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# Some fixed point results of generalized $(\phi, \psi)$ -contractive mappings in ordered $b$ -metric spaces

Belay Mitiku<sup>1\*</sup>, Kalyani Karusala<sup>2</sup> and Seshagiri Rao Namana<sup>1</sup>

## Abstract

**Objectives:** The aim of this paper is to establish some fixed point, coincidence point and, coupled coincidence and coupled common fixed point results for generalized  $(\phi, \psi)$ -contractive mappings in partially ordered  $b$ -metric spaces. Our results generalize, extend and unify most of the fundamental metrical fixed point theorems in the existing literature. Few examples are illustrated to justify our results.

**Result:** The existence and uniqueness theorems for a fixed point and coincidence point, coupled coincidence point and coupled common fixed points for two mappings satisfying generalized  $(\phi, \psi)$ -contractive conditions in complete partially ordered  $b$ -metric spaces are proved. These results generalize several comparable results in the existing literature.

**Keywords:** Partially ordered  $b$ -metric space, Fixed point, Coupled coincidence point, Coupled common fixed point, Compatible, Mixed  $F$ -monotone

**Mathematics Subject Classification:** Primary: 47H10, Secondary: 54H25

## Introduction and preliminaries

In analysis, the Banach contraction principle is one of the most important results and plays a crucial role in various fields of applied mathematics and scientific applications. It has been generalized and improved in many different directions. One of the most influential generalization is  $b$ -metric space also called metric type space by some authors, introduced and studied by Bakhtin [1] in 1989. later, Czerwik [2] generalized the Banach contraction principle in the context of complete  $b$ -metric spaces. After that many researchers have carried out further studies in  $b$ -metric space and their topological properties, some of which are in [3–18] and the references therein.

The concept of coupled fixed points for certain mappings in ordered spaces was first introduced by Bhaskar and Lakshmikantham [19] and applied their results to study the existence and uniqueness of solutions for boundary value problems. While the concept of coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings having monotone property in partially ordered complete metric spaces was first initiated by Lakshmikantham and Ćirić [20]. Since then, several authors have obtained fixed point, common fixed point, coupled coincidence point and coupled fixed point results for generalized contraction mappings in partially ordered metric spaces and partially ordered  $b$ -metric spaces, the readers may refer to [4, 21–42].

In this article some fixed point, coincidence, coupled coincidence and coupled common fixed points theorems are proved for mappings satisfying generalized  $(\phi, \psi)$ -contractions in complete partially ordered  $b$ -metric spaces. These results generalize and extend the results of

\*Correspondence: belaymida@yahoo.com

<sup>1</sup> Department of Applied Mathematics, School of Applied Natural Sciences, Adama Science and Technology University, Post Box No.1888, Adama, Ethiopia

Full list of author information is available at the end of the article



[19, 20, 40–42] and several comparable results in the literature. Three examples are given to support our results.

The following definitions and results will be needed in what follows.

**Definition 1** [27] A map  $d : P \times P \rightarrow [0, +\infty)$ , where  $P$  is a non-empty set is said to be a  $b$ -metric, if it satisfies the properties given below for any  $v, \xi, \mu \in P$  and for some real number  $s \geq 1$ ,

- (a)  $d(v, \xi) = 0$  if and only if  $v = \xi$ ,
- (b)  $d(v, \xi) = d(\xi, v)$ ,
- (c)  $d(v, \xi) \leq s(d(v, \mu) + d(\mu, \xi))$ .

Then  $(P, d, s)$  is known as a  $b$ -metric space. If  $(P, \preceq)$  is still a partially ordered set, then  $(P, d, s, \preceq)$  is called a partially ordered  $b$ -metric space.

**Definition 2** [27] Let  $(P, d, s, \preceq)$  be a  $b$ -metric space. Then

- (1) A sequence  $\{v_n\}$  is said to converge to  $v$ , if  $\lim_{n \rightarrow +\infty} d(v_n, v) = 0$  and written as  $\lim_{n \rightarrow +\infty} v_n = v$ .
- (2)  $\{v_n\}$  is said to be a Cauchy sequence in  $P$ , if  $\lim_{n, m \rightarrow +\infty} d(v_n, v_m) = 0$ .
- (3)  $(P, d, s)$  is said to be complete, if every Cauchy sequence in it is convergent.

**Definition 3** If the metric  $d$  is complete then  $(P, d, s, \preceq)$  is called complete partially ordered  $b$ -metric space.

**Definition 4** [34] Let  $(P, \preceq)$  be a partially ordered set and let  $f, S : P \rightarrow P$  are two mappings. Then

- (1)  $S$  is called a monotone nondecreasing, if  $S(v) \preceq S(\xi)$  for all  $v, \xi \in P$  with  $v \preceq \xi$ .
- (2) An element  $v \in P$  is called a coincidence (common fixed) point of  $f$  and  $S$ , if  $f v = S v$  ( $f v = S v = v$ ).
- (3)  $f$  and  $S$  are called commuting, if  $f S v = S f v$ , for all  $v \in P$ .
- (4)  $f$  and  $S$  are called compatible, if any sequence  $\{v_n\}$  with  $\lim_{n \rightarrow +\infty} f v_n = \lim_{n \rightarrow +\infty} S v_n = \mu$ , for  $\mu \in P$  then  $\lim_{n \rightarrow +\infty} d(S f v_n, f S v_n) = 0$ .
- (5) A pair of self maps  $(f, S)$  is called weakly compatible, if  $f S v = S f v$ , when  $S v = f v$  for some  $v \in P$ .
- (6)  $S$  is called monotone  $f$ -nondecreasing, if

$$f v \preceq f \xi \Rightarrow S v \preceq S \xi, \text{ for any } v, \xi \in P.$$

- (7) A non empty set  $P$  is called well ordered set, if very two elements of it are comparable i.e.,  $v \preceq \xi$  or  $\xi \preceq v$ , for  $v, \xi \in P$ .

**Definition 5** [20, 26] Let  $(P, d, \preceq)$  be a partially ordered set and let  $S : P \times P \rightarrow P$  and  $f : P \rightarrow P$  be two mappings. Then

- (1)  $S$  has the mixed  $f$ -monotone property, if  $S$  is non-decreasing  $f$ -monotone in its first argument and is non increasing  $f$ -monotone in its second argument, that is for any  $v, \xi \in P$

$$v_1, v_2 \in P, f v_1 \preceq f v_2 \Rightarrow S(v_1, \xi) \preceq S(v_2, \xi) \text{ and } \xi_1, \xi_2 \in P, f \xi_1 \preceq f \xi_2 \Rightarrow S(v, \xi_1) \succeq S(v, \xi_2).$$

Suppose, if  $f$  is an identity mapping then  $S$  is said to have the mixed monotone property.

- (2) An element  $(v, \xi) \in P \times P$  is called a coupled coincidence point of  $S$  and  $f$ , if  $S(v, \xi) = f v$  and  $S(\xi, v) = f \xi$ . Note that, if  $f$  is an identity mapping then  $(v, \xi)$  is said to be a coupled fixed point of  $S$ .
- (3) Element  $v \in P$  is called a common fixed point of  $S$  and  $f$ , if  $S(v, v) = f v = v$ .
- (4)  $S$  and  $f$  are commutative, if for all  $v, \xi \in P$ ,  $S(f v, f \xi) = f(S v, S \xi)$ .
- (5)  $S$  and  $f$  are said to be compatible, if

$$\lim_{n \rightarrow +\infty} d(f(S(v_n, \xi_n)), S(f v_n, f \xi_n)) = 0 \text{ and } \lim_{n \rightarrow +\infty} d(f(S(\xi_n, v_n)), S(f \xi_n, f v_n)) = 0,$$

whenever  $\{v_n\}$  and  $\{\xi_n\}$  are any two sequences in  $P$  such that  $\lim_{n \rightarrow +\infty} S(v_n, \xi_n) = \lim_{n \rightarrow +\infty} f v_n = v$  and  $\lim_{n \rightarrow +\infty} S(\xi_n, v_n) = \lim_{n \rightarrow +\infty} f \xi_n = \xi$ , for any  $v, \xi \in P$ .

The following result can be used for the convergence of a sequence in  $b$ -metric space.

