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Analytic Nakagami fading parameter estimation in dependent noise channel using copula

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Abstract

In this paper, the probability density function (PDF) estimation is introduced in the framework of estimating the Nakagami fading parameter. This approach provides an analytic procedure for finding the fading parameter. Using the copula theory, an accurate PDF estimate is obtained even when the desired signal is corrupted in a noisy environment. In the real world, the noise samples could be highly dependent on the main signal. Copula-based models are a general set of statistical models defined for any multivariate random variable. Thus, they depict the statistical behavior of a received signal including two dependent terms, representative of the desired signal and noise. Previous works in the Nakagami parameter determination have mainly examined estimation based on either a noiseless sample model or an independent trivial noisy one. In this paper, we consider a more comprehensive situation about the noise destruction and our investigation is done in low signal-to-noise ratios. The parametric bootstrap method approves the accuracy of the analytically estimated PDF, and simulation results show that the new estimator has superior performance over conventional estimators.

Keywords: Nakagami fading; PDF estimation; Dependent noise; Copula theory

1 Introduction

The Nakagami-m distribution is considered as one of the most important models among all the statistical ones that have been proposed to characterize the fading envelope due to multipath fading in wireless communications [1]. With a simple exponential family form, the Nakagami-m distribution often leads to closed-form analytical results. The Nakagami fading exploits Nakagami probability density function (PDF) for the envelope of received signal which possesses two parameters: scale and shape parameters. The latter is more important and called the fading parameter or m-parameter. Determining m is a problem in Nakagami PDF estimation.

The most prominent conventional procedures used for the estimation of the Nakagami fading parameter, m, are based on either maximum likelihood estimation or moment-based estimators [2,3]. Among maximum likelihood (ML)-based conventional methods, the Greenwood-Durand estimator is well known in m-parameter estimation [4]. There are also analytic and bootstrap bias-corrected ML estimators for estimating m that improve conventional ML estimators [5]. On the other hand, the inverse normalized variance and generalized method of moments (GMM) are the moment-based procedures, in which the latter presents the Nakagami parameter estimation in noisy environment with acceptable performance [4].

However, all aforementioned approaches either do not take into account noisy cases or consider trivial noises. In this paper, we intend to estimate the Nakagami parameter in a dependent noise environment based on the PDF of received signal and the copula concept. We present a comprehensive noise model that is not confined to restrictive assumptions, such as uncorrelatedness or independence. Signal-dependent noise is used in the paper which is a more realistic assumption in practical applications such as image transmission [6], radar [7], and wireless communications [8]. The copula theory is one of the best methods used for modeling the dependency in conventional works

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[9]. The copula theory is an elegant concept introduced by Sklar in order to find the link between a joint PDF and its marginal PDFs [10]. Then, copula functions have been extensively studied and a comprehensive discussion of their mathematical properties has been presented [11,12].

Now, a new copula-based method is presented to estimate the Nakagami fading parameter in the faded received signal contaminated by dependent noise. The novelty of our approach lies on determining the analytic PDF of the received data under the assumption of dependent noise by using the copula theory in order to estimate the parameter of the Nakagami-*m* fading model. To measure the accuracy of the estimation approach, the result is examined by using a goodness of fit test called parametric bootstrap algorithm [13].

This paper is organized as follows: In Section 2, we recall basic facts and definitions on copulas and the dependency problem. Section 3 includes the main signal and noise model used in the paper. The introduced models are such comprehensive ones which precisely correspond to actual signals. Using copula, the PDF of the received signal is estimated in Section 4, and the fading parameter is determined. Finally, simulation results are given in Section 5, and some conclusions are drawn in Section 6.

2 Copula

One popular method of modeling the dependencies is the copula approach. The word copula is a Latin noun which means 'a link, tie, or bond' and was first employed by Sklar in mathematical and statistical problems [10]. Mathematically, copula is a function that combines univariate PDFs to obtain a joint PDF with a particular dependency structure. In this paper, the fading parameter is estimated based on the PDF of the received signal, given that the received signal has been corrupted by a signal-dependent noise. Due to the signal-dependent nature of noise, we are required to determine the PDF of a signal that is composed of two dependent components. Thus, the copula concept is a tool that is compatible with our problem, and it facilitates the PDF estimation procedure. The foundation theorem for copula was introduced by Sklar which states that for a given joint multivariate PDF and the relevant marginal PDFs, there exists a copula function that relates them. In a bivariate case, Sklar's theorem is as follows:

Let F_{xy} be a joint CDF with margins F_x and F_y. Then, there exists a function $C : [0, 1]^2 \to [0, 1]$ such that:

$$F_{xy}(x, y) = C\left(F_x(x), F_y(y)\right). \tag{1}$$

If F_x and F_y are continuous, then C is unique; otherwise, C is uniquely determined on the (range of F_x) $\times$ (range of F_y).

Conversely, if C is a copula and F_x and F_y are CDFs, then the function F_{xy} defined by (1) is a joint CDF with margins F_x and F_y.

The proof of the theorem can be found in [12]. Since C is a rather particular type of function, it possesses some inherent properties. A thorough description of these properties is found in [12]. The mentioned properties state that a copula is itself a CDF, defined on $[0, 1]^2$, with uniform margins.

Building multivariate CDFs by applying the copula approach provides a suitable flexibility because it allows to choose separately the margins and their dependence relationship [14]. For any copula function, there is a corresponding copula density function, which is the mixed partial derivative of function C, and can be given by:

$$c(u, v) = \frac{f_{\mathrm{xy}}(u, v)}{f_{\mathrm{x}}(u) f_{\mathrm{y}}(v)}, \tag{2}$$

where f_{xy}, f_{x}, and f_{y} are the PDFs related to the CDFs presented in (1). Equation (2) can be expressed in an equivalent and more suitable form:

$$f_{\mathrm{xy}}(x, y) = c(u, v) f_{\mathrm{x}}(x) f_{\mathrm{y}}(y), \tag{3}$$

where u, v are related to x, y through the marginal CDFs:

$$\begin{aligned} u &= F_{\mathrm{x}}(x), \\ v &= F_{\mathrm{y}}(y). \end{aligned} \tag{4}$$

The copulas have two main families. One of them is the family of elliptical copulas. The most common elliptical copulas are normal and Student's *t*. The key advantage of an elliptical copula is that one can specify different levels of dependency between the margins. Another important class of copulas is known as the Archimedean copulas. Archimedean copulas are popular because they are constructed easily and allow modeling the dependence in arbitrarily high dimensions with only one parameter, governing the strength of dependence [12].

