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Multiple solutions of boundary value problems with ϕ-Laplacian operators and under a Wintner-Nagumo growth condition

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Dedicated to Professor Jean Mawhin

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Abstract

In this paper, we establish the existence of multiple solutions to second-order differential equations with ϕ-Laplacian satisfying periodic, Dirichlet or Neumann boundary conditions. The right-hand side is a Carathéodory function satisfying a growth condition of Wintner-Nagumo type. The existence of upper and lower solutions is assumed. The proofs rely on the fixed point index theory.
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1 Introduction

In this paper, we consider boundary value problems for second-order nonlinear differential equations with ϕ-Laplacian of the form:

$$\big(\phi\big(u'(t)\big)\big)' = f\big(t, u(t), u'(t)\big) \quad \text{a.e. } t \in [0, T],$$
$$u \in \mathcal{B},$$
(1.1)

where $\mathcal{B}$ denotes the Dirichlet, periodic, or Neumann boundary conditions:

$$u(0) = \mu, \qquad u(T) = v;$$
(1.2)

$$u(0) = u(T), \qquad \phi\big(u'(0)\big) = \phi\big(u'(T)\big);$$
(1.3)

$$\phi\big(u'(0)\big) = \mu, \qquad \phi\big(u'(T)\big) = v.$$
(1.4)

Here, $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, and $\phi : \mathbb{R} \to \mathbb{R}$ is a bijective increasing homeomorphism.

Such problems have been studied by many authors. The method of upper and lower solutions was widely used to obtain existence results; see, for instance, [1–16] and the references therein.

Many multiplicity results were obtained with the method of strict lower and upper solutions and for problems with the right member being a Carathéodory map not de-

pending on the derivative, $f(t, u)$. To our knowledge, De Coster [7] was the first to obtain multiplicity results for this problem with the Dirichlet boundary condition. Ben-Naoum and De Coster [1] considered the case where ϕ is the p-Laplacian with Sturm-Liouville boundary condition. Bereanu and Mawhin [2] established the existence of multiple periodic solutions in the case where $f(t, u) = e(t) - g(u) + c$ and ϕ can have a bounded domain or a bounded range. Zhang *et al.* [16] considered the periodic problem with the right member being a continuous map depending also on the derivative $f(t, u, u')$. A Nagumo growth condition was imposed on f. The case where $f(t, u, u')$ is a Carathéodory map and $\phi(u') = u'$ was studied by El Khattabi [8] under a linear growth condition, and by Goudreau [11] under a Wintner-Nagumo growth condition.

Existence results were established for problem (1.1) under a growth condition of Nagumo type of the form:

$$\left| f(t, x, y) \right| \le \theta(|y|) k(t), \tag{1.5}$$

with

$$\min\left\{ \int_{\phi(d)}^{\infty} \frac{\phi^{-1}(s)^{(p-1)/p}}{\theta(\phi^{-1}(s))} \, ds, \int_{-\infty}^{\phi(-d)} \frac{\phi^{-1}(s)^{(p-1)/p}}{\theta(\phi^{-1}(s))} \, ds \right\} > c,$$

for c, d suitable constants. In particular, the existence of a solution was obtained by O'Regan [13] when the right member has the form $q(t)f(t, u, u')$ with f continuous, and with either Dirichlet or mixed boundary conditions. His result was extended in [14] and in [15] for a Carathéodory function f. Cabada and Pouso [3] considered also a Carathéodory map f, and they established the existence of a solution to the problem with Neumann or periodic boundary conditions. More general boundary conditions or more general operators ϕ were considered in [4–6, 9, 12]. All those results rely on the Schauder fixed point theorem.

In this paper, we consider a Carathéodory map f satisfying a growth condition different from (1.5). Namely, we impose the growth condition of Wintner-Nagumo type

$$\left| f(t, x, y) \right| \le \psi(|y|) \left(l(t) + c(t)|y|^{(p-1)/p} \right).$$

Using the method of upper and lower solutions and the fixed point index theory, we establish existence and multiplicity results for problem (1.1) with Dirichlet, Neumann or periodic boundary value conditions. Our proofs rely on the fixed point index theory. This theory is particularly convenient for the Neumann problem where we use the contraction property of the fixed point index to reduce the computation of the fixed point index in an affine space.