**Lemma 6** [26] Let  $(P, d, s, \preceq)$  be a  $b$ -metric space with  $s > 1$  and suppose that  $\{v_n\}$  and  $\{\xi_n\}$  are  $b$ -convergent to  $v$  and  $\xi$  respectively. Then

$$\frac{1}{s^2} d(v, \xi) \leq \liminf_{n \rightarrow +\infty} d(v_n, \xi_n) \leq \limsup_{n \rightarrow +\infty} d(v_n, \xi_n) \leq s^2 d(v, \xi).$$

In particular, if  $v = \xi$ , then  $\lim_{n \rightarrow +\infty} d(v_n, \xi_n) = 0$ . Moreover, for each  $\tau \in P$ , we have

$$\frac{1}{s} d(v, \tau) \leq \liminf_{n \rightarrow +\infty} d(v_n, \tau) \leq \limsup_{n \rightarrow +\infty} d(v_n, \tau) \leq s d(v, \tau).$$

Throughout this paper, we introduce the following distance functions.

A self mapping  $\phi$  defined on  $[0, +\infty)$  is said to be an altering distance function, if it satisfies the following conditions:

- (i)  $\phi$  is continuous,
- (ii)  $\phi$  is nondecreasing,
- (iii)  $\phi(t) = 0$  if and only if  $t = 0$ .

Let us denote the set of all altering distance functions on  $[0, +\infty)$  by  $\Phi$ .

Similarly,  $\Psi$  denote the set of all functions  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  satisfying the following conditions:

- (i)  $\psi$  is lower semi-continuous,
- (ii)  $\psi(t) = 0$  if and only if  $t = 0$ .

Let  $(P, d, s, \preceq)$  be a partially ordered  $b$ -metric space with parameter  $s > 1$  and let  $S : P \rightarrow P$  be a mapping. Set

$$M(v, \xi) = \max \left\{ \frac{d(\xi, S\xi)[1 + d(v, Sv)]}{1 + d(v, \xi)}, \frac{d(v, Sv) d(\xi, S\xi)}{1 + d(Sv, S\xi)}, \frac{d(v, Sv) d(v, S\xi)}{1 + d(v, S\xi) + d(\xi, Sv)}, d(v, \xi) \right\}, \tag{1}$$

and

$$N(v, \xi) = \max \left\{ \frac{d(\xi, S\xi)[1 + d(v, Sv)]}{1 + d(v, \xi)}, d(v, \xi) \right\}. \tag{2}$$

Now, we introduced the following definition.

**Definition 7** Let  $(P, d, s, \preceq)$  be a partially ordered  $b$ -metric space with parameter  $s > 1$  and  $\phi \in \Phi, \psi \in \Psi$ . We say that  $S : P \rightarrow P$  is a generalized  $(\phi, \psi)$ -contractive mapping if it satisfies

$$\phi(sd(Sv, S\xi)) \leq \phi(M(v, \xi)) - \psi(N(v, \xi)), \tag{3}$$

for any  $v, \xi \in P$  with  $v \preceq \xi$ .

**Main text**

In this section, we prove some fixed point results of mappings satisfying generalized  $(\phi, \psi)$ -contraction conditions in the context of partially ordered  $b$ -metric spaces. The first result in this paper is the following fixed point theorem.

**Theorem 8** Suppose that  $(P, d, s, \preceq)$  be a complete partially ordered  $b$ -metric space with  $s > 1$ . Let  $S : P \rightarrow P$  be an almost generalized  $(\phi, \psi)$ -contractive mapping, and be continuous, nondecreasing mapping with regards to  $\preceq$ . If there exists  $v_0 \in P$  with  $v_0 \preceq Sv_0$ , then  $S$  has a fixed point in  $P$ .

**Proof** For some  $v_0 \in P$  such that  $Sv_0 = v_0$ , then we have the result. Assume that  $v_0 \prec Sv_0$ , then construct a sequence  $\{v_n\} \subset P$  by  $v_{n+1} = Sv_n$ , for  $n \geq 0$ . Since  $S$  is nondecreasing, we obtain by induction that

$$v_0 \prec Sv_0 = v_1 \preceq \dots \preceq v_n \preceq Sv_n = v_{n+1} \preceq \dots \tag{4}$$

If for some  $n_0 \in \mathbb{N}$  such that  $v_{n_0} = v_{n_0+1}$  then from (4),  $v_{n_0}$  is a fixed point of  $S$  and we have nothing to prove. Suppose that  $v_n \neq v_{n+1}$ , for all  $n \geq 1$ . Since  $v_n \succ v_{n-1}$ , for any  $n \geq 1$  and then from contraction condition (3), we have

$$\begin{aligned} \phi(d(v_n, v_{n+1})) &= \phi(d(Sv_{n-1}, Sv_n)) \\ &\leq \phi(sd(Sv_{n-1}, Sv_n)) \\ &\leq \phi(M(v_{n-1}, v_n)) - \psi(N(v_{n-1}, v_n)). \end{aligned} \tag{5}$$

From (5), we get

$$d(v_n, v_{n+1}) = d(Sv_{n-1}, Sv_n) \leq \frac{1}{s} M(v_{n-1}, v_n), \tag{6}$$

where

$$\begin{aligned} M(v_{n-1}, v_n) &= \max \left\{ \frac{d(v_n, Sv_n)[1 + d(v_{n-1}, Sv_{n-1})]}{1 + d(v_{n-1}, v_n)}, \frac{d(v_{n-1}, Sv_{n-1}) d(v_n, Sv_n)}{1 + d(Sv_{n-1}, Sv_n)}, \right. \\ &\quad \left. \frac{d(v_{n-1}, Sv_{n-1}) d(v_{n-1}, Sv_n)}{1 + d(v_{n-1}, Sv_n) + d(v_n, Sv_{n-1})}, d(v_{n-1}, v_n) \right\} \\ &= \max \left\{ \frac{d(v_n, v_{n+1})[1 + d(v_{n-1}, v_n)]}{1 + d(v_{n-1}, v_n)}, \frac{d(v_{n-1}, v_n) d(v_n, v_{n+1})}{1 + d(v_n, v_{n+1})}, \right. \\ &\quad \left. \frac{d(v_{n-1}, v_n) d(v_{n-1}, v_{n+1})}{1 + d(v_{n-1}, v_{n+1}) + d(v_n, v_n)} d(v_{n-1}, v_n) \right\} \\ &\leq \max \{d(v_n, v_{n+1}), d(v_{n-1}, v_n)\} \end{aligned} \tag{7}$$

If  $\max\{d(v_n, v_{n+1}), d(v_{n-1}, v_n)\} = d(v_n, v_{n+1})$  for some  $n \geq 1$ , then from (6) follows

$$d(v_n, v_{n+1}) \leq \frac{1}{s}d(v_n, v_{n+1}), \tag{8}$$

which is a contradiction. This means that  $\max\{d(v_n, v_{n+1}), d(v_{n-1}, v_n)\} = d(v_{n-1}, v_n)$  for  $n \geq 1$ . Hence, we obtain from (6) that

$$d(v_n, v_{n+1}) \leq \frac{1}{s}d(v_{n-1}, v_n). \tag{9}$$

Since,  $\frac{1}{s} \in (0, 1)$  then the sequence  $\{v_n\}$  is a Cauchy sequence by [3, 5, 8, 10]. Since  $P$  is complete, then there exists some  $\mu \in P$  such that  $v_n \rightarrow \mu$ .

Further, the continuity of  $S$  implies that

$$S\mu = S(\lim_{n \rightarrow +\infty} v_n) = \lim_{n \rightarrow +\infty} Sv_n = \lim_{n \rightarrow +\infty} v_{n+1} = \mu. \tag{10}$$

Therefore,  $\mu$  is a fixed point of  $S$  in  $P$ . □

Last result is still valid for  $S$  not necessarily continuous, assuming an additional hypothesis on  $P$ .

**Theorem 9** *Let  $(P, d, s, \leq)$  be a complete partially ordered  $b$ -metric space with  $s > 1$ . Assume that  $P$  satisfies*

if a nondecreasing sequence  $\{v_n\} \rightarrow \mu$  in  $P$ ,  
then  $v_n \leq \mu$  for all  $n \in \mathbb{N}$ , i.e.,  $\mu = \sup v_n$ .