In this paper, the bivariate normal and Clayton copulas are applied as the representative of both families. The normal copula is given by:

$$\begin{aligned} C_\rho(u, v) = &\int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \\ &\times \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2\left(1-\rho^2\right)}\right\} dx\, dy, \end{aligned} \tag{5}$$

where ρ is the normal copula parameter and Φ^{-1} is the inverse of the univariate standard normal CDF. It is called the normal copula because, similar to bivariate normal distribution, it also enforces dependency by using pairwise correlations among the variables. However, in the normal copula, the marginal distributions are arbitrary.

On the other hand, the Clayton copula is given by:

$$C_\alpha(u, v) = \left[-1 + u^{-\alpha} + v^{-\alpha}\right]^{-1/\alpha}, \quad \alpha > 0, \tag{6}$$

where α is the Clayton copula parameter.

3 Signal model

In this section, a new model is introduced for the received signal in which noise and fading are statistically embedded. By estimating the PDF of the received signal, the nonlinear minimum mean square error (NMMSE) estimator principle is invoked to estimate the fading parameter. To transmit information over a communication channel, there are different methods to modulate the information. Quadrature amplitude modulation, phase shift keying, frequency shift keying, and continuous phase modulation are some prominent modulation methods. All these methods use a sinusoidal function as the carrier signal. Hence, we base signaling assumption on a sinusoidal transmission entering a multipath environment infested by generally a non-Gaussian noise. Thus, the procedure is extensible to all of the above types of modulation schemes.

Suppose that the main transmitted signal is in the following form:

$$s_c(t) = A\,\cos(\omega_c t + \theta), \tag{7}$$

where $A > 0$ and θ are the amplitude and the phase of the transmitted signal, respectively, and ω_c is the carrier frequency. Let us assume $s_c(t)$ enters a multipath channel with L distinct paths. Then, the output of this channel is expressed as follows:

$$\begin{aligned} s(t) &= \sum_{i=1}^{L} a_i A \cos\left(\omega_c t + \omega_{d_i} t + \theta + \theta_i\right) \\ &= \Re\left\{\sum_{i=1}^{L} a_i A \exp\left(j\omega_{d_i} t + j\theta + j\theta_i\right)\exp\left(j\omega_c t\right)\right\}, \end{aligned} \tag{8}$$

where a_i, ω_{d_i}, and θ_i are the attenuation factor, Doppler shift, and the phase on the ith path, respectively, and $\Re(\cdot)$ denotes the real part of its argument.

Define:

$$s_\ell(t) \triangleq \sum_{i=1}^{L} a_i \exp\left(j\omega_{d_i} t + j\theta + j\theta_i\right). \tag{9}$$

According to (8) and (9),

$$\begin{aligned} s(t) &= \Re\left\{s_\ell(t)\, A \exp\left(j\omega_c t\right)\right\} \\ &= R(t)\; A\cos\left(\omega_c t + \Theta(t)\right). \end{aligned} \tag{10}$$

The envelope R and the phase Θ are given by:

$$R(t) = \sqrt{I^2(t) + Q^2(t)} \tag{11}$$

$$\Theta(t) = \arctan\left(\frac{Q(t)}{I(t)}\right), \tag{12}$$

where $I(t)$ and $Q(t)$ are the inphase and quadrature components of signal $s(t)/A$. When L is large, the inphase and quadrature components will be normally distributed. Therefore, the phase Θ is uniformly distributed over $[0, 2\pi)$.

Due to various fading channels, i.e., short-term, long-term, and mixed fading, different PDF models are proposed for $R(t)$. Some new models, the so-called 'multiplicative' fading models, have also been developed recently [15]. In short-term, mixed, and multiplicative fading models, the Nakagami-m distribution is an accurate versatile PDF. Therefore, we use Nakagami or m-distribution to describe the fading envelope in multipath environments. It has the following PDF:

$$f_{\mathrm{R}}(r_o) = \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m {r_o}^{2m-1}\exp\left(-\frac{m{r_o}^2}{\Omega}\right), \quad r_o \geqslant 0, \tag{13}$$

where $\Gamma(\cdot)$ is the gamma function, $\Omega = E\{R^2\}$ is the scale parameter, and the shape parameter is expressed as follows:

$$m = \frac{\{E[R^2]\}^2}{\mathrm{Var}[R^2]} = \frac{\Omega^2}{E[(R^2-\Omega)^2]}, \tag{14}$$

with the constraint $m \geqslant 0.5$. The corresponding cumulative distribution function (CDF) can be expressed as:

$$F_{\mathrm{R}}(r_o) = P\left(\frac{m{r_o}^2}{\Omega}, m\right), \tag{15}$$

where $P(\cdot,\cdot)$ is the incomplete gamma function. Nakagami is a general fading distribution that reduces to the Rayleigh for $m = 1$ and to the one-sided normal distribution for $m = 0.5$. It also approximates the Rician and lognormal distributions.

The scale parameter Ω is defined as the second-order moment of the Nakagami-m fading envelope and is calculated straightforward. Thus, we focus on estimating the primary parameter, i.e., the shape parameter m.

On the other hand, when the envelope $R(t)$ has Nakagami distribution, we can assume that the phase $\Theta(t)$ is uniformly distributed over $[0, 2\pi)$ [16].

In order to start the formulation of our problem, firstly, let us suppose that $s(t)$ in (8) is received without any additive noise. So, the received signal is $s(t)$, and the PDF of $s(t)$ is needed. Based on (10), we find the PDF of the product of two random variables (RVs) at time t, i.e., $R(t)$ and $A\cos(\omega_c t + \Theta(t))$.

