR}) + \log_2(\lambda/N_t)\right] f(\lambda)\, d\lambda. \tag{55}$$

Therefore, since the parameter λ in (55) can not be greater than M and the parameter SNR tends to infinity, we have:

$$\log_2 (\mathrm{SNR}) >> \log_2 (\lambda/N_t). \quad (56)$$

Thus, the last relationship in (54) could be given by:

$$C_t = 2\int_0^M \log_2 (\mathrm{SNR}) f(\lambda)\, d\lambda$$
$$= \log_2 (\mathrm{SNR}) \int_0^M 2f(\lambda)\, d\lambda = 2\log_2 (\mathrm{SNR}). \quad (57)$$

Thus, the capacity is obtained for asymptotic values of SNR.

Although we discussed a MIMO system with two transmitters and two receivers, it is able to be generalized to arbitrary number of transmitters and receivers. For example, in a 3×3 MIMO channel, we should estimate two correlation parameters from the samples of each receiver in (30), and in capacity prediction, matrix $\mathbf{H}$ in (33) is 3×3, and the copula density function in (35) has nine variables. Note that the procedure is the same as the case 2×2 MIMO channel. However, more transmitters and receivers lead to complicated mathematical calculations that could be sometimes cumbersome. In the next section, there are some simulations to approve the results related to both parameter estimation and capacity calculation.

6 Simulation and result

It is essential to assess the proposed approach by employing some simulations. The simulations should cover both discussions, channel parameter estimation and channel capacity prediction. At first, the ability of the proposed algorithm in Nakagami-m and correlation parameter estimation is evaluated. Suppose we have a 2×2 MIMO channel as the communication system to transfer the cosine signal $2\cos(2\pi f_c t)$ with $f_c = 100$ MHz. Two adjacent antennas send this signal to two receiving antennas which are far from each other. This arrangement for transmitters and receivers leads to having two different environments from the transmitters to each one of the receivers (Figure 1). Thus, we suppose the path to the first receiver has a Nakagami behavior with parameter m_1, and the second one is affected by a Nakagami model with parameter m_2. On the other hand, since at each receiver, we have the sum of two signals from two near transmitters, these two signals are correlated. In the simulation, we suppose the correlation parameter between two signals in the first receiver is ρ_1 and in the second one is ρ_2.

Now, the fading and correlation parameters, i.e., m_1 and ρ_1, in the path to the first receiver are estimated based on the PDF of the received signal $q_1(t)$ in (17), and the results are depicted in Figures 2 and 3. In Figures 2 and 3, the actual values of m_1 and ρ_1 are 4 and 0.5, respectively. The simulation is done for the sample size $N = 10,000$ and SNR values from 0 to +10 dB.

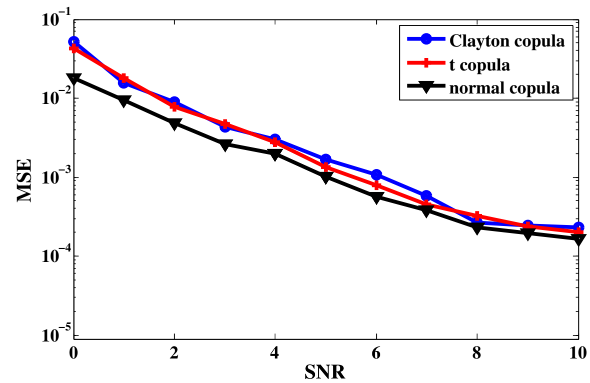

Figure 2 MSE of m_1 estimation by using three copulas for ρ_1 =0.5.

Figures 4 and 5 contain the results related to the estimation of parameters m_2 and ρ_2 in the path to the second receiver based on the PDF of the received signal $q_2(t)$ in (17). In Figures 4 and 5, the actual values of m_2 and ρ_2 are 2 and 0.1, respectively. The sample size and SNR values are the same as Figures 2 and 3.

The index of performance, in all four figures, is presented by mean square error (MSE). All estimations are done with three kinds of copula, i.e., normal, Clayton, and t copula. The comparison between the copulas guarantees that the simulation results are reliable based on all mentioned copulas. However, for example, in our simulation, the normal copula has almost better fit with the correlation model compared with other copulas. Thus, when more accuracy is required, copula goodness-of-fit testing is done and the optimized selection about the various kinds of copula is performed [18].