2 Preliminaries

In what follows, we denote $I = [0, T]$. The space of continuous functions $\mathcal{C}(I)$, and the space of continuously derivable functions $\mathcal{C}^1(I)$ are equipped with the usual norms $\| \cdot \|_0$ and $\|u\|_1 = \max\{\|u\|_0, \|u'\|_0\}$, respectively. We denote the usual norm in $L^p(I)$ by $\|u\|_{L^p}$, where $1 \le p \le \infty$. We set

$$W(I) = \left\{ u \in \mathcal{C}^1(I) : \phi(u') \text{ is absolutely continuous and } \left(\phi(u') \right)' \in L^1(I) \right\}.$$

Definition 2.1 A map $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is a *Carathéodory function* if

(i) $f(t, \cdot, \cdot)$ is continuous for almost every $t \in I$;

(ii) $f(\cdot, x, y)$ is measurable for all $(x, y) \in \mathbb{R}^2$;

(iii) for all $R > 0$, there exists $h_R \in L^1(I)$ such that $|f(t, x, y)| \le h_R(t)$ for all $(x, y) \in \mathbb{R}^2$ such that $|x| \le R$, $|y| \le R$, and for almost every $t \in I$.

Definition 2.2 We say that $\alpha \in W(I)$ is a *lower solution of* (1.1) if

$$\big(\phi\big(\alpha'(t)\big)\big)' \ge f\big(t, \alpha(t), \alpha'(t)\big), \quad \text{a.e. } t \in I; \tag{2.1}$$

and, in addition,

(i) if $\mathcal{B}$ denotes Dirichlet boundary condition (1.2), it satisfies

$$\alpha(0) \le \mu \quad \text{and} \quad \alpha(T) \le \nu;$$

(ii) if $\mathcal{B}$ denotes periodic boundary condition (1.3), it satisfies

$$\alpha(0) = \alpha(T) \quad \text{and} \quad \phi\big(\alpha'(0)\big) \ge \phi\big(\alpha'(T)\big);$$

(iii) if $\mathcal{B}$ denotes Neumann boundary condition (1.4), it satisfies

$$\phi\big(\alpha'(0)\big) \ge \mu \quad \text{and} \quad \phi\big(\alpha'(T)\big) \le \nu.$$

Similarly, we define an *upper solution* of (1.1) if the previous conditions are satisfied with the reversed inequalities.

Definition 2.3 We say that $\alpha \in W(I)$ is a *strict lower solution of* (1.1) if for any $t_0 \in]0, T[$, there exist $\varepsilon > 0$ and $\mathcal{U}_{t_0}$, a neighborhood of t_0 in I, such that for almost every $t \in \mathcal{U}_{t_0}$ and all $(x, y) \in [\alpha(t), \alpha(t) + \varepsilon] \times [\alpha'(t) - \varepsilon, \alpha'(t) + \varepsilon]$,

$$\big(\phi\big(\alpha'(t)\big)\big)' \ge f(t, x, y); \tag{2.2}$$

and, in addition,

(i) if $\mathcal{B}$ denotes Dirichlet boundary condition (1.2), it satisfies

$$\alpha(0) < \mu \quad \text{and} \quad \alpha(T) < \nu;$$

(ii) if $\mathcal{B}$ denotes periodic boundary condition (1.3), it satisfies

$$\alpha(0) = \alpha(T) \quad \text{and} \quad \phi\big(\alpha'(0)\big) > \phi\big(\alpha'(T)\big);$$

(iii) if $\mathcal{B}$ denotes Neumann boundary condition (1.4), it satisfies

$$\phi\big(\alpha'(0)\big) > \mu \quad \text{and} \quad \phi\big(\alpha'(T)\big) < \nu.$$

Similarly, we define a *strict upper solution* of (1.1) if the previous conditions are satisfied with the reversed inequalities.

We will use the following general assumptions.

(H_ϕ) The map $\phi : \mathbb{R} \to \mathbb{R}$ is a bijective increasing homeomorphism.