Assume that Z_1 and Z_2 are two RVs. If $Z = Z_1 Z_2$, then the PDF of the RV Z is:

$$f_{\mathrm{Z}}(z) = \int_{-\infty}^{\infty} \frac{1}{|\eta|} f_{\mathrm{Z_1 Z_2}}(\eta, z/\eta)\, d\eta, \tag{16}$$

where η is an auxiliary variable. The signal $s(t)$ in (10) is considered as the product of the two following signals:

$$\begin{aligned} s_1(t) &\triangleq R(t), \\ s_2(t) &\triangleq A\,\cos\left(\omega_c t + \Theta(t)\right). \end{aligned} \tag{17}$$

The PDF of $s_1(t)$ is the Nakagami distribution presented in (13) and the signal $s_2(t)$ has the following PDF:

$$f_{s_2}(s_2) = \frac{1}{\pi\sqrt{A^2 - s_2^2}}, \quad |s_2| < A. \tag{18}$$

Since the envelope $R(t)$ and phase $\Theta(t)$ are independent processes [16], hence, $s_1(t)$ and $s_2(t)$ are also independent. So, by using (16), the PDF of $s(t)$ is determined analytically:

$$\begin{aligned} f_s(s) &= \frac{2}{\pi\,\Gamma(m)}\left(\frac{m}{\Omega}\right)^m \int_{s/A}^{\infty} \frac{\eta^{2m-1}}{\sqrt{A^2\eta^2 - s^2}} \exp\left(-\frac{m\eta^2}{\Omega}\right) d\eta \\ &= \frac{2\left(\frac{m}{\Omega}\right)^m}{\pi\, A^{2m}\Gamma(m)} \exp\left(\frac{-ms^2}{\Omega A^2}\right) \\ &\quad \times \int_0^{\infty} \left(\kappa^2 + s^2\right)^{m-1} \exp\left(-\frac{m\kappa^2}{\Omega A^2}\right)\, d\kappa, \end{aligned} \tag{19}$$

where κ is an auxiliary variable. A detailed proof for (18) and (19) is provided in Appendix 1. By using binomial expansion of $(\kappa^2 + s^2)^{m-1}$, (19) is expressed for integer m as:

$$\begin{aligned} f_s(s) &= \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{\Gamma\left(k + 1/2\right)}{\pi\,\Gamma(m)} \left(\frac{m}{\Omega A^2}\right)^{m-k-1/2} \\ &\quad \times s^{2(m-1-k)} \exp\left(\frac{-ms^2}{\Omega A^2}\right). \end{aligned} \tag{20}$$

In (19) and (20), we obtained an expression for the PDF of the received signal $s(t)$ in a noiseless scenario. Next, we intend to derive an expression if a noise term is also present along with $s(t)$.

There are signal-dependent noise (SDN) channels in which the noise characteristics depend highly on the transmitted signal [8]. The application for such dependency is also in radar and sonar systems [17]. In this paper, we introduce a compound channel that models both fading effects and signal-dependent noise behavior. The cascaded channel modeling is used which was presented in [15]. There are several sub-channels in [15] that are all representative of Nakagami fading. Two cascaded Nakagami fading sub-channels were a special case expressed mathematically in [15]. We also introduce a channel constructed from two sub-channels, the first one being a Nakagami fading channel similar to [15] and its output being the faded signal expressed in (10). However, unlike the proposed method in [15], we assume the second sub-channel as an SDN channel. This assumption stems from the fact that the statistical model for a cascaded channel is not unique. A general relationship between the input and output of an SDN channel is given by [18]:

$$S_o(t) = S_i(t) + f(S_i(t))\,\mathcal{N}_u(t), \tag{21}$$

where $S_i(\cdot)$ and $S_o(\cdot)$ are the input and output of the SDN channel, respectively, $f(\cdot)$ is an arbitrary nonlinear function, and $\mathcal{N}_u(t)$ is a normal random process with zero mean and unit variance. Regardless of the kind of function $f(\cdot)$, our proposed method would be applicable, if the PDF of function $f(\cdot)$ is computable. In this paper, for simplicity, we define:

$$f(S_i(t)) \triangleq |S_i(t)|^{\gamma}, \tag{22}$$

where the parameter γ is between 0 and 1. It is well known that the model defined in (22) mimics many types of random variates [18,19]. This second sub-channel signifies the statistical importance of signal characteristics which could be modified by a random medium denoted by $\mathcal{N}_u(t)$.

There is also an additive independent noise $\nu(t)$ that complements our new modeling. It is a new model that simultaneously covers fading and noise dependency. This channel is shown in Figure 1. The signal s_c in Figure 1 is the main transmitted signal in (7). Let us define:

$$n(t) \triangleq |s(t)|^{\gamma}\,\mathcal{N}_u(t) + \nu(t), \tag{23}$$

where $\nu(t)$ is an independent zero-mean normally distributed random process whose variance is a random variate, and $s(t)$ is the faded signal in (10). $n(t)$ changes the received signal from faded signal in (10) to $r(t)$ in the following form:

$$r(t) = R(t)\, A\cos(\omega_c t + \Theta(t)) + n(t). \tag{24}$$

The dependency between the faded signal and noise helps to have a more actual model for the received signal.

The samples of signal $\nu(t)$ can be at hand, for example, by using an out-of-band measurement of the noise. In this way, the data transmission is not interrupted and the samples of $\nu(t)$ can be measured as accurately as needed. In other words, in (23), during the times that out-of-band measurements are performed, the presence of the signal is contemplated as $s(t) = 0$; therefore, this allows us to estimate the parameters of $\nu(t)$. Alternatively, the samples can also be obtained by sending a 'zero' signal and sampling the output of the receiver.

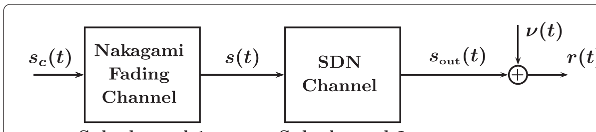

Figure 1 The proposed channel model. The output of SDN channel is $s_{\text{out}} = s(t) + |s(t)|^{\gamma}\,\boldsymbol{\mathcal{N}}_u(t)$.

Now, since we have the samples of the noise $\nu(t)$, it is possible to estimate its variance given the samples. It is proven in Appendix 2 that if the variance of the normally distributed noise $\nu(t)$ becomes a random variate, the resulted noise process $\nu(t)$ will have the K-distribution that is presented in the following:

$$f_\nu(\nu) = \frac{1}{b\sqrt{\pi}\Gamma\left(\frac{N_1}{2}\right)}\left(\frac{|\nu|}{2b}\right)^{\frac{N_1-1}{2}} \times K_{\frac{N_1-1}{2}}\left(\frac{|\nu|}{b}\right), \quad -\infty < \nu < \infty, \tag{25}$$

for $b > 0$ and $N_1 > 1$, where K is a modified Bessel function of the second kind. The K-distribution is a useful PDF for modeling reverberation in sonar [20] and clutter in radar [21]. Its characteristic function is [20]:

$$\psi_\nu(\omega) = \frac{1}{\left(1+b^2\omega^2\right)^{\frac{N_1}{2}}}. \tag{26}$$

The power of signal $\nu(t)$ is:

$$\sigma_\nu^2 = E\left[\nu^2\right] = E[\tau]\,E\left[N^2\right] = N_1 b^2. \tag{27}$$

The new presented model helps to ensure that final results are reliable even in an actual non-Gaussian environment.