*Let  $S : P \rightarrow P$  be a nondecreasing mapping such that the contraction condition (3) is satisfied. If there exists  $v_0 \in P$  with  $v_0 \leq Sv_0$ , then  $S$  has a fixed point in  $P$ .*

**Proof** From the proof of Theorem 8, we construct a nondecreasing Cauchy sequence  $\{v_n\}$ , which converges to  $\mu$  in  $P$ . Therefore from the hypotheses, we have  $v_n \leq \mu$  for any  $n \in \mathbb{N}$ , implies that  $\mu = \sup v_n$ .

Now, we prove that  $\mu$  is a fixed point of  $S$ , that is  $S\mu = \mu$ . Suppose that  $S\mu \neq \mu$ . Let

$$M(v_n, \mu) = \max \left\{ \frac{d(\mu, S\mu)[1 + d(v_n, Sv_n)]}{1 + d(v_n, \mu)}, \frac{d(v_n, Sv_n) d(\mu, S\mu)}{1 + d(Sv_n, S\mu)}, \frac{d(v_n, Sv_n) d(v_n, S\mu)}{1 + d(v_n, S\mu) + d(\mu, Sv_n)}, d(v_n, \mu) \right\}, \tag{11}$$

and

$$N(v_n, \mu) = \max \left\{ \frac{d(\mu, S\mu)[1 + d(v_n, Sv_n)]}{1 + d(v_n, \mu)}, d(v_n, \mu) \right\}. \tag{12}$$

Letting  $n \rightarrow +\infty$  and from the fact that  $\lim_{n \rightarrow +\infty} v_n = \mu$ , we get

$$\lim_{n \rightarrow +\infty} M(v_n, \mu) = \max\{d(\mu, S\mu), 0, 0, 0\} = d(\mu, S\mu), \tag{13}$$

and

$$\lim_{n \rightarrow +\infty} N(v_n, \mu) = \max\{d(\mu, S\mu), 0\} = d(\mu, S\mu). \tag{14}$$

We know that  $v_n \leq \mu$ , for all  $n$  then from contraction condition (3), we get

$$\begin{aligned} \phi(d(v_{n+1}, S\mu)) &= \phi(d(Sv_n, S\mu)) \leq \phi(sd(Sv_n, S\mu)) \\ &\leq \phi(M(v_n, \mu)) - \psi(N(v_n, \mu)). \end{aligned} \tag{15}$$

Letting  $n \rightarrow +\infty$  and use of (13) and (14), we get

$$\phi(d(\mu, S\mu)) \leq \phi(d(\mu, S\mu)) - \psi(d(\mu, S\mu)) < \phi(d(\mu, S\mu)), \tag{16}$$

which is a contradiction under (16). Thus,  $S\mu = \mu$ , that is  $S$  has a fixed point  $\mu$  in  $P$ . □

Now we give the sufficient condition for the uniqueness of the fixed point exists in Theorems 8 and 9.

every pair of elements has a lower bound or an upper bound. (17)

This condition is equivalent to,

for every  $v, \xi \in P$ ,  
there exists  $w \in P$  which is comparable to  $v$  and  $\xi$ .

**Theorem 10** *In addition to the hypotheses of Theorem 8 (or Theorem 9), condition (17) provides uniqueness of the fixed point of  $S$  in  $P$ .*

**Proof** From Theorem 8 (or Theorem 9), we conclude that  $S$  has a nonempty set of fixed points. Suppose that  $v^*$  and  $\xi^*$  be two fixed points of  $S$  then, we claim that  $v^* = \xi^*$ . Suppose that  $v^* \neq \xi^*$ , then from the hypotheses we have

$$\begin{aligned} \phi(d(Sv^*, S\xi^*)) &\leq \phi(sd(Sv^*, S\xi^*)) \\ &\leq \phi(M(v^*, \xi^*)) - \psi(N(v^*, \xi^*)). \end{aligned} \tag{18}$$

Consequently, we get

$$d(v^*, \xi^*) = d(Sv^*, S\xi^*) \leq \frac{1}{s}M(v^*, \xi^*), \tag{19}$$

for any  $v, \xi \in P$  with  $f v \leq f \xi$ , where  $M_f(v, \xi)$  and  $N_f(v, \xi)$  are given by (22) and (23) respectively.

where

$$M(v^*, \xi^*) = \max \left\{ \frac{d(\xi^*, S\xi^*)[1 + d(v^*, Sv^*)]}{1 + d(v^*, \xi^*)}, \frac{d(v^*, Sv^*) d(\xi^*, S\xi^*)}{1 + d(Sv^*, S\xi^*)}, \frac{d(v^*, Sv^*) d(v^*, S\xi^*)}{1 + d(v^*, S\xi^*) + d(\xi^*, Sv^*)}, d(v^*, \xi^*) \right\}$$

$$= \max \left\{ \frac{d(\xi^*, \xi^*)[1 + d(v^*, v^*)]}{1 + d(v^*, \xi^*)}, \frac{d(v^*, v^*) d(\xi^*, \xi^*)}{1 + d(v^*, \xi^*)}, \frac{d(v^*, v^*) d(v^*, \xi^*)}{1 + d(v^*, \xi^*) + d(\xi^*, v^*)}, d(v^*, \xi^*) \right\}$$

$$= \max\{0, 0, 0, d(v^*, \xi^*)\}$$

$$= d(v^*, \xi^*)$$

**Theorem 12** Suppose that  $(P, d, s, \leq)$  be a complete par-

From (19), we obtain that

$$d(v^*, \xi^*) \leq \frac{1}{s}d(v^*, \xi^*) < d(v^*, \xi^*), \tag{21}$$

which is a contradiction. Hence,  $v^* = \xi^*$ . This completes the proof.  $\square$

Let  $(P, d, s, \leq)$  be a partially ordered  $b$ -metric space with parameter  $s > 1$ , and let  $S, f : P \rightarrow P$  be two mappings. Set

$$M_f(v, \xi) = \max \left\{ \frac{d(f\xi, S\xi)[1 + d(fv, Sv)]}{1 + d(fv, f\xi)}, \frac{d(fv, Sv) d(f\xi, S\xi)}{1 + d(Sv, S\xi)}, \frac{d(fv, Sv) d(fv, S\xi)}{1 + d(fv, S\xi) + d(f\xi, Sv)}, d(fv, f\xi) \right\}, \tag{22}$$

and

$$N_f(v, \xi) = \max \left\{ \frac{d(f\xi, S\xi)[1 + d(fv, Sv)]}{1 + d(fv, f\xi)}, d(fv, f\xi) \right\}. \tag{23}$$

Now, we introduce the following definition.

**Definition 11** Let  $(P, d, s, \leq)$  be a partially ordered  $b$ -metric space with parameter  $s > 1$ . The mapping  $S : P \rightarrow P$  is called a generalized  $(\phi, \psi)$ -contraction mapping with respect to  $f : P \rightarrow P$  for some  $\phi \in \Phi$  and  $\psi \in \Psi$ , if

tially ordered  $b$ -metric space with  $s > 1$ . Let  $S : P \rightarrow P$  be an almost generalized  $(\phi, \psi)$ -contractive mapping with respect to  $f : P \rightarrow P$  and,  $S$  and  $f$  are continuous such that  $S$  is a monotone  $f$ -non decreasing mapping, compatible with  $f$  and  $SP \subseteq fP$ . If for some  $v_0 \in P$  such that  $f v_0 \leq S v_0$ , then  $S$  and  $f$  have a coincidence point in  $P$ .