(H_f) The map $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is Carathéodory.

($H_\mathcal{B}$) There exist $\alpha, \beta \in W(I)$, respectively lower and upper solutions of (1.1), such that $\alpha(t) \le \beta(t)$ for all $t \in I$.

(WN) There exist $k > 0$, $p \in\,]1, \infty]$, $c \in L^p(I, [0, \infty[)$, $l \in L^1(I, [0, \infty[)$ and $\psi : [0, \infty[\to [k, \infty[$ such that

$$\frac{1}{\psi \circ \phi^{-1}} \in L^1_{loc}(\mathbb{R}), \qquad \int_{-\infty}^{\infty} \frac{ds}{\psi(|\phi^{-1}(s)|)} = \infty,$$

and

$$\left|f(t, x, y)\right| \le \psi(|y|)\left(l(t) + c(t)|y|^{(p-1)/p}\right)$$

$$\forall x \in \left[\alpha(t), \beta(t)\right], y \in \mathbb{R}, \text{ and a.e. } t \in I,$$

with $(p - 1)/p = 1$ if $p = \infty$.

In what follows, ($H_\mathcal{B}$) will be replaced by (H_D), (H_N) or (H_P) if $\mathcal{B}$ denotes (1.2), (1.3) or (1.4), the Dirichlet, periodic or Neumann boundary conditions, respectively.

We present some properties of operators that will be used later. Here is a particular case of Lemmas 2.3 and 2.6 in [10].

Lemma 2.4 *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function and $\sigma : I \times \mathbb{R}^2 \to \mathbb{R}$ such that*

(i) *for any $u \in \mathcal{C}^1(I)$, the map $t \mapsto \sigma(t, u(t), u'(t))$ is measurable;*

(ii) *for any sequence $\{u_n\}$ converging to u_0 in $\mathcal{C}^1(I)$, there exists $c > 0$ such that*

$$\left\| \sigma(t, u_n(t), u'_n(t)) \right\| \le c \quad \text{a.e. } t \in I \text{ and } \forall n \ge 0;$$

and

$$\sigma(t, u_n(t), u'_n(t)) \to \sigma(t, u_0(t), u'_0(t)) \quad \text{a.e. } t \in I.$$

Then the operator $N_g : \mathcal{C}^1(I) \to \mathcal{C}(I)$ defined by

$$N_g(t) = \int_0^t g(s, u(s), u'(s))\, ds,$$

with

$$g(t, u(t), u'(t)) = f(t, \sigma(t, u(t), u'(t))),$$

is continuous and completely continuous. Moreover, for every $u \in \mathcal{C}^1(I)$, $N_g(u)$ is absolutely continuous and

$$\frac{d}{dt} N_g(u)(t) = g(t, u(t), u'(t)) \quad \text{a.e. } t \in I.$$

Observe that the previous lemma holds with $\sigma(t, x, y) = (x, y)$. In this case, $N_g = N_f$.

Now, we present some results related to the homeomorphism ϕ. The first one is a lemma due to Manásevich and Mawhin [17].

Lemma 2.5 *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *satisfy* (H_ϕ), *and let* $\mathcal{G} : \mathbb{R} \times \mathcal{C}(I) \to \mathbb{R}$ *be defined by*

$$\mathcal{G}(a, h) = \int_0^T \phi^{-1}(a + h(t)) \, dt.$$

Then the following statements hold:

(1) *For any* $c \in \mathbb{R}$ *and any* $h \in \mathcal{C}(I)$, *the equation*

$$\mathcal{G}(a, h) = c$$

has a unique solution $\overline{a}_c(h)$.

(2) *For any* $c \in \mathbb{R}$, *the function* $\overline{a}_c : \mathcal{C}(I) \to \mathbb{R}$ *defined in* (1) *is continuous, and it sends bounded sets into bounded sets.*

It is easy to show the following result.

Lemma 2.6 *Assume that* $\phi : \mathbb{R} \to \mathbb{R}$ *satisfies* (H_ϕ). *Let* $\Phi : \mathbb{R} \times \mathcal{C}(I) \to \mathcal{C}^1(I)$ *be defined by*

$$\Phi(r, h)(t) = r + \int_0^t \phi^{-1}(h(s)) \, ds.$$

Then Φ *is continuous.*

We will use the following maximum principle type result.