The PDF of the dependent noise $n(t)$ is obtained analytically using (23). Since the signals $s(t)$, $\mathcal{N}_u(t)$, and $\nu(t)$ are independent, the PDF of $n(t)$ is calculated in the following form:

$$f_{\mathsf{n}}(n) = \frac{1}{\gamma\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(f_{\mathsf{s}}\left(\eta^{\frac{1}{\gamma}}\right) + f_{\mathsf{s}}\left(-\eta^{\frac{1}{\gamma}}\right)\right) \times f_\nu(\nu)\exp\left(-\frac{1}{2}\left(\frac{n-\nu}{\eta}\right)^2\right)\left|\eta^{\frac{1}{\gamma}-2}\right| d\eta d\nu, \tag{28}$$

where $f_{\mathsf{s}}(s)$ and $f_\nu(\nu)$ are obtained from (19) and (25), respectively, and η is an auxiliary variable.

Up to now, both PDF of the faded signal and the dependent noise were estimated analytically. In the next section, we determine the total PDF of the received signal.

4 Parameter estimation

Our final priority is to compute the conglomerate PDF of the faded signal and noise. The received signal in (24) can be expressed as:

$$r(t) = s(t) + n(t). \tag{29}$$

In order to compute the PDF for $r(t)$, it is often supposed that the signals $s(t)$ and $n(t)$ are independent, so the PDF of $r(t)$ is simply obtained using the convolution of the PDFs $f_{\mathsf{s}}(s)$ and $f_{\mathsf{n}}(n)$. However, in the actual noisy background, it is more realistic to assume that there is a dependency between the faded signal and the noise. Now is the time that the copula theory can be used and it presents an effective role on the final estimated PDF. The PDF of $r(t)$ is presented as:

$$f_{\mathsf{r}}(r) = \int_{-\infty}^{\infty} f_{\mathsf{sn}}(r-n, n)\, dn. \tag{30}$$

So, the joint PDF f_{sn} should be estimated. If we appeal to the copula theory for determining the joint PDF, the PDFs of the signals $s(t)$ and $n(t)$ should be considered as the marginal density functions in the copula discussion. From (3), the joint PDF is exhibited as:

$$f_{\mathsf{sn}}(s,n) = f_{\mathsf{s}}(s) f_{\mathsf{n}}(n)\, c(F_{\mathsf{s}}(s), F_{\mathsf{n}}(n); \theta_c), \tag{31}$$

where $F_{\mathsf{s}}(s)$ and $F_{\mathsf{n}}(n)$ are the marginal CDFs of the faded signal and noise, and θ_c is the dependence parameter in the copula density function. Therefore, the joint PDF is equal to the product of the marginal PDFs and the copula density function, i.e., the term $c(F_{\mathsf{s}}(s), F_{\mathsf{n}}(n); \theta_c)$.

As mentioned, two copula functions are used. The normal copula is selected among the elliptical group, and the Clayton copula is also employed from another family, i.e., Archimedean. Using the copula functions presented in (5) and (6), the related copula density functions can be extracted [22]:

$$c_\rho(F_{\mathsf{s}}(s), F_{\mathsf{n}}(n); \rho) = \frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{s^2 - 2\rho\, sn + n^2}{2\left(1-\rho^2\right)}\right\} \times \exp\left(\frac{s^2}{2}\right)\exp\left(\frac{n^2}{2}\right) \tag{32}$$

and

$$c_\alpha(F_{\mathsf{s}}(s), F_{\mathsf{n}}(n); \alpha) = (1+\alpha)\left(F_{\mathsf{s}}(s)F_{\mathsf{n}}(n)\right)^{-\alpha-1} \times\left[-1 + F_{\mathsf{s}}(s)^{-\alpha} + F_{\mathsf{n}}(n)^{-\alpha}\right]^{-2-1^{1/\alpha}}, \times \alpha > 0. \tag{33}$$

The parameters ρ and α are the same dependence parameter, θ_c, in the normal and Clayton copula density functions, respectively. We use Pearson's correlation ρ_p as a measure of dependency in this paper. It is also called linear correlation. The normal copula parameter ρ is the same as the linear correlation ρ_p, but the relationship between Clayton copula parameter α and linear correlation ρ_p is given by:

$$\alpha = \frac{4\sin^{-1}(\rho_p)}{\pi - 2\sin^{-1}(\rho_p)}. \tag{34}$$

Until now, we have determined the PDF of the received signal analytically. However, it is necessary to validate the estimated PDF. The possible discrepancy between the hypothesized PDF model and the observed data is measured by the so-called goodness of fit statistics. In order to decide whether the observed discrepancy is substantial, performing a statistical test is required. Bootstrapping is an ideal procedure to estimate the accuracy of a parameter estimator [13]. This method calculates confidence intervals for parameters. To assess a parameter estimator using the bootstrap method, we examine whether the estimated parameter is in a corresponding confidence interval or not. The bootstrap method, and also the confidence interval that the bootstrap method presents for the fading parameter in our work, is described in Appendix 3. The results in Appendix 3 about using the bootstrap method for calculating the 95% confidence interval of the fading parameter are satisfactory.

After being sure about the reliability of the received signal PDF, the fading parameter is estimated by using the analytic PDF. This estimation is performed by using the NMMSE estimator principle. Utilizing the NMMSE estimator, a value of the fading parameter m is found such that it minimizes the difference between the obtained analytic PDF in (30) and the statistical one that is calculated based on the samples of the received signal in the sequel:

$$\hat{f}_{\mathrm{r}}(r) = \frac{1}{Nh} \sum_{i=0}^{N-1} \Psi\left(\frac{r - r_i}{h}\right), \tag{35}$$

where Ψ is the kernel function that must integrate to 1. The parameter h is called the window width or bandwidth of the kernel. N is the number of the received samples, and r_i is the value of the ith sample. So, the fading parameter is obtained by using the NMMSE estimator as follows:

$$\hat{m} = \arg\min_{m} \left| f_{\mathrm{r}}(r) - \hat{f}_{\mathrm{r}}(r) \right|^2. \tag{36}$$

In the next section, some simulations approve the obtained results.