**Proof** By following the proof of a Theorem 2.2 in [30], we construct two sequences  $\{v_n\}$  and  $\{\xi_n\}$  in  $P$  such that

$$\xi_n = S v_n = f v_{n+1} \text{ for all } n \geq 0, \tag{25}$$

for which

$$f v_0 \leq f v_1 \leq \dots \leq f v_n \leq f v_{n+1} \leq \dots \tag{26}$$

Again from [30], we have to show that

$$d(\xi_n, \xi_{n+1}) \leq \lambda d(\xi_{n-1}, \xi_n), \tag{27}$$

for all  $n \geq 1$  and where  $\lambda \in [0, \frac{1}{s})$ . Now from (24) and use of (25) and (26), we have

$$\begin{aligned} \phi(sd(\xi_n, \xi_{n+1})) &= \phi(sd(Sv_n, Sv_{n+1})) \\ &\leq \phi(M_f(v_n, v_{n+1})) - \psi(N_f(v_n, v_{n+1})), \end{aligned} \tag{28}$$

where

$$\lim_{n \rightarrow +\infty} Sv_n = \lim_{n \rightarrow +\infty} fv_{n+1} = \mu.$$

$$\begin{aligned} M_f(v_n, v_{n+1}) &= \max \left\{ \frac{d(fv_{n+1}, Sv_{n+1})[1 + d(fv_n, Sv_n)]}{1 + d(fv_n, fv_{n+1})}, \frac{d(fv_n, Sv_n) d(fv_{n+1}, Sv_{n+1})}{1 + d(Sv_n, Sv_{n+1})}, \right. \\ &\quad \left. \frac{d(fv_n, Sv_n) d(fv_n, Sv_{n+1})}{1 + d(fv_n, Sv_{n+1}) + d(fv_{n+1}, Sv_n)}, d(fv_n, fv_{n+1}) \right\} \\ &= \max \left\{ \frac{d(\xi_n, \xi_{n+1})[1 + d(\xi_{n-1}, \xi_n)]}{1 + d(\xi_{n-1}, \xi_n)}, \frac{d(\xi_{n-1}, \xi_n) d(\xi_n, \xi_{n+1})}{1 + d(\xi_n, \xi_{n+1})}, \right. \\ &\quad \left. \frac{d(\xi_{n-1}, \xi_n) d(\xi_{n-1}, \xi_{n+1})}{1 + d(\xi_{n-1}, \xi_{n+1}) + d(\xi_n, \xi_n)}, d(\xi_{n-1}, \xi_n) \right\} \\ &\leq \max \{d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1})\} \end{aligned}$$

and

Thus by the compatibility of  $S$  and  $f$ , we obtain that

$$\begin{aligned} N_f(v_n, v_{n+1}) &= \max \left\{ \frac{d(fv_{n+1}, Sv_{n+1})[1 + d(fv_n, Sv_n)]}{1 + d(fv_n, fv_{n+1})}, d(fv_n, fv_{n+1}) \right\} \\ &= \max \left\{ \frac{d(\xi_n, \xi_{n+1})[1 + d(\xi_{n-1}, \xi_n)]}{1 + d(\xi_{n-1}, \xi_n)}, d(\xi_{n-1}, \xi_n) \right\} \\ &= \max \{d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1})\} \end{aligned}$$

Therefore from equation (28), we get

$$\begin{aligned} \phi(sd(\xi_n, \xi_{n+1})) &\leq \phi(\max\{d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1})\}) \\ &\quad - \psi(\max\{d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1})\}). \end{aligned} \tag{29}$$

If  $0 < d(\xi_{n-1}, \xi_n) \leq d(\xi_n, \xi_{n+1})$  for some  $n \in \mathbb{N}$ , then from (29) we get

$$\begin{aligned} \phi(sd(\xi_n, \xi_{n+1})) &\leq \phi(d(\xi_n, \xi_{n+1})) \\ &\quad - \psi(d(\xi_n, \xi_{n+1})) < \phi(d(\xi_n, \xi_{n+1})), \end{aligned} \tag{30}$$

or equivalently

$$sd(\xi_n, \xi_{n+1}) \leq d(\xi_n, \xi_{n+1}). \tag{31}$$

This is a contradiction. Hence from (29) we obtain that

$$sd(\xi_n, \xi_{n+1}) \leq d(\xi_{n-1}, \xi_n). \tag{32}$$

Thus equation (27) holds, where  $\lambda \in [0, \frac{1}{s})$ . Therefore from (27) and Lemma 3.1 of [16], we conclude that  $\{\xi_n\} = \{Sv_n\} = \{fv_{n+1}\}$  is a Cauchy sequence in  $P$  and then converges to some  $\mu \in P$  as  $P$  is complete such that

$$\lim_{n \rightarrow +\infty} d(f(Sv_n), S(fv_n)) = 0, \tag{33}$$

and from the continuity of  $S$  and  $f$ , we have

$$\lim_{n \rightarrow +\infty} f(Sv_n) = f\mu, \quad \lim_{n \rightarrow +\infty} S(fv_n) = S\mu. \tag{34}$$

Further by use of triangular inequality and from equations (33) and (34), we get

$$\begin{aligned} \frac{1}{s}d(S\mu, f\mu) &\leq d(S\mu, S(fv_n)) \\ &\quad + sd(S(fv_n), f(Sv_n)) + sd(f(Sv_n), f\mu). \end{aligned} \tag{35}$$

Finally, we arrive at  $d(Sv, fv) = 0$  as  $n \rightarrow +\infty$  in (35). Therefore,  $v$  is a coincidence point of  $S$  and  $f$  in  $P$ .  $\square$

Relaxing the continuity of  $f$  and  $S$  in Theorem 12, we obtain the following result.

**Theorem 13** In Theorem 12, assume that  $P$  satisfies

for any nondecreasing sequence  $\{fv_n\} \subset P$  with  $\lim_{n \rightarrow +\infty} fv_n = fv$  in  $fP$ , where  $fP$  is a closed subset of  $P$  implies that  $fv_n \leq fv, fv \leq f(fv)$  for  $n \in \mathbb{N}$ .

If there exists  $v_0 \in P$  such that  $fv_0 \leq Sv_0$ , then the weakly compatible mappings  $S$  and  $f$  have a coincidence point in  $P$ . Moreover,  $S$  and  $f$  have a common fixed point, if  $S$  and  $f$  commute at their coincidence points.

**Proof** The sequence,  $\{\xi_n\} = \{Sv_n\} = \{fv_{n+1}\}$  is a Cauchy sequence from the proof of Theorem 12. Since  $fP$  is closed, then there is some  $\mu \in P$  such that

$$\lim_{n \rightarrow +\infty} Sv_n = \lim_{n \rightarrow +\infty} fv_{n+1} = f\mu.$$

Thus by the hypotheses, we have  $fv_n \leq f\mu$  for all  $n \in \mathbb{N}$ . Now, we have to prove that  $\mu$  is a coincidence point of  $S$  and  $f$ .

From equation (24), we have

$$\phi(sd(Sv_n, Sv)) \leq \phi(M_f(v_n, v)) - \psi(N_f(v_n, v)), \tag{36}$$

where

$$\begin{aligned} M_f(v_n, \mu) &= \max \left\{ \frac{d(f\mu, S\mu)[1 + d(fv_n, Sv_n)]}{1 + d(fv_n, f\mu)}, \frac{d(fv_n, Sv_n) d(f\mu, S\mu)}{1 + d(Sv_n, S\mu)}, \right. \\ &\quad \left. \frac{d(fv_n, Sv_n) d(fv_n, S\mu)}{1 + d(fv_n, S\mu) + d(f\mu, Sv_n)}, d(fv_n, f\mu) \right\} \\ &\rightarrow \max\{d(f\mu, S\mu), 0, 0, 0\} \\ &= d(f\mu, S\mu) \text{ as } n \rightarrow +\infty, \end{aligned}$$

and

$$N_f(v_n, \mu) = \max \left\{ \frac{d(f\mu, S\mu)[1 + d(fv_n, Sv_n)]}{1 + d(fv_n, f\mu)}, d(fv_n, f\mu) \right\} \rightarrow \max\{d(f\mu, S\mu), 0\} = d(f\mu, S\mu) \text{ as } n \rightarrow +\infty.$$

Therefore equation (36) becomes

$$\begin{aligned} \phi(s \lim_{n \rightarrow +\infty} d(Sv_n, Sv)) &\leq \phi(d(f\mu, S\mu)) \\ &- \psi(d(f\mu, S\mu)) < \phi(d(f\mu, S\mu)). \end{aligned} \tag{37}$$

Consequently, we get

$$\lim_{n \rightarrow +\infty} d(Sv_n, Sv) < \frac{1}{s} d(f\mu, S\mu). \tag{38}$$

Further by triangular inequality, we have

$$\frac{1}{s} d(f\mu, S\mu) \leq d(f\mu, Sv_n) + d(Sv_n, S\mu), \tag{39}$$

then (38) and (39) lead to contradiction, if  $f\mu \neq S\mu$ . Hence,  $f\mu = S\mu$ . Let  $f\mu = S\mu = \rho$ , that is  $S$  and  $f$  commute at  $\rho$ , then  $S\rho = S(f\mu) = f(S\mu) = f\rho$ . Since  $f\mu = f(f\mu) = f\rho$ , then by equation (36) with  $f\mu = S\mu$  and  $f\rho = S\rho$ , we get

$$\begin{aligned} \phi(sd(S\mu, S\rho)) &\leq \phi(M_f(\mu, \rho)) - \psi(N_f(\mu, \rho)) \\ &< \phi(d(S\mu, S\rho)), \end{aligned} \tag{40}$$

or equivalently,

$$sd(S\mu, S\rho) \leq d(S\mu, S\rho),$$

which is a contradiction, if  $S\mu \neq S\rho$ . Thus,  $S\mu = S\rho = \rho$ . Hence,  $S\mu = f\rho = \rho$ , that is  $\rho$  is a common fixed point of  $S$  and  $f$ .  $\square$