In the second part of the simulation, the channel capacity should be calculated. The results are depicted in

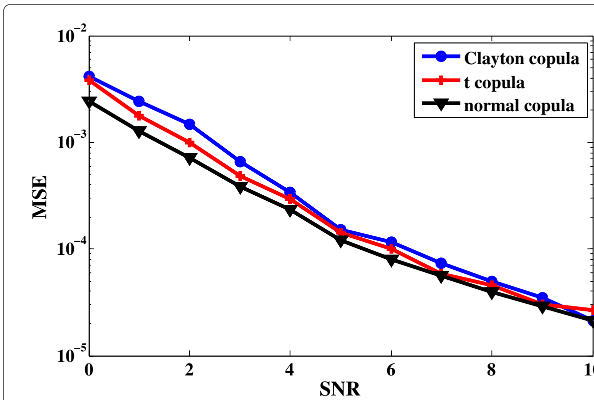

Figure 3 MSE of ρ_1 estimation by using three copulas for actual value $\rho_1 = 0.5$.

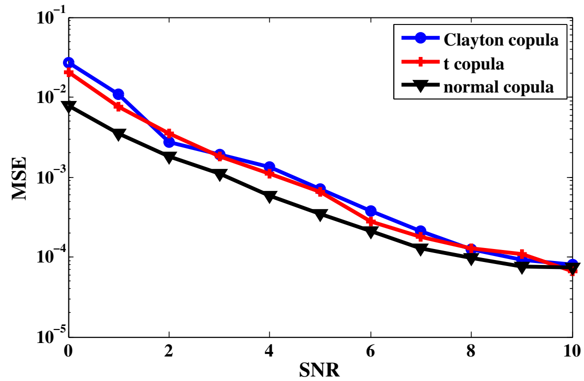

Figure 4 MSE of m_2 estimation by using three copulas for $\rho_2 = 0.1$.

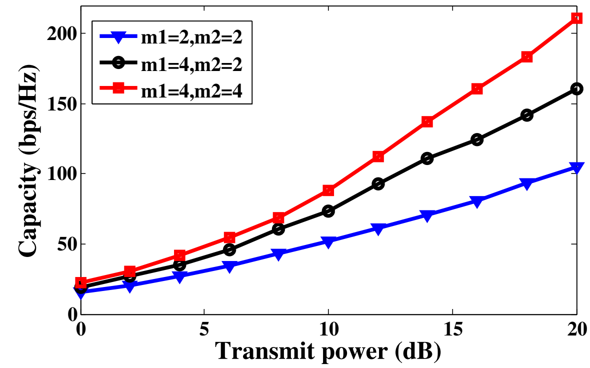

Figure 6 The channel capacity for the ρ_1=0.1 and ρ_2 =0.1.

Figures 6 and 7. In Figure 6, the correlation parameters are supposed to be $\rho_1 = 0.1$ and $\rho_2 = 0.1$. The variance of the independent zero-mean normally distributed noise is assumed to be $\sigma_n^2 = 1$, and the total transmit power P is from 0 to 20 dB. The capacity in three cases is compared in Figure 6. The three cases are a) $m_1 = 2, m_2 = 2$, b) $m_1 = 4, m_2 = 2$, and c) $m_1 = 4, m_2 = 4$. As can be seen, the channel capacity increases when the fading parameters are raised.

Figure 7 includes the same simulations for the correlation parameters $\rho_1 = 0.5$ and $\rho_2 = 0.5$, where there is also capacity increasing when either the total transmit power or fading parameter increases. It is also obvious that the capacity in Figure 7 is totaly less than the capacity in the same cases in Figure 6. This is because of the larger correlation between the transmitted signals. Fortunately, the proposed procedure presents the value of channel capacity in the Nakagami-m MIMO system by using the copula concept even when there is a large correlation between the signals.

To have a comparison with conventional methods, our method is examined when $m_1 = m_2 = 1$. Since the Nakagami-m correlated MIMO channel is equivalent to a Rayleigh correlated MIMO channel for $m_1 = m_2 = 1$, it could be compared with the method of [7] in Figure 8. The simulation is done for the correlation parameters $\rho_1 = 0.5$ and $\rho_2 = 0.5$. Figure 8 indicates that the conventional and new results are almost equal in a similar environment. The small difference between the results at low values of transmit power is due to a trivial error in PDF estimation at low values of SNR. If the sample size is considered larger, there is no difference anymore. Thus, our proposed approach covers conventional methods in addition to presenting a new improved algorithm for a more reliable channel environment.