5 Simulation and results

The efficiency of the proposed method for fading parameter estimation based on the estimated analytic PDF is evaluated using some simulations. It is essential that the proposed method be compared with other prominent conventional methods. The moment-based estimation [2], enhanced moment-based method [3], ML-based estimation [5], and GMM procedure [4] are known as the conventional methods. The mentioned methods are compared with the two proposed copula-based estimators. The difference between the two copula-based methods is due to the type of copula being used.

Figure 2 depicts the discrepancy between the performance of different methods in determining the fading parameter for the linear correlation $\rho_p = 0.1$. Since correlation refers to any of a broad class of statistical relationships involving dependence, we use the linear correlation in the simulation to create dependency between the faded signal and noise. The actual value of m in Figure 2 is 2.5, and the examination is done for the sample size $N = 10{,}000$ and signal-to-noise ratio (SNR) values from -10 to $+10$ dB. The index of performance is presented by mean square error (MSE). In all simulations, the noise power consists of 50% dependent and 50% independent noise power. In Figure 2, besides conventional methods, two copula estimators, normal and Clayton, are also compared. As shown, both copula estimators outperform the conventional approaches. In addition, the normal copula yields slightly better results than the Clayton estimator, because the construction of the Clayton copula is simple, and the PDFs of our signals have more consistency with the normal copula. However, when less complexity is preferable even with a little more error, the Clayton copula is a plausible option. Figure 2 also depicts the corresponding Cramér-Rao lower bound (CRLB), computed numerically, to benchmark the MSE of the estimators. Both proposed copula-based estimators have small deviations from the CRLB. The deviations are even reduced by increasing the SNR.

In Figure 3, the simulation is repeated for $\rho_p = 0.9$. As expected, greater dependency causes almost incorrect results in the conventional methods, especially in low SNRs. Nevertheless, the proposed copula-based methods are powerful even in low SNRs and highly dependent noise scenario. In Figure 3, all methods are also compared against CRLB.

Note that since the results of the two moment-based [2] and enhanced moment-based methods [3], in Figures 2 and 3, are not reliable for SNR values from -10 to 0 dB, we depict them only for SNR values from 0 to $+10$ dB.

Another comparison is presented in Figures 4 and 5, in which the estimation of different values of m from 0.5 to 10 is examined for a constant SNR $= +5$ dB and two values of correlation, i.e., $\rho_p = 0.1$ and $\rho_p = 0.9$. Again, the sample size N is 10,000, and the index of performance is presented by MSE. Figures 4 and 5 also approve that the two copula-based methods have convincing results for various values of fading parameter m.

In addition to simulation, the statistical accuracy of the proposed copula-based estimator is examined by the bootstrap method which is introduced in Appendix 3.

6 Conclusion

In this work, we propose a new estimation method based on the PDF and copula function to estimate the Nakagami-m fading parameter in wireless communications. The copula concept, which was originally proposed in econometrics literature, provides an analytic approach for finding the PDF of the received signal. It models the dependency between the faded signal and noise, and facilitates the separation of dependent noise from the desired part of the received signal. Therefore, in this paper, a novel approach is employed to estimate the Nakagami-m parameter that analyzes the noise behavior much better

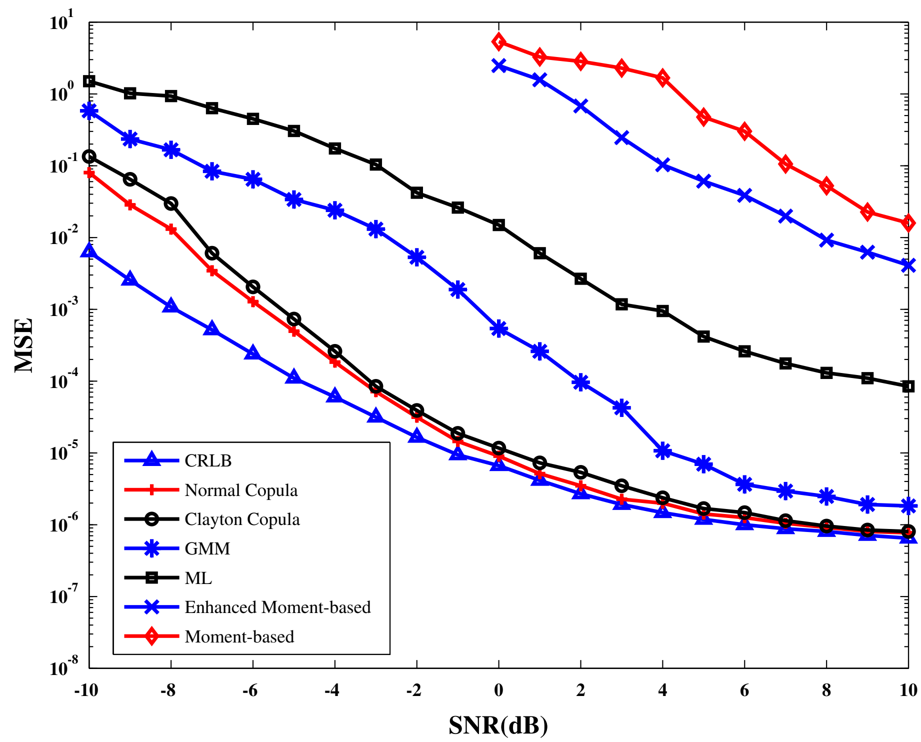

Figure 2 Comparison between MSE of m estimation in two proposed methods and other conventional approaches for $\rho_p = 0.1$.