Lemma 2.7 *Assume that* (H_ϕ) *is satisfied. Let* $v, w \in W(I)$ *be such that*

$$(\phi(v'(t)))' \le (\phi(w'(t)))' \quad a.e. \ on \ \{t \in I : v(t) < w(t)\}.$$

Assume that one of the following conditions holds:

(i) $v(0) \ge w(0)$, $v(T) \ge w(T)$;

(ii) $\phi(v'(0)) \le \phi(w'(0))$, $\phi(v'(T)) \ge \phi(w'(T))$;

(iii) $v(0) = v(T)$, $w(0) = w(T)$, $\phi(v'(0)) - \phi(v'(T)) \le \phi(w'(0)) - \phi(w'(T))$.

Then $v(t) \ge w(t)$ *for all* $t \in I$, *or there exists* $c > 0$ *such that* $v(t) = w(t) - c$ *for all* $t \in I$.

Proof Assume that

$$A = \{t \in I : v(t) < w(t)\} \ne \emptyset.$$

Condition (i) (resp. (ii) or (iii)) implies that one of the following statements holds:

there exists $[t_0, t_1] \subset A$ such that

$$v(t_0) - w(t_0) < v(t_1) - w(t_1) \quad \text{and} \quad \phi(v'(t_0)) \le \phi(w'(t_0)); \tag{2.3}$$

there exists $[t_0, t_1] \subset A$ such that

$$v(t_0) - w(t_0) > v(t_1) - w(t_1) \quad \text{and} \quad \phi(v'(t_1)) \geq \phi(w'(t_1)); \tag{2.4}$$

or

there exists $c > 0$ such that $\quad v(t) = w(t) - c \quad \forall t \in I.$ $\hspace{2cm}$ (2.5)

Indeed, if (2.5) does not hold, let $\tau, \rho \in I$ be such that

$$v(\tau) - w(\tau) = \min\{v(t) - w(t) : t \in I\} < v(\rho) - w(\rho),$$

and

$$v(t) - w(t) < 0 \quad \text{for all } t \text{ between } \tau \text{ and } \rho.$$

If $\tau \in {]}0, T{[}$, then $\phi(v'(\tau)) = \phi(w'(\tau))$ and (2.3) or (2.4) hold.

If $\tau \in \{0, T\}$, then (i) does not hold. If (ii) holds, then

$$\phi(v'(0)) \leq \phi(w'(0)) \quad \text{and} \quad \phi(v'(T)) \geq \phi(w'(T)).$$

Similarly, if (iii) holds, since

$$v(0) - w(0) = v(T) - w(T) = v(\tau) - w(\tau),$$

one has $v'(0) \geq w'(0)$ and $v'(T) \leq w'(T)$. Using (iii) and the fact that ϕ is increasing, we get

$$\phi(v'(0)) - \phi(w'(0)) \geq 0 \geq \phi(v'(T)) - \phi(w'(T)) \geq \phi(v'(0)) - \phi(w'(0)).$$

Hence, (2.3) or (2.4) are satisfied.

Therefore, by assumption,

$$(\phi(v'(t)))' \leq (\phi(w'(t)))' \quad \text{a.e. } t \in [t_0, t_1]. \tag{2.6}$$

If (2.3) holds, for every $t \in {]}t_0, t_1]$,

$$\phi(v'(t)) - \phi(v'(t_0)) = \int_{t_0}^{t} (\phi(v'(s)))' \, ds \leq \int_{t_0}^{t} (\phi(w'(s)))' \, ds$$
$$= \phi(w'(t)) - \phi(w'(t_0)) \leq \phi(w'(t)) - \phi(v'(t_0)).$$

Since ϕ is increasing, we deduce that $v - w$ is nonincreasing in $[t_0, t_1]$. This is a contradiction. Similarly, we obtain a contradiction if (2.4) holds.