**Definition 14** Let  $(P, d, s, \leq)$  be a complete partially

ordered  $b$ -metric space with  $s > 1$ ,  $\phi \in \Phi$  and  $\psi \in \Psi$ . A mapping  $S : P \times P \rightarrow P$  is said to be an almost gen-

eralized  $(\phi, \psi)$ -contractive mapping with respect to  $f : P \rightarrow P$  such that

$$\begin{aligned} \phi(s^k d(S(v, \xi), S(\rho, \tau))) &\leq \phi(M(v, \xi, \rho, \tau)) \\ &- \psi(N(v, \xi, \rho, \tau)), \end{aligned} \tag{41}$$

for all  $v, \xi, \rho, \tau \in P$  with  $fv \leq f\rho$  and  $f\xi \geq f\tau$ ,  $k > 2$  where



$$M_f(v, \xi, \rho, \tau) = \max \left\{ \frac{d(f\rho, S(\rho, \tau))[1 + d(fv, S(v, \xi))]}{1 + d(fv, f\rho)}, \frac{d(fv, S(v, \xi)) d(f\rho, S(\rho, \tau))}{1 + d(S(v, \xi), S(\rho, \tau))}, \right. \\ \left. \frac{d(fv, S(v, \xi)) d(fv, S(\rho, \tau))}{1 + d(fv, S(\rho, \tau)) + d(f\rho, S(v, \xi))}, d(fv, f\rho) \right\}, \tag{42}$$

and

$$N_f(v, \xi, \rho, \tau) = \max \left\{ \frac{d(f\rho, S(\rho, \tau))[1 + d(fv, S(v, \xi))]}{1 + d(fv, f\rho)}, d(fv, f\rho) \right\}.$$

**Theorem 15** Let  $(P, d, s, \preceq)$  be a complete partially ordered  $b$ -metric space with  $s > 1$ . Suppose that  $S : P \times P \rightarrow P$  be an almost generalized  $(\phi, \psi)$ -contractive mapping with respect to  $f : P \rightarrow P$  and,  $S$  and  $f$  are continuous functions such that  $S$  has the mixed  $f$ -monotone property and commutes with  $f$ . Also assume that  $S(P \times P) \subseteq f(P)$ . Then  $S$  and  $f$  have a coupled coincidence point in  $P$ , if there exists  $(v_0, \xi_0) \in P \times P$  such that  $f v_0 \preceq S(v_0, \xi_0)$  and  $f \xi_0 \succeq S(\xi_0, v_0)$ .

**Proof** From the hypotheses and following the proof of Theorem 2.2 of [30], we construct two sequences  $\{v_n\}$  and  $\{\xi_n\}$  in  $P$  such that

$$f v_{n+1} = S(v_n, \xi_n), \quad f \xi_{n+1} = S(\xi_n, v_n), \quad \text{for all } n \geq 0.$$

In particular,  $\{f v_n\}$  is nondecreasing and  $\{f \xi_n\}$  is non-increasing sequences in  $P$ . Now from (41) by replacing  $v = v_n, \xi = \xi_n, \rho = v_{n+1}, \tau = \xi_{n+1}$ , we get

Similarly by taking  $v = \xi_{n+1}, \xi = v_{n+1}, \rho = v_n, \tau = v_n$  in (41), we get

$$\phi(s^k d(f \xi_{n+1}, f \xi_{n+2})) \leq \phi(\max\{d(f \xi_n, f \xi_{n+1}), d(f \xi_{n+1}, f \xi_{n+2})\}) - \psi(\max\{d(f \xi_n, f \xi_{n+1}), d(f \xi_{n+1}, f \xi_{n+2})\}). \tag{47}$$

From the fact that  $\max\{\phi(c), \phi(d)\} = \phi\{\max\{c, d\}\}$  for all  $c, d \in [0, +\infty)$ . Then combining (46) and (47), we get

$$\phi(s^k \delta_n) \leq \phi(\max\{d(f v_n, f v_{n+1}), d(f v_{n+1}, f v_{n+2}), d(f \xi_n, f \xi_{n+1}), d(f \xi_{n+1}, f \xi_{n+2})\}) - \psi(\max\{d(f v_n, f v_{n+1}), d(f v_{n+1}, f v_{n+2}), d(f \xi_n, f \xi_{n+1}), d(f \xi_{n+1}, f \xi_{n+2})\}), \tag{48}$$

where

$$\delta_n = \max\{d(f v_{n+1}, f v_{n+2}), d(f \xi_{n+1}, f \xi_{n+2})\}. \tag{49}$$

Let us denote,

$$\Delta_n = \max\{d(f v_n, f v_{n+1}), d(f v_{n+1}, f v_{n+2}), d(f \xi_n, f \xi_{n+1}), d(f \xi_{n+1}, f \xi_{n+2})\}. \tag{50}$$

Hence from equations (46)-(49), we obtain

$$\phi(s^k d(f v_{n+1}, f v_{n+2})) = \phi(s^k d(S(v_n, \xi_n), S(v_{n+1}, \xi_{n+1}))) \leq \phi(M_f(v_n, \xi_n, v_{n+1}, \xi_{n+1})) - \psi(N_f(v_n, \xi_n, v_{n+1}, \xi_{n+1})), \tag{43}$$

where

$$s^k \delta_n \leq \Delta_n. \tag{51}$$

Next, we prove that

$$M_f(v_n, \xi_n, v_{n+1}, \xi_{n+1}) \leq \max\{d(f v_n, f v_{n+1}), d(f v_{n+1}, f v_{n+2})\} \tag{44}$$

and

$$N_f(v_n, \xi_n, v_{n+1}, \xi_{n+1}) = \max\{d(f v_n, f v_{n+1}), d(f v_{n+1}, f v_{n+2})\}. \tag{45}$$

Therefore from (43), we have

$$\delta_n \leq \lambda \delta_{n-1}, \tag{52}$$

for all  $n \geq 1$  and where  $\lambda = \frac{1}{s^k} \in [0, 1)$ .

$$\phi(s^k d(f v_{n+1}, f v_{n+2})) \leq \phi(\max\{d(f v_n, f v_{n+1}), d(f v_{n+1}, f v_{n+2})\}) - \psi(\max\{d(f v_n, f v_{n+1}), d(f v_{n+1}, f v_{n+2})\}). \tag{46}$$



Suppose that if  $\Delta_n = \delta_n$  then from (51), we get  $s^k \delta_n \leq \delta_n$  which leads to  $\delta_n = 0$  as  $s > 1$  and hence (52) holds. If  $\Delta_n = \max\{d(fv_n, fv_{n+1}), d(f\xi_n, f\xi_{n+1})\}$ , i.e.,  $\Delta_n = \delta_{n-1}$  then (51) follows (52).