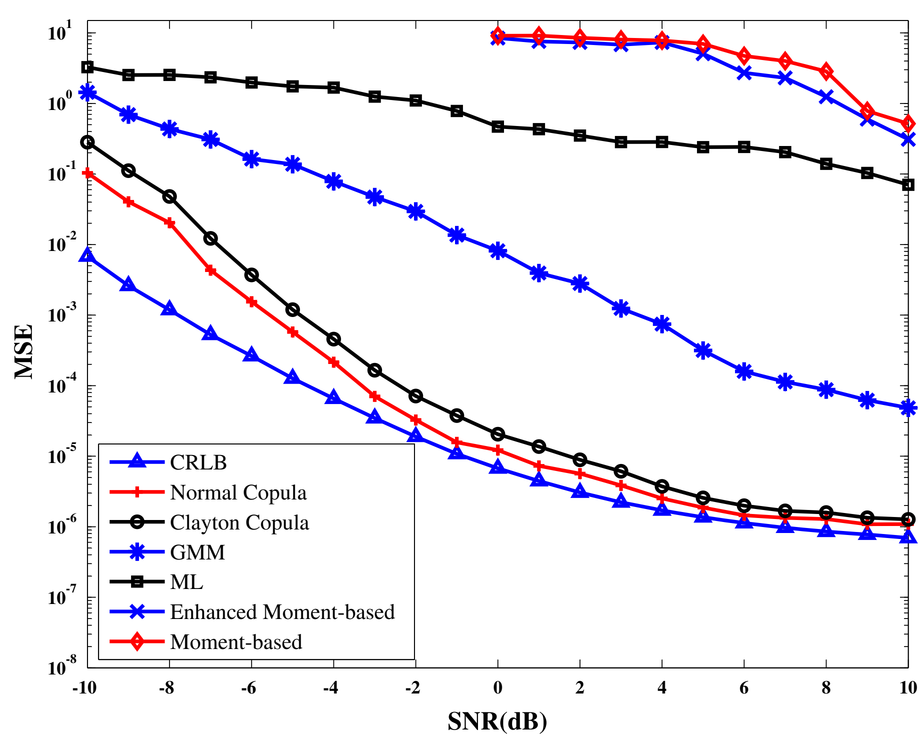

Figure 3 Comparison between MSE of m estimation in two proposed methods and other conventional approaches for $\rho_p = 0.9$.

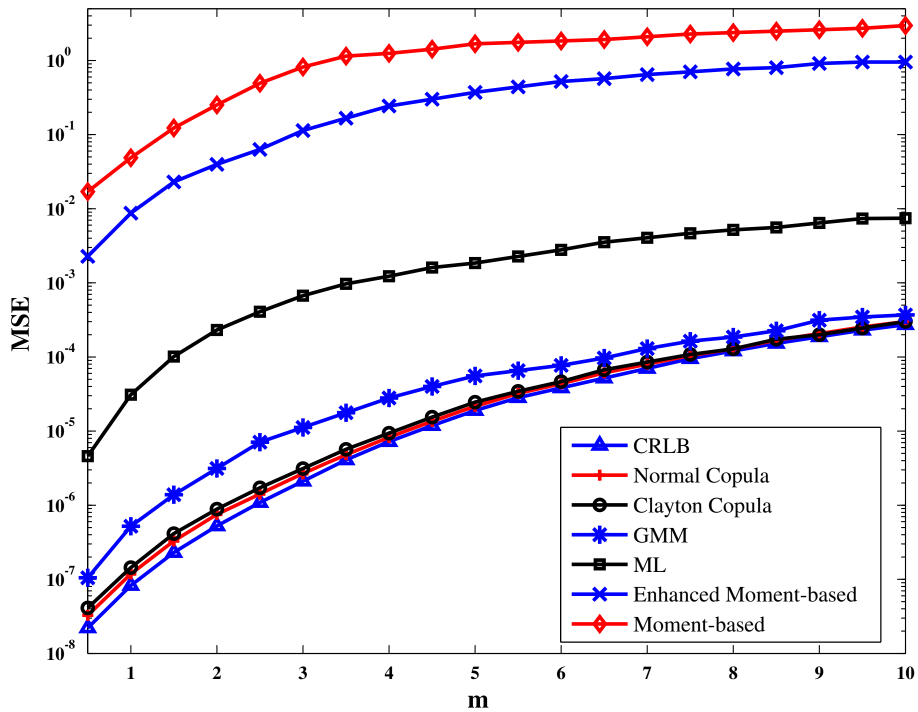

Figure 4 Comparison between MSE of m estimation in two proposed methods and other conventional approaches for $\rho_p = 0.1$.

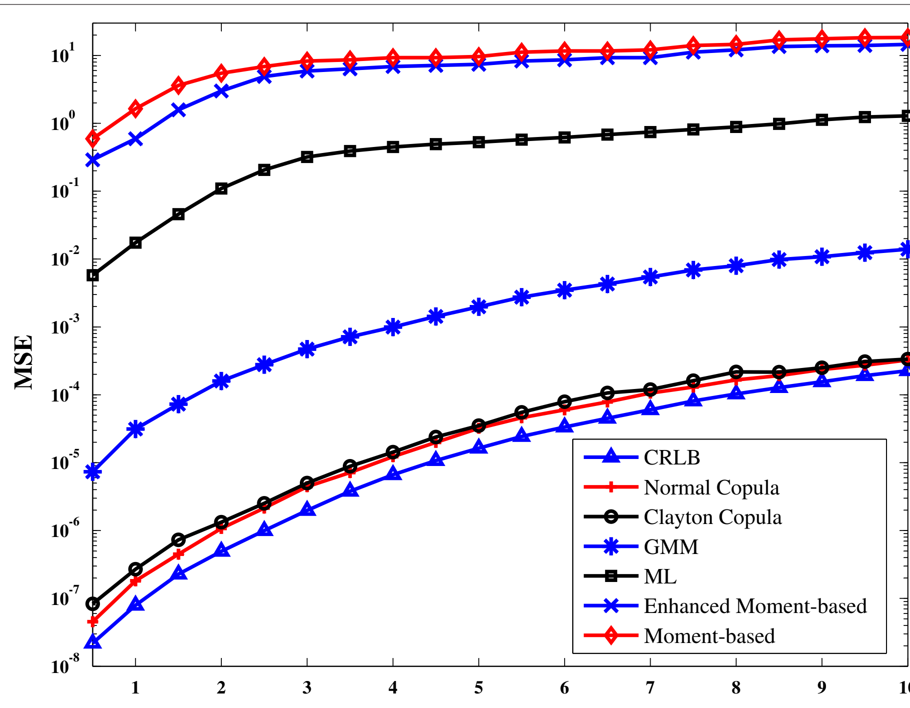

Figure 5 Comparison between MSE of m estimation in two proposed methods and other conventional approaches for $\rho_p = 0.9$.

than other conventional procedures. Moreover, a comprehensive model is also suggested for the noise behavior that is a suitable representation for the actual noisy environment. The presented copula-based method has precise results in low SNRs, while no other conventional methods can be reliable in such SNRs even in independent noise background. The parametric bootstrapping method is used to test the accuracy of the estimations. In addition to goodness of fit tests, simulation results also show the validity of the estimations.