7 Conclusions

In this paper, a new approach is proposed to estimate simultaneously the fading parameters in every route in a MIMO system and also the correlation parameter between these routes. The proposed method is based on the PDF estimation and the copula theory. The copula concept facilitates the PDF estimation when we are faced

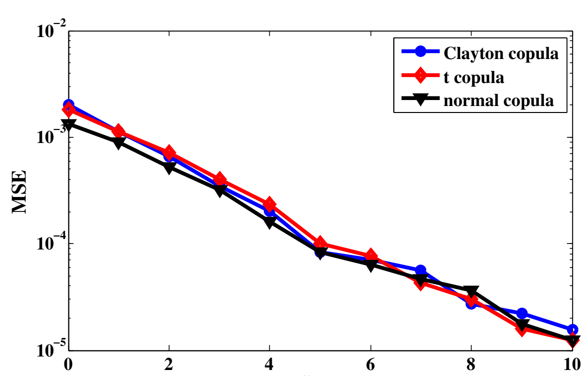

Figure 5 MSE of ρ_2 estimation by using three copulas for actual value $\rho_2 = 0.1$.

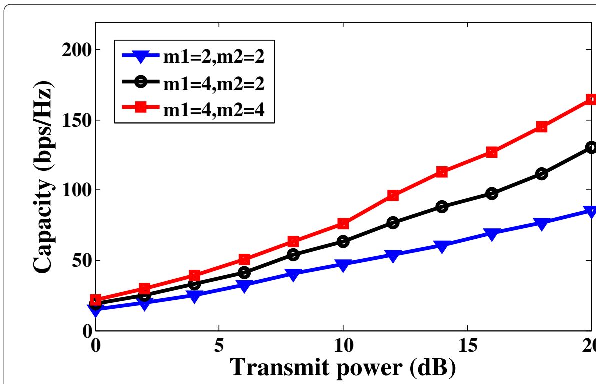

Figure 7 The channel capacity for the ρ_1=0.5 and ρ_2 =0.5.

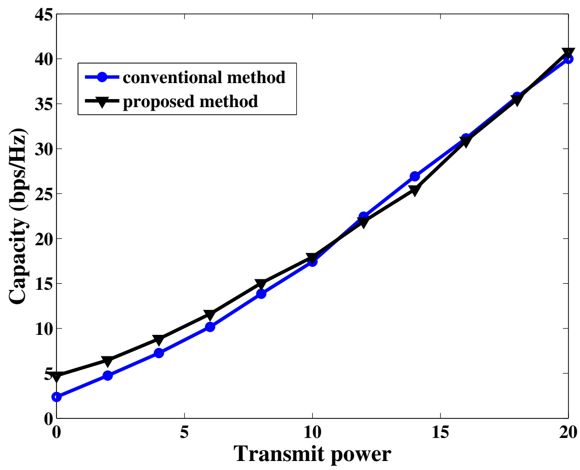

Figure 8 Comparison between the channel capacity prediction in two proposed and conventional methods for the ρ_1 =0.5 and ρ_2 =s0.5.

with the correlation between some parameters. Hence, the combination of PDF estimation and copula concept creates a novel method to identify a correlated MIMO system with Nakagami-m fading. Moreover, we calculate the capacity of the ergodic MIMO channel by using the estimated parameters. Precise estimated parameters result in a suitable prediction for the channel capacity. Some simulations are also presented to depict the validity of our proposed procedure in both fading parameter estimation and channel capacity prediction.

Appendix

A The copula relationships

The Clayton copula function is given by:

$$C\left(u_1,\ldots,u_n\right)=\left(1-n+\sum_{i=1}^{n}u_i^{-\alpha}\right)^{-\frac{1}{\alpha}}, \tag{58}$$

where $\alpha > 0$ is the Clayton copula parameter.

The t copula function is also given by:

$$C\left(u_1,\ldots,u_n\right)=\int_{-\infty}^{t_\nu^{-1}(u_1)}\cdots\int_{-\infty}^{t_\nu^{-1}(u_n)}\frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^n\,|\mathbf{R}|}}\times\left(1+\frac{\mathbf{x}^T\mathbf{R}^{-1}\mathbf{x}}{\nu}\right)^{-\frac{\nu+n}{2}}d\mathbf{x}, \tag{59}$$

where $t_\nu^{-1}(.)$ denotes the inverse function of a standard univariate t_ν distribution, matrix $\mathbf{R}$ is the correlation matrix, $\mathbf{x}$ is a vector that is defined as $\mathbf{x} \triangleq [x_1,\ldots,x_n]$, and ν is the t copula parameter that is called degrees of freedom.