Now from (51), we obtain that  $\delta_n \leq \lambda^n \delta_0$  and hence,

$$d(fv_{n+1}, fv_{n+2}) \leq \lambda^n \delta_0 \text{ and } d(f\xi_{n+1}, f\xi_{n+2}) \leq \lambda^n \delta_0. \tag{53}$$

Therefore from Lemma 3.1 of [16], the sequences  $\{fv_n\}$  and  $\{f\xi_n\}$  are Cauchy sequences in  $P$ . Hence, by following the remaining proof of Theorem 2.2 of [4], we can show that  $S$  and  $f$  have a coincidence point in  $P$ .  $\square$

**Corollary 16** Let  $(P, d, s, \leq)$  be a complete partially ordered  $b$ -metric space with  $s > 1$ , and  $S : P \times P \rightarrow P$  be a continuous mapping such that  $S$  has a mixed monotone property. Suppose there exists  $\phi \in \Phi$  and  $\psi \in \Psi$  such that

$$\begin{aligned} &\phi(s^k d(S(v, \xi), S(\rho, \tau))) \\ &\leq \phi(M_f(v, \xi, \rho, \tau)) - \psi(N_f(v, \xi, \rho, \tau)), \end{aligned}$$

for all  $v, \xi, \rho, \tau \in P$  with  $v \leq \rho$  and  $\xi \geq \tau$ ,  $k > 2$  and where

$$M_f(v, \xi, \rho, \tau) = \max \left\{ \frac{d(\rho, S(\rho, \tau))[1 + d(v, S(v, \xi))]}{1 + d(v, \rho)}, \frac{d(v, S(v, \xi)) d(\rho, S(\rho, \tau))}{1 + d(S(v, \xi), S(\rho, \tau))}, \frac{d(v, S(v, \xi)) d(v, S(\rho, \tau))}{1 + d(v, S(\rho, \tau)) + d(\rho, S(v, \xi))}, d(v, \rho) \right\},$$

and

$$N_f(v, \xi, \rho, \tau) = \max \left\{ \frac{d(\rho, S(\rho, \tau))[1 + d(v, S(v, \xi))]}{1 + d(v, \rho)}, d(v, \rho) \right\}.$$

Then  $S$  has a coupled fixed point in  $P$ , if there exists  $(v_0, \xi_0) \in P \times P$  such that  $v_0 \leq S(v_0, \xi_0)$  and  $\xi_0 \geq S(\xi_0, v_0)$ .

**Proof** Set  $f = I_P$  in Theorem 15.  $\square$

**Corollary 17** Let  $(P, d, s, \leq)$  be a complete partially ordered  $b$ -metric space with  $s > 1$ , and  $S : P \times P \rightarrow P$  be a continuous mapping such that  $S$  has a mixed monotone property. Suppose there exists  $\psi \in \Psi$  such that

$$d(S(v, \xi), S(\rho, \tau)) \leq \frac{1}{s^k} M_f(v, \xi, \rho, \tau) - \frac{1}{s^k} \psi(N_f(v, \xi, \rho, \tau)),$$

for all  $v, \xi, \rho, \tau \in P$  with  $v \leq \rho$  and  $\xi \geq \tau$ ,  $k > 2$  where

$$M_f(v, \xi, \rho, \tau) = \max \left\{ \frac{d(\rho, S(\rho, \tau))[1 + d(v, S(v, \xi))]}{1 + d(v, \rho)}, \frac{d(v, S(v, \xi)) d(\rho, S(\rho, \tau))}{1 + d(S(v, \xi), S(\rho, \tau))}, \frac{d(v, S(v, \xi)) d(v, S(\rho, \tau))}{1 + d(v, S(\rho, \tau)) + d(\rho, S(v, \xi))}, d(v, \rho) \right\},$$

and

$$N_f(v, \xi, \rho, \tau) = \max \left\{ \frac{d(\rho, S(\rho, \tau))[1 + d(v, S(v, \xi))]}{1 + d(v, \rho)}, d(v, \rho) \right\}.$$

If there exists  $(v_0, \xi_0) \in P \times P$  such that  $v_0 \leq S(v_0, \xi_0)$  and  $\xi_0 \geq S(\xi_0, v_0)$ , then  $S$  has a coupled fixed point in  $P$ .

**Theorem 18** In addition to Theorem 15, if for all  $(v, \xi), (r, s) \in P \times P$ , there exists  $(c^*, d^*) \in P \times P$  such that  $(S(c^*, d^*), S(d^*, c^*))$  is comparable to  $(S(v, \xi), S(\xi, v))$  and to  $(S(r, s), S(s, r))$ , then  $S$  and  $f$  have a unique coupled common fixed point in  $P \times P$ .

**Proof** From Theorem 15, we know that there exists atleast one coupled coincidence point in  $P$  for  $S$  and  $f$ . Assume that  $(v, \xi)$  and  $(r, s)$  are two coupled coinci-

dence points of  $S$  and  $f$ , i.e.,  $S(v, \xi) = fv$ ,  $S(\xi, v) = f\xi$  and  $S(r, s) = fr$ ,  $S(s, r) = fs$ . Now, we have to prove that

$$fv = fr \text{ and } f\xi = fs.$$

From the hypotheses, there exists  $(c^*, d^*) \in P \times P$  such that  $(S(c^*, d^*), S(d^*, c^*))$  is comparable to  $(S(v, \xi), S(\xi, v))$  and to  $(S(r, s), S(s, r))$ . Suppose that

$$\begin{aligned} (S(v, \xi), S(\xi, v)) &\leq (S(c^*, d^*), S(d^*, c^*)) \text{ and} \\ (S(r, s), S(s, r)) &\leq (S(c^*, d^*), S(d^*, c^*)). \end{aligned}$$

Let  $c_0^* = c^*$  and  $d_0^* = d^*$  and then choose  $(c_1^*, d_1^*) \in P \times P$  as

$$fc_1^* = S(c_0^*, d_0^*), fd_1^* = S(d_0^*, c_0^*) \quad (n \geq 1).$$

By repeating the same procedure above, we can obtain two sequences  $\{fc_n^*\}$  and  $\{fd_n^*\}$  in  $P$  such that

$$fc_{n+1}^* = S(c_n^*, d_n^*), fd_{n+1}^* = S(d_n^*, c_n^*) \quad (n \geq 0).$$

Similarly, define the sequences  $\{fv_n\}$ ,  $\{f\xi_n\}$  and  $\{fr_n\}$ ,  $\{fs_n\}$  as above in  $P$  by setting  $v_0 = v$ ,  $\xi_0 = \xi$  and  $r_0 = r$ ,  $s_0 = s$ . Further, we have that

$$\begin{aligned} & \phi(\max\{d(fv, fc_{n+1}^*), d(f\xi, fd_{n+1}^*)\}) \\ & \leq \phi(\max\{d(fv, fc_n^*), d(f\xi, fd_n^*)\}) \\ & \quad - \psi(\max\{d(fv, fc_n^*), d(f\xi, fd_n^*)\}) \\ & < \phi(\max\{d(fv, fc_n^*), d(f\xi, fd_n^*)\}). \end{aligned} \tag{59}$$

Hence by the property of  $\phi$ , we get

$$fv_n \rightarrow S(v, \xi), f\xi_n \rightarrow S(\xi, v), fr_n \rightarrow S(r, s), fs_n \rightarrow S(s, r) \quad (n \geq 1). \tag{54}$$

Since,  $(S(v, \xi), S(\xi, v)) = (fv, f\xi) = (fv_1, f\xi_1)$  is comparable to  $(S(c^*, d^*), S(d^*, c^*)) = (fc^*, fd^*) = (fc_1^*, fd_1^*)$  and hence we get  $(fv_1, f\xi_1) \leq (fc_1^*, fd_1^*)$ . Thus, by induction we obtain that

$$(fv_n, f\xi_n) \leq (fc_n^*, fd_n^*) \quad (n \geq 0). \tag{55}$$

Therefore from (41), we have

$$\begin{aligned} & \max\{d(fv, fc_{n+1}^*), d(f\xi, fd_{n+1}^*)\} < \max\{d(fv, fc_n^*), d(f\xi, fd_n^*)\}, \\ & \text{which shows that } \max\{d(fv, fc_n^*), d(f\xi, fd_n^*)\} \text{ is a decreasing} \\ & \text{sequence and by a result there exists } \gamma \geq 0 \text{ such that} \\ & \lim_{n \rightarrow +\infty} \max\{d(fv, fc_n^*), d(f\xi, fd_n^*)\} = \gamma. \end{aligned}$$