Therefore, $v(t) \geq w(t)$ for all $t \in I$, or (2.5) is satisfied. $\qquad \square$

Lemma 2.8 *Assume* (H_ϕ) *and* (H_f). *Let* $\alpha, \beta \in W(I)$ *be respectively strict lower and upper solutions of* (1.1) *such that* $\alpha(t) < \beta(t)$ *for all* $t \in I$. *If* $u \in W(I)$ *is a solution of* (1.1) *such that* $\alpha(t) \leq u(t) \leq \beta(t)$ *for all* $t \in I$, *then* $\alpha(t) < u(t) < \beta(t)$ *for all* $t \in I$.

Proof Let $u \in W(I)$ be a solution of (1.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in I$. Assume that

$$A = \{t \in I : \alpha(t) = u(t)\} \neq \emptyset.$$

First, we claim that $0, T \notin A$. This is obviously the case if $\mathcal{B}$ denotes Dirichlet boundary condition (1.2). If $\mathcal{B}$ denotes periodic boundary condition (1.3), then $u - \alpha$ attains a minimum at 0 and T. So, by (H_ϕ) and Definition 2.3(ii),

$$\phi(u'(T)) - \phi(\alpha'(T)) \leq 0 \leq \phi(u'(0)) - \phi(\alpha'(0)) < \phi(u'(T)) - \phi(\alpha'(T)),$$

a contradiction. If $\mathcal{B}$ denotes Neumann boundary condition (1.4), then $u - \alpha$ attains a minimum at 0 or at T. So, by (H_ϕ) and Definition 2.3(iii),

$$0 \leq \phi(u'(0)) - \phi(\alpha'(0)) < \mu - \mu \quad \text{or} \quad 0 \geq \phi(u'(T)) - \phi(\alpha'(T)) > \nu - \nu,$$

a contradiction.

Let $t_0 = \max A \in]0, T[$. So, $\phi(u'(t_0)) = \phi(\alpha'(t_0))$. By Definition 2.3, there exist $\varepsilon > 0$ and $\mathcal{U}_{t_0}$, a neighborhood of t_0, such that $(\phi(\alpha'(t)))' \geq f(t, x, y)$ a.e. $t \in \mathcal{U}_{t_0}$ and all x, y such that $\alpha(t) \leq x \leq \alpha(t) + \varepsilon$ and $|y - \alpha'(t)| \leq \varepsilon$. Since $u \in \mathcal{C}^1(I)$, there exists $t_1 \in]t_0, T[$ such that $u(t) \in]\alpha(t), \alpha(t) + \varepsilon]$ and $|u'(t) - \alpha'(t)| \leq \varepsilon$ for all $t \in]t_0, t_1]$. Since $t_0 = \max A$, there exists $t \in]t_0, t_1]$ such that $u'(t) > \alpha'(t)$. Using the fact that ϕ is increasing, we deduce that

$$\phi(\alpha'(t)) - \phi(\alpha'(t_0)) < \phi(u'(t)) - \phi(u'(t_0)) = \int_{t_0}^{t} f(s, u(s), u'(s)) \, ds$$

$$\leq \int_{t_0}^{t} (\phi(\alpha'(s)))' \, ds = \phi(\alpha'(t)) - \phi(\alpha'(t_0)).$$

This is a contradiction. Therefore, $\alpha(t) < u(t)$ for all $t \in I$.

Similarly, we show that $\beta(t) > u(t)$ for all $t \in I$. □

The following result establishes the existence of an *a priori* bound on the derivative of functions satisfying a suitable inequality.

Lemma 2.9 *Assume that* (H_ϕ) *is satisfied. Let* $k > 0$ *and* $\psi : [0, \infty[\to [k, \infty[$ *be such that*

$$\frac{1}{\psi \circ \phi^{-1}} \in L^1_{loc}(\mathbb{R}) \quad \text{and} \quad \int_0^{\pm\infty} \frac{ds}{\psi(|\phi^{-1}(s)|)} = \pm\infty. \tag{2.7}$$

Then, for every $d_0 \geq 0$, $p \in]1, \infty]$, $c_0 \in L^p(I, [0, \infty[)$, $l_0 \in L^1(I, [0, \infty[)$, *and* $B \subset \mathcal{C}(I)$ *bounded, there exists* $M_0 > d_0$ *such that for every*

$$u \in \{u \in W(I) : u \in B, \min\{|u'(t)| : t \in I\} \leq d_0\}$$

satisfying

$$\left|(\phi(u'(t)))'\right| \leq \psi(|u'(t)|)(l_0(t) + c_0(t)|u'(t)|^{(p-1)/p}) \quad \text{a.e. } t \in I,$$

one has $\|u'\|_0 < M_0$.