Appendices

Appendix 1: the proof for (18) and (19)

Firstly, we show how (18) is derived. Then, the proof of (19) is presented.

The proof for (18)

It is provable that if a random variable Θ has the PDF $f_{\Theta}(\Theta)$, the PDF of $s_2 \triangleq g(\Theta)$ is determined by [23]:

$$f_{s_2}(s_2) = \frac{f_{\Theta}(\Theta_1)}{|g'(\Theta_1)|} + \ldots + \frac{f_{\Theta}(\Theta_n)}{|g'(\Theta_n)|} + \ldots, \tag{37}$$

where Θ_is are the real roots of the equation $s_2 = g(\Theta)$, and $g'(\Theta)$ is the derivative of $g(\Theta)$.

Suppose that $g(\Theta) \triangleq A \cos(\omega_c t + \Theta)$. Thus,

$$s_2 = A \cos(\omega_c t + \Theta). \tag{38}$$

We should get the PDF $f_{s_2}(s_2)$ based on (37). If $|s_2| > A$, then (38) has no solutions; hence, $f_{s_2}(s_2) = 0$. If $|s_2| < A$, then it has infinitely many solutions:

$$\Theta_n = \cos^{-1}\left(\frac{s_2}{A}\right) - \omega_c t \qquad n = \ldots, -1, 0, 1, \ldots. \tag{39}$$

Since $g'(\Theta_n) = -A \sin(\omega_c t + \Theta) = -\sqrt{A^2 - s_2^2}$, (37) yields:

$$f_{s_2}(s_2) = \frac{1}{\sqrt{A^2 - s_2^2}} \sum_{n=-\infty}^{\infty} f_{\Theta}(\Theta_n), \qquad |s_2| < A. \tag{40}$$

In this paper, Θ is uniformly distributed over $[0, 2\pi)$. Thus, only two solutions, which are in the interval $[0, 2\pi)$, are acceptable. The function $f_{\Theta}(\Theta)$ equals $\frac{1}{2\pi}$ for these two values, and it equals 0 for any Θ_n outside the interval $[0, 2\pi)$. Therefore, the proof of (18) is concluded.

The proof for (19)

To prove (19), we substitute both PDFs $f_{s_1}(s_1)$ and $f_{s_2}(s_2)$ in (16):

$$f_s(s) = \int_{-\infty}^{\infty} \frac{1}{|\eta|} f_{s_1 s_2}(\eta, s/\eta)\, d\eta. \tag{41}$$

Since $s_1(t)$ and $s_2(t)$ are independent, we have:

$$f_s(s) = \int_{-\infty}^{\infty} \frac{1}{|\eta|} f_{s_1}(\eta) f_{s_2}(s/\eta)\, d\eta. \tag{42}$$

By using (13) and (18), the PDF $f_s(s)$ is obtained as:

$$\begin{aligned} f_s(s) &= \int_{-\infty}^{\infty} \frac{1}{|\eta|} \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m \eta^{2m-1} \\ &\quad \times \exp\left(-\frac{m\eta^2}{\Omega}\right) \frac{1}{\pi\sqrt{A^2 - \frac{s^2}{\eta^2}}} d\eta, \qquad \eta \geqslant 0,\ \left|\frac{s}{\eta}\right| < A \\ &= \frac{2}{\pi\Gamma(m)} \left(\frac{m}{\Omega}\right)^m \int_{-\infty}^{\infty} \frac{|\eta|\eta^{2m-1}}{|\eta|\sqrt{A^2\eta^2 - s^2}} \exp\left(-\frac{m\eta^2}{\Omega}\right) d\eta, \\ &\quad \times \eta > \frac{s}{A} \\ &= \frac{2}{\pi\Gamma(m)} \left(\frac{m}{\Omega}\right)^m \int_{s/A}^{\infty} \frac{\eta^{2m-1}}{\sqrt{A^2\eta^2 - s^2}} \exp\left(-\frac{m\eta^2}{\Omega}\right) d\eta. \end{aligned} \tag{43}$$

Thus, the first equality of (19) is proven. The latter equation of (19) is obtained by setting $\kappa^2 \triangleq A^2\eta^2 - s^2$, and the proof of (19) is concluded.

Appendix 2: *K*-distribution

In this section, it is proven that if the variance of the normally distributed noise $\nu(t)$ becomes a random variate, the noise $\nu(t)$ will have the K-distribution instead of normal distribution.

As mentioned, the variance of noise $\nu(t)$ is estimated given samples of $\nu(t)$. The samples can be broken up into small segments of length N_1 such that the samples in each segment have almost identical variance.

Thus, the estimate of variance based on these N_1 normally distributed samples is:

$$\hat{\sigma}_\nu^2 = \frac{1}{N_1} \sum_{i=1}^{N_1} \nu_i^2, \tag{44}$$

which has chi-squared behavior with N_1 degrees of freedom.

With this estimation, $\nu(t)$ is now considered as the product of two random variates. The first variate has a normal distribution but with a constant variance. The other one is seen as the random nature of variance and has chi-squared behavior. The two distributions can construct the PDF of ν based on the following form:

$$\nu = \sqrt{\tau}\mathcal{N}, \tag{45}$$

where τ has chi-squared distribution with N_1 degrees of freedom, and $\mathcal{N}$ is normally distributed with mean zero and standard deviation b.

Since the gamma distribution, $\Gamma(k, \beta)$, has the following PDF:

$$f_\mu(\mu) = \begin{cases} \frac{\beta^k}{\Gamma(k)} \mu^{k-1} \exp(-\beta\mu), & \mu > 0 \\ 0, & \mu \leqslant 0 \end{cases}, \tag{46}$$

for $k, \beta > 0$, the N_1 chi-squared distribution could be a gamma distribution with the shape parameter $k = \frac{N_1}{2}$ and the scale parameter $\beta = \frac{1}{2}$. Thus, the PDF of $\Lambda \triangleq \sqrt{\tau}$ is determined in the following form:

$$f_\Lambda(\lambda) = \frac{1}{2^{\frac{N_1}{2}-1}\Gamma\left(\frac{N_1}{2}\right)} \lambda^{N_1-1} \exp\left(-\frac{\lambda^2}{2}\right), \quad \lambda \geqslant 0, \tag{47}$$

where $\Gamma(\cdot)$ is the gamma function.