From (59) taking upper limit as  $n \rightarrow +\infty$ , we get

$$\begin{aligned} \phi(d(fv, fc_{n+1}^*)) & \leq \phi(s^k d(fv, fc_{n+1}^*)) = \phi(s^k d(S(v, \xi), S(c_n^*, d_n^*))) \\ & \leq \phi(M_f(v, \xi, c_n^*, d_n^*)) - \psi(N_f(v, \xi, c_n^*, d_n^*)), \end{aligned} \tag{56}$$

where

$$\phi(\gamma) \leq \phi(\gamma) - \psi(\gamma), \tag{60}$$

$$\begin{aligned} M_f(v, \xi, c_n^*, d_n^*) & = \max \left\{ \frac{d(fc_n^*, S(c_n^*, d_n^*))[1 + d(fv, S(v, \xi))]}{1 + d(fv, fc_n^*)}, \frac{d(fv, S(v, \xi)) d(fc_n^*, S(c_n^*, d_n^*))}{1 + d(S(v, \xi), S(c_n^*, d_n^*))}, \right. \\ & \quad \left. \frac{d(fv, S(v, \xi)) d(fv, S(c_n^*, d_n^*))}{1 + d(fv, S(c_n^*, d_n^*)) + d(fc_n^*, S(v, \xi))}, d(fv, fc_n^*) \right\} \\ & = \max\{0, 0, 0, d(fv, fc_n^*)\} \\ & = d(fv, fc_n^*) \end{aligned}$$

and

from which we get  $\psi(\gamma) = 0$ , implies that  $\gamma = 0$ . Thus,

$$\begin{aligned} N_f(v, \xi, c_n^*, d_n^*) & = \max \left\{ \frac{d(fc_n^*, S(c_n^*, d_n^*))[1 + d(fv, S(v, \xi))]}{1 + d(fv, fc_n^*)}, d(fv, fc_n^*) \right\} \\ & = d(fv, fc_n^*). \end{aligned}$$

Thus from (56),

$$\lim_{n \rightarrow +\infty} \max\{d(fv, fc_n^*), d(f\xi, fd_n^*)\} = 0.$$

$$\phi(d(fv, fc_{n+1}^*)) \leq \phi(d(fv, fc_n^*)) - \psi(d(fv, fc_n^*)). \tag{57}$$

Consequently, we get

As by the similar process, we can prove that

$$\lim_{n \rightarrow +\infty} d(fv, fc_n^*) = 0 \text{ and } \lim_{n \rightarrow +\infty} d(f\xi, fd_n^*) = 0. \tag{61}$$

$$\phi(d(f\xi, fd_{n+1}^*)) \leq \phi(d(f\xi, fd_n^*)) - \psi(d(f\xi, fd_n^*)). \tag{58}$$

By similar argument, we get

From (57) and (58), we have

$$\lim_{n \rightarrow +\infty} d(fr, fc_n^*) = 0 \text{ and } \lim_{n \rightarrow +\infty} d(fs, fd_n^*) = 0. \tag{62}$$

Therefore from (61) and (62), we get  $f v = f r$  and  $f \xi = f s$ . Since  $f v = S(v, \xi)$  and  $f \xi = S(\xi, v)$ , then by the commutativity of  $S$  and  $f$ , we have

$$\begin{aligned} f(f v) &= f(S(v, \xi)) = S(f v, f \xi) \text{ and} \\ f(f \xi) &= f(S(\xi, v)) = S(f \xi, f v). \end{aligned} \tag{63}$$

Let  $f v = a^*$  and  $f \xi = b^*$  then (63) becomes

$$f(a^*) = S(a^*, b^*) \text{ and } f(b^*) = S(b^*, a^*), \tag{64}$$

which shows that  $(a^*, b^*)$  is a coupled coincidence point of  $S$  and  $f$ . It follows that  $f(a^*) = f r$  and  $f(b^*) = f s$  that is  $f(a^*) = a^*$  and  $f(b^*) = b^*$ . Thus from (64), we get  $a^* = f(a^*) = S(a^*, b^*)$  and  $b^* = f(b^*) = S(b^*, a^*)$ . Therefore,  $(a^*, b^*)$  is a coupled common fixed point of  $S$  and  $f$ .

For the uniqueness let  $(u^*, v^*)$  be another coupled common fixed point of  $S$  and  $f$ , then we have  $u^* = f u^* = S(u^*, v^*)$  and  $v^* = f v^* = S(v^*, u^*)$ . Since  $(u^*, v^*)$  is a coupled common fixed point of  $S$  and  $f$ , then we obtain  $f u^* = f v = a^*$  and  $f v^* = f \xi = b^*$ . Thus,  $u^* = f u^* = f a^* = a^*$  and  $v^* = f v^* = f b^* = b^*$ . Hence the result.  $\square$

**Theorem 19** *In addition to the hypotheses of Theorem 18, if  $f v_0$  and  $f \xi_0$  are comparable, then  $S$  and  $f$  have a unique common fixed point in  $P$ .*

**Proof** From Theorem 18,  $S$  and  $f$  have a unique coupled common fixed point  $(v, \xi) \in P$ . Now, it is enough to prove that  $v = \xi$ . From the hypotheses, we have  $f v_0$  and  $f \xi_0$  are comparable then we assume that  $f v_0 \preceq f \xi_0$ . Hence by induction we get  $f v_n \preceq f \xi_n$  for all  $n \geq 0$ , where  $\{f v_n\}$  and  $\{f \xi_n\}$  are from Theorem 15.

Now by use of Lemma 6, we get

$$\begin{aligned} \phi(s^{k-2} d(v, \xi)) &= \phi(s^k \frac{1}{s^2} d(v, \xi)) \leq \limsup_{n \rightarrow +\infty} \phi(s^k d(v_{n+1}, \xi_{n+1})) \\ &= \limsup_{n \rightarrow +\infty} \phi(s^k d(S(v_n, \xi_n), S(\xi_n, v_n))) \\ &\leq \limsup_{n \rightarrow +\infty} \phi(M_f(v_n, \xi_n, \xi_n, v_n)) - \liminf_{n \rightarrow +\infty} \psi(N_f(v_n, \xi_n, \xi_n, v_n)) \\ &\leq \phi(d(v, \xi)) - \liminf_{n \rightarrow +\infty} \psi(N_f(v_n, \xi_n, \xi_n, v_n)) \\ &< \phi(d(v, \xi)), \end{aligned}$$

which is a contradiction. Thus,  $v = \xi$ , i.e.,  $S$  and  $f$  have a common fixed point in  $P$ .  $\square$

**Remark 20**

*It is well known that  $b$ -metric space is a metric space when  $s = 1$ . So, from the result of Jachymski [39], the condition*

$$\begin{aligned} \phi(d(S(v, \xi), S(\rho, \tau))) &\leq \phi(\max\{d(f v, f \rho), \\ &d(f \xi, f \tau)\}) - \psi(\max\{d(f v, f \rho), d(f \xi, f \tau)\}) \end{aligned}$$

is equivalent to,

$$d(S(v, \xi), S(\rho, \tau)) \leq \varphi(\max\{d(f v, f \rho), d(f \xi, f \tau)\}),$$

where  $\phi \in \Phi$ ,  $\psi \in \Psi$  and  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is continuous,  $\varphi(t) < t$  for all  $t > 0$  and  $\varphi(t) = 0$  if and only if  $t = 0$ . So, in view of above our results generalize and extend the results of [19, 20, 40–42] and several other comparable results.

**Corollary 21** *Suppose  $(P, d, s, \preceq)$  be a complete partially ordered  $b$ -metric space with parameter  $s > 1$ . Let  $S : P \rightarrow P$  be a continuous, nondecreasing map with regards to  $\preceq$  such that there exists  $v_0 \in P$  with  $v_0 \preceq S v_0$ . Suppose that*

$$\phi(sd(Sv, S\xi)) \leq \phi(M(v, \xi)) - \psi(M(v, \xi)), \tag{65}$$

where  $M(v, \xi)$  and the conditions upon  $\phi, \psi$  are same as in Theorem 8. Then  $S$  has a fixed point in  $P$ .

**Proof** Set  $N(v, \xi) = M(v, \xi)$  in a contraction condition (3) and apply Theorem 8, we have the required proof.  $\square$

**Note 22** Similarly by removing the continuity of a nondecreasing mapping  $S$  and taking a nondecreasing sequence  $\{v_n\}$  as above in Theorem 9, we can obtain a fixed point for  $S$  in  $P$ . Also one can obtains the uniqueness of a fixed point of  $S$  by using condition (17) in  $P$  and following the proof of Theorem 10.

**Note 23** By following the proofs of Theorem 12 and 13, we can find the coincidence point for  $S$  and  $f$  in  $P$ . Similarly, from Theorem 15, 18 and 19, one can obtain a coupled coincidence point and its uniqueness, and a unique common fixed point for mappings  $S$  and  $f$  in  $P \times P$  satisfying an almost generalized contraction condition (65), where  $M_f(v, \xi)$ ,  $M_f(v, \xi, \rho, \tau)$  and the conditions upon  $\phi, \psi$  are same as above defined.