Proof Let

$$m = \max\{u(t_1) - u(t_2) : t_1, t_2 \in I, u \in B\}.$$

Assumptions (H_ϕ) and (2.7) imply that there exists $M_0 > d_0$ such that

$$\phi(M_0) > \phi(d_0) \geq \phi(-d_0) > \phi(-M_0),$$

and

$$\max\left\{\int_{\phi(d_0)}^{\phi(M_0)} \frac{ds}{\psi(|\phi^{-1}(s)|)}, \int_{\phi(-M_0)}^{\phi(-d_0)} \frac{ds}{\psi(|\phi^{-1}(s)|)}\right\}$$
$$> \|l_0\|_{L^1} + m^{(p-1)/p}\|c_0\|_{L^p}. \tag{2.8}$$

Assume that there exists

$$u \in \{u \in W(I) : u \in B, \min\{|u'(t)| : t \in I\} \leq d_0,$$
$$\text{and } |(\phi(u'(t)))'| \leq \psi(|u'(t)|)(l_0(t) + c_0(t)|u'(t)|^{(p-1)/p}) \text{ a.e. } t \in I\}$$

such that $\|u'\|_0 \geq M_0$. If $\max\{u'(t) : t \in I\} \geq M_0$, there exist $t_0, t_1 \in I$ such that $\phi(u'(t_0)) = \phi(d_0)$, $\phi(u'(t_1)) = \phi(M_0)$, $\phi(u'(t)) \in]\phi(d_0), \phi(M_0)[$ and $u'(t) > 0$ for all t between t_0 and t_1. Without loss of generality, we assume that $t_0 < t_1$. Then, by assumption,

$$\frac{(\phi(u'(t)))'}{\psi(u'(t))} \leq l_0(t) + c_0(t)(u'(t))^{(p-1)/p} \quad \text{a.e. } t \in [t_0, t_1].$$

Integrating from t_0 to t_1 and using the Hölder inequality and the change of variable formula in an integral give us

$$\int_{\phi(d_0)}^{\phi(M_0)} \frac{ds}{\psi(|\phi^{-1}(s)|)} = \int_{t_0}^{t_1} \frac{(\phi(u'(t)))'}{\psi(u'(t))} dt$$
$$\leq \int_{t_0}^{t_1} l_0(t) + c_0(t)(u'(t))^{(p-1)/p} dt$$
$$\leq \|l_0\|_{L^1} + \|c_0\|_{L^p}(u(t_1) - u(t_0))^{(p-1)/p}$$
$$\leq \|l_0\|_{L^1} + m^{(p-1)/p}\|c_0\|_{L^p}.$$

This contradicts (2.8).

Similarly, if $\min\{u'(t) : t \in I\} \leq -M_0$, there exist $t_0, t_1 \in I$ such that $\phi(u'(t_0)) = \phi(-d_0)$, $\phi(u'(t_1)) = \phi(-M_0)$, $\phi(u'(t)) \in]\phi(-M_0), \phi(-d_0)[$ and $u'(t) < 0$ for all t between t_0 and t_1. Arguing as above leads to a contradiction. □