Since ν is the product of $\mathcal{N}$ and Λ, using (16) and noting that $\mathcal{N}$ and Λ are independent, the PDF of ν is expressed as follows:

$$f_\nu(\nu) = \int_{-\infty}^{\infty} \frac{1}{|\eta|} f_\Lambda(\eta) f_\mathcal{N}\left(\frac{\nu}{\eta}\right) d\eta, \tag{48}$$

where η is an auxiliary variable. According to (48), since $f_\nu(-\nu) = f_\nu(\nu)$, f_ν is an even function. Therefore, it is sufficient that the PDF of ν is calculated for $\nu > 0$. Substituting the PDF of $\mathcal{N}$ and Λ in (48), we have:

$$f_\nu(\nu) = \frac{2^{-\frac{N_1-1}{2}}}{\Gamma\left(\frac{N_1}{2}\right)\sqrt{\pi} b} \int_0^\infty \eta^{N_1-2} \exp\left(-\frac{1}{2}\left(\eta^2 + \frac{\nu^2}{b^2\eta^2}\right)\right) d\eta. \tag{49}$$

On the other hand, for real number a and complex number z, we have (see p. 21 in [24]):

$$\pi H_k^{(1)}(az) = -i \exp\left(-^{i\pi k}/2\right) a^k \times \int_0^\infty \exp\left(i\frac{1}{2}z\left(\zeta + \frac{a^2}{\zeta}\right)\right) \zeta^{-k-1} d\zeta, \tag{50}$$

where $Im(z) > 0$, $Im(a^2 z) > 0$, $H_k^{(1)}(\cdot)$ is the Hankel function of the first kind, i is the imaginary unit, and ζ is an auxiliary variable. Suppose that $a = \frac{\nu}{b} > 0$, $z = +i$, $\zeta = \eta^2$, and $k = -\frac{N_1-1}{2}$. Comparing (49) and (50) results to:

$$f_\nu(\nu) = \frac{1}{b\sqrt{\pi}\Gamma\left(\frac{N_1}{2}\right)} \left(\frac{\nu}{2b}\right)^{\frac{N_1-1}{2}} \frac{i\pi}{2} \exp\left(-\frac{i\pi}{2}\frac{N_1-1}{2}\right) \times H^{(1)}_{-\frac{N_1-1}{2}}\left(\frac{\nu}{b} i\right), \qquad 0 < \nu < \infty. \tag{51}$$

Based on the following two relationships (see pp. 358 and 375 in [25]):

$$\begin{aligned} H_{-\epsilon}^{(1)}(\varrho) &= e^{\epsilon\pi i} H_\epsilon^{(1)}(\varrho), \\ K_\epsilon(\varrho) &= \frac{\pi}{2} i^{\epsilon+1} H_\epsilon^{(1)}(i\varrho), \end{aligned} \tag{52}$$

where ϵ and ϱ are two arbitrary parameters, (51) turns to:

$$f_\nu(\nu) = \frac{1}{b\sqrt{\pi}\Gamma\left(\frac{N_1}{2}\right)} \left(\frac{\nu}{2b}\right)^{\frac{N_1-1}{2}} \times K_{\frac{N_1-1}{2}}\left(\frac{\nu}{b}\right), \quad 0 < \nu < \infty. \tag{53}$$

Since the function $f_\nu(\nu)$ is even, the calculation of $f_\nu(\nu)$ for negative values of ν is a straightforward matter. Therefore, the proof is concluded.

Table 1 95% Confidence interval for fading parameter *m* in normal copula-based estimator for $\rho_p = 0.9$ ($B = 100$)

SNR	Estimated *m*	95% Confidence interval
−10	2.8269	(2.3207,3.1723)
−5	2.4841	(2.4646,2.5612)
0	2.4935	(2.4732,2.5017)
+5	2.4988	(2.4899,2.5101)
+10	2.5010	(2.4987,2.5029)

Appendix 3: bootstrapping

Let us consider $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ as a random sample of the received signal $r(t)$ with the PDF $f_{\mathrm{r}}(r)$ in (30). The sample is used to estimate the certain fading parameter, m, associated with the PDF $f_{\mathrm{r}}(r)$. A statistic, $T = T(\mathbf{r})$, might be used to estimate m from the data. In this paper, the mentioned statistic is introduced in (36). The bootstrap method determines a measure of the statistical accuracy of the estimator $T(\mathbf{r})$ in estimating the parameter m. The method can statistically quantify the error between m and the statistic T.

In bootstrapping, B resamples $\mathbf{r}^{*(1)}, \ldots, \mathbf{r}^{*(B)}$ are produced with replacement from $\mathbf{r}$. Then, the statistics $T^{*(1)}, \ldots, T^{*(B)}$, obtained from the resamples, are used to calculate the confidence interval for the fading parameter. More detailed information can be seen in [13]. To assess the parameter estimator T using the bootstrap method, we examine whether the estimated parameter m is in corresponding confidence interval or not. Tables 1 and 2 show the results of using the bootstrap method to calculate the 95% confidence interval for the fading parameter in both normal and Clayton copula-based estimators in

Table 2 95% Confidence interval for fading parameter *m* in Clayton copula-based estimator for $\rho_p = 0.9$ ($B = 100$)

SNR	Estimated *m*	95% Confidence interval
−10	2.0476	(1.9835,2.8142)
−5	2.5349	(2.4271,2.5410)
0	2.4947	(2.4878,2.5193)
+5	2.5015	(2.5010,2.5097)
+10	2.5011	(2.5000,2.5017)

$\rho_p = 0.9$. A normal or a Clayton copula-based estimator refers to the estimator in (36), in which the function $f_r(r)$ is calculated by using the normal or Clayton copula, respectively. In the tables, the results are given for several SNRs. The number of resamples is $B = 100$, and the actual value of fading parameter is $m = 2.5$. The tables also provide the estimated value of m besides the confidence interval in each SNR.

Competing interests
The authors declare that they have no competing interests.

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