**Corollary 24** Suppose that  $(P, d, s, \preceq)$  be a complete partially ordered  $b$ -metric space with  $s > 1$ . Let  $S : P \rightarrow P$  be a continuous, nondecreasing mapping with regards to  $\preceq$ . If there exists  $k \in [0, 1)$  and for any  $v, \xi \in P$  with  $v \preceq \xi$  such that

$$d(Sv, S\xi) \leq \frac{k}{s} \max \left\{ \frac{d(\xi, S\xi)[1 + d(v, Sv)]}{1 + d(v, \xi)}, \frac{d(v, Sv) d(\xi, S\xi)}{1 + d(Sv, S\xi)}, \frac{d(v, Sv) d(v, S\xi)}{1 + d(v, S\xi) + d(\xi, Sv)}, d(v, \xi) \right\}. \tag{66}$$

If there exists  $v_0 \in P$  with  $v_0 \preceq Sv_0$ , then  $S$  has a fixed point in  $P$ .

**Proof** Set  $\phi(t) = t$  and  $\psi(t) = (1 - k)t$ , for all  $t \in (0, +\infty)$  in Corollary 21.  $\square$

**Note 25** Relaxing the continuity of a map  $S$  in Corollary 24, one can obtain a fixed point for  $S$  on taking a nondecreasing sequence  $\{v_n\}$  in  $P$  by following the proof of Theorem 9.

We illustrate the usefulness of the obtained results in different cases such as continuity and discontinuity of a metric  $d$  in a space  $P$ .

**Example 26**

Define a metric  $d : P \rightarrow P$  as below and  $\leq$  is an usual order on  $P$ , where  $P = \{1, 2, 3, 4, 5, 6\}$

$$\begin{aligned} d(v, \xi) &= d(v, \xi) = 0, & \text{if } v, \xi &= 1, 2, 3, 4, 5, 6 \text{ and } v = \xi, \\ d(v, \xi) &= d(v, \xi) = 3, & \text{if } v, \xi &= 1, 2, 3, 4, 5 \text{ and } v \neq \xi, \\ d(v, \xi) &= d(v, \xi) = 12, & \text{if } v &= 1, 2, 3, 4 \text{ and } v = 6, \\ d(v, \xi) &= d(v, \xi) = 20, & \text{if } v &= 5 \text{ and } \xi = 6. \end{aligned}$$

Define a map  $S : P \rightarrow P$  by  $S1 = S2 = S3 = S4 = S5 = 1, S6 = 2$  and let  $\phi(t) = \frac{t}{2}, \psi(t) = \frac{t}{4}$  for  $t \in [0, +\infty)$ . Then  $S$  has a fixed point in  $P$ .

**Proof** It is apparent that,  $(P, d, s, \preceq)$  is a complete partially ordered  $b$ -metric space for  $s = 2$ . Consider the possible cases for  $v, \xi$  in  $P$ :

**Case 1.** Suppose  $v, \xi \in \{1, 2, 3, 4, 5\}, v < \xi$  then  $d(Sv, S\xi) = d(1, 1) = 0$ . Hence,

$$\phi(2d(Sv, S\xi)) = 0 \leq \phi(M(v, \xi)) - \psi(M(v, \xi)).$$

**Case 2.** Suppose that  $v \in \{1, 2, 3, 4, 5\}$  and  $\xi = 6$ , then  $d(Sv, S\xi) = d(1, 2) = 3, M(6, 5) = 20$  and  $M(v, 6) = 12$ , for  $v \in \{1, 2, 3, 4\}$ . Therefore, we have the following inequality,

$$\phi(2d(Sv, S\xi)) \leq \frac{M(v, \xi)}{4} = \phi(M(v, \xi)) - \psi(M(v, \xi)).$$

Thus, condition (65) of Corollary 21 holds. Furthermore, the remaining assumptions in Corollary 21 are fulfilled. Hence,  $S$  has a fixed point in  $P$  as Corollary 21 is appropriate to  $S, \phi, \psi$  and  $(P, d, s, \preceq)$ .  $\square$

**Example 27** A metric  $d : P \rightarrow P$ , where  $P = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\}$  with usual order  $\leq$  is as follows

$$d(v, \xi) = \begin{cases} 0 & , \text{if } v = \xi \\ 1 & , \text{if } v \neq \xi \in \{0, 1\} \\ |v - \xi| & , \text{if } v, \xi \in \left\{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \geq 1\right\} \\ 6 & , \text{otherwise.} \end{cases}$$

A map  $S : P \rightarrow P$  be such that  $S0 = 0, S\frac{1}{n} = \frac{1}{12n}$  for all  $n \geq 1$  and let  $\phi(t) = t, \psi(t) = \frac{4t}{5}$  for  $t \in [0, +\infty)$ . Then,  $S$  has a fixed point in  $P$ .

**Proof** It is obvious that for  $s = \frac{12}{5}, (P, d, s, \preceq)$  is a complete partially ordered  $b$ -metric space and also by definition,  $d$  is discontinuous  $b$ -metric space. Now for  $v, \xi \in P$  with  $v < \xi$ , then we have the following cases:

**Case 1.** If  $v = 0$  and  $\xi = \frac{1}{n}, n \geq 1$ , then  $d(Sv, S\xi) = d(0, \frac{1}{12n}) = \frac{1}{12n}$  and  $M(v, \xi) = \frac{1}{n}$  or  $M(v, \xi) = \{1, 6\}$ . Therefore, we have

$$\phi\left(\frac{12}{5}d(Sv, S\xi)\right) \leq \frac{M(v, \xi)}{5} = \phi(M(v, \xi)) - \psi(M(v, \xi)).$$

**Case 2.** If  $v = \frac{1}{m}$  and  $\xi = \frac{1}{n}$  with  $m > n \geq 1$ , then

$$\begin{aligned} d(Sv, S\xi) &= d\left(\frac{1}{12m}, \frac{1}{12n}\right) \text{ and} \\ M(v, \xi) &\geq \frac{1}{n} - \frac{1}{m} \text{ or } M(v, \xi) = 6. \end{aligned}$$

Therefore,

$$\phi\left(\frac{12}{5}d(Sv, S\xi)\right) \leq \frac{M(v, \xi)}{5} = \phi(M(v, \xi)) - \psi(M(v, \xi)).$$

Hence, condition (65) of Corollary 21 and remaining assumptions are satisfied. Thus,  $S$  has a fixed point in  $P$ .  $\square$

**Example 28** Let  $P = C[a, b]$  be the set of all continuous functions. Let us define a  $b$ -metric  $d$  on  $P$  by

$$d(\theta_1, \theta_2) = \sup_{t \in C[a, b]} \{|\theta_1(t) - \theta_2(t)|^2\}$$

for all  $\theta_1, \theta_2 \in P$  with partial order  $\leq$  defined by  $\theta_1 \leq \theta_2$  if  $a \leq \theta_1(t) \leq \theta_2(t) \leq b$ , for all  $t \in [a, b]$ ,  $0 \leq a < b$ . Let  $S : P \rightarrow P$  be a mapping defined by  $S\theta = \frac{\theta}{5}$ ,  $\theta \in P$  and the two altering distance functions by  $\phi(t) = t$ ,  $\psi(t) = \frac{t}{3}$ , for any  $t \in [0, +\infty]$ . Then  $S$  has a unique fixed point in  $P$ .

**Proof** By the hypotheses, it is clear that  $(P, d, s, \leq)$  is a complete partially ordered  $b$ -metric space with parameter  $s = 2$  and fulfill all conditions of Corollary 21 and Note 22. Furthermore for any  $\theta_1, \theta_2 \in P$ , the function  $\min(\theta_1, \theta_2)(t) = \min\{\theta_1(t), \theta_2(t)\}$  is also continuous and the conditions of Corollary 21 and Note 22 are satisfied. Hence,  $S$  has a unique fixed point  $\theta = 0$  in  $P$ .  $\square$

## Limitations

In this manuscript, some fixed point, coincidence point, coupled coincidence point and coupled common fixed point results for mappings satisfying generalized  $(\phi, \psi)$ -contraction conditions in complete partially ordered  $b$ -metric spaces are proved. These results generalize and extend some known results in the existing literature. Few examples are presented at the end to support our results.

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## Author details

<sup>1</sup> Department of Applied Mathematics, School of Applied Natural Sciences, Adama Science and Technology University, Post Box No.1888, Adama, Ethiopia.

<sup>2</sup> Department of Mathematics, Vignar's Foundation for Science, Technology and Research, Vadlamudi, Andhra Pradesh 522213, India.

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