B The proof for (21)

Define the signal $Q_2(t)$ as the following:

$$Q_2(t)=A\cos\left(\omega_c t+\Theta_{k1}(t)\right)\triangleq g(\Theta_{k1}(t)). \tag{60}$$

It is provable that the PDF $f_{Q_2}(Q_2)$ is determined by [19]:

$$f_{Q_2}(Q_2)=\frac{f_\Theta(\Theta_1)}{|g'(\Theta_1)|}+\ldots+\frac{f_\Theta(\Theta_n)}{|g'(\Theta_n)|}+\ldots, \tag{61}$$

where Θ_is are the real roots of the equation $Q_2 = g(\Theta)$, $g'(\Theta)$ is the derivative of $g(\Theta)$, and $f_\Theta(\Theta)$ is the PDF of the signal phase that is presented in (10).

Using (60) and (61), we have:

$$f_{Q_2}(Q_2)=\frac{1}{\sqrt{A^2-Q_2^2}}\sum_{n=-\infty}^{\infty}f_\Theta(\Theta_n),\quad |Q_2|<A. \tag{62}$$

In this paper, Θ is distributed over $[0,2\pi)$. Thus, only two solutions, which are in the interval $[0,2\pi)$, are acceptable:

$$\begin{aligned}\Theta_1&=\cos^{-1}\left(\frac{Q_2}{A}\right)-\omega_c t,\\ \Theta_2&=2\pi-\cos^{-1}\left(\frac{Q_2}{A}\right)-\omega_c t.\end{aligned} \tag{63}$$

If the function $f_\Theta(\Theta)$ is obtained by (10) for these two values, the PDF $f_{Q_2}(Q_2)$ could be calculated as:

$$f_{Q_2}(Q_2)=\frac{2^{-m}\Gamma(m)}{A^{2m-2}\Gamma^2(m/2)\sqrt{A^2-Q_2^2}}\sum_{i=1}^{2}|\sin(2\Theta_i)|. \tag{64}$$

By using (64), the proof of (21) is concluded.

Competing interests

The authors declare that they have no competing interests.

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References

1. IE Telatar, Capacity of multi-antenna Gaussian channels. Eur Trans Telecommun. **10**(6), 585–95 (1999)
2. GJ Foschini, Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas. Bell Labs Tech J. **1**(2), 41–59 (1996)
3. AJ Paulraj, CB Papadias, Space-time processing for wireless communications. IEEE Signal Process Mag. **14**(6), 49–83 (1997)
4. SM Kay, *Fundamentals of statistical signal processing: estimation theory*. (Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1998)
5. WC Jakes, DC Cox (eds.), *Microwave Mobile Communications* (Wiley-IEEE Press, New York, 1994)
6. Z Xu, S Sfar, RS Blum, Analysis of MIMO systems with receive antenna selection in spatially correlated Rayleigh fading channels. IEEE Trans Veh Technol. **58**, 251–62 (2009)

7. M Kang, MS Alouini, Capacity of correlated MIMO Rayleigh channels. IEEE Trans Wireless Commun. **5**, 143–55 (2006)
8. T Aulin, Characteristics of a digital mobile channel type. IEEE Trans Veh Technol. **30**, 45–53 (1981)
9. M Yacoub, G Fraidenraich, JS Filho, Nakagami-m phase-envelope joint distribution. IEEE Elec Lett. **41**(5), 259–61 (2005)
10. O Longoria-Gandara, R Parra-Michel, Estimation of correlated MIMO channels using partial channel state information and DPSS. IEEE Trans Wireless Commun. **10**(11), 3711–9 (2011)
11. A Sklar, Fonctions de repartition a n dimensions et leurs marges. Publ Inst Statist Univ Paris. **87**, 229–31 (1959)
12. RB Nelsen, *An Introduction to Copulas*, 2nd edition. (Springer, New York, 2006)
13. RT Clemen, T Reilly, Correlations and copulas for decision and risk analysis. Manage Sci. **45**(2), 208–24 (1999)
14. SK Sharma, S Chatzinotas, B. Ottersten, SNR estimation for multi-dimensional cognitive receiver under correlated channel/noise. IEEE Trans Wireless Commun. **12**(12), 6392–405 (2013)
15. MH Gholizadeh, H Amindavar, JA Ritcey, Analytic Nakagami fading parameter estimation in dependent noise channel using copula. EURASIP J Adv Signal Proc. **2013**, 129 (2013)
16. K Zyczkowski, M Kus, Random unitary matrices. J Phys. **A27**, 4235–45 (1994)
17. A Edelman, Eigenvalues and condition numbers of random matrices. SIAM J Matrix Analy App. **9**(4), 543–60 (1988)
18. D Berg, Copula goodness-of-fit testing: an overview and power comparison. The Eur J Finance. **15**, 675–701 (2009)
19. A Papoulis, SU Pillai, *Probability, Random Variables and Stochastic Processes*, 4th edition. (McGraw-Hill, New York, 2002)

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