3 The Dirichlet problem

In this section, we consider problem (1.1) with the Dirichlet boundary condition. In order to establish the existence of a solution to (1.1), (1.2), we consider the following family of

problems defined for $\lambda \in [0, 1]$:

$$\big(\phi\big(u'(t)\big)\big)' = \lambda \widehat{f}\big(t, u(t), u'(t)\big), \quad \text{a.e. } t \in I,$$

$$u(0) = \mu, \qquad u(T) = \nu, \tag{3.1_λ}$$

where $\widehat{f} : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\widehat{f}(t, x, y) = \begin{cases} f(t, \beta(t), \beta'(t)) - \widehat{h}(t)(x - \beta(t)) & \text{if } x > \beta(t), \\ f(t, x, y) & \text{if } \alpha(t) \le x \le \beta(t), \\ f(t, \alpha(t), \alpha'(t)) + \widehat{g}(t)(x - \alpha(t)) & \text{if } x < \alpha(t); \end{cases} \tag{3.2}$$

with $\widehat{g}, \widehat{h} \in L^1(I)$ chosen such that

$$\widehat{g}(t) > \max\big\{0, f\big(t, \alpha(t), \alpha'(t)\big)\big\} \quad \text{and} \quad \widehat{h}(t) < \min\big\{0, f\big(t, \beta(t), \beta'(t)\big)\big\}. \tag{3.3}$$

We show that the solutions to these problems are *a priori* bounded.

Proposition 3.1 *Assume that* (H_ϕ), (H_f), (H_D) *and* (WN) *hold. Then there exists* $M > \max\{\|\alpha\|_0, \|\beta\|_0\}$ *such that any solution* u *of* (3.1_λ) *satisfies* $\|u\|_1 < M$.

Proof Fix $M_1 > 0$ such that

$$-M_1 \le \min_{t \in I} \alpha(t) - 1 \quad \text{and} \quad M_1 \ge \max_{t \in I} \beta(t) + 1.$$

We claim that any solution u of (3.1_λ) is such that $\|u\|_0 \le M_1$. Indeed, by (H_D),

$$-M_1 < u(0) = \mu < M_1 \quad \text{and} \quad -M_1 < u(T) = \nu < M_1.$$

From the definition of $\widehat{f}$, one has, almost everywhere on $\{t \in I : u(t) < -M_1\}$,

$$\begin{aligned} \big(\phi\big(u'(t)\big)\big)' &= \lambda\big(f\big(t, \alpha(t), \alpha'(t)\big) + \widehat{g}(t)\big(u(t) - \alpha(t)\big)\big) \\ &\le \lambda\big(f\big(t, \alpha(t), \alpha'(t)\big) - \widehat{g}(t)\big) \\ &\le 0. \end{aligned}$$

Similarly,

$$\big(\phi\big(u'(t)\big)\big)' \ge 0 \quad \text{a.e. on } \big\{t \in I : u(t) > M_1\big\}.$$

It follows from Lemma 2.7 that $-M_1 \le u(t) \le M_1$ for all $t \in I$.

We look for an *a priori* bound on the derivative of any solution u of (3.1_λ). Let

$$l_1(t) = \max\big\{\big|f\big(t, \alpha(t), \alpha'(t)\big)\big| + 2M_1\big|\widehat{g}(t)\big|, \big|f\big(t, \beta(t), \beta'(t)\big)\big| + 2M_1\big|\widehat{h}(t)\big|\big\}.$$

Observe that, by (WN),

$$\left|\left(\phi\left(u'(t)\right)\right)'\right| \leq \max\left\{l_1(t), \psi\left(\left|u'(t)\right|\right)\left(l(t) + c(t)\left|u'(t)\right|^{(p-1)/p}\right)\right\}$$

$$\leq \psi\left(\left|u'(t)\right|\right)\left(l_0(t) + c(t)\left|u'(t)\right|^{(p-1)/p}\right) \quad \text{a.e. } t \in I,$$

with

$$l_0(t) = \max\left\{l(t), \frac{l_1(t)}{k}\right\}. \tag{3.4}$$

It follows from Lemma 2.9 applied with $d_0 = (\nu - \mu)/T$ that there exists M_0 such that any solution u of (3.1_λ) satisfies $\|u'\|_0 < M_0$.

Finally, set $M = \max\{M_0, 1 + M_1\}$. We have that $\|u\|_1 < M$ for any solution u of (3.1_λ). □

Proposition 3.2 *Assume that* (H_ϕ), (H_f), (H_D) *and* (WN) *hold. Then, for every* $\lambda \in [0,1]$, *problem* (3.1_λ) *has at least one solution.*

Proof Let us define $N_{\widehat{f}} : \mathcal{C}^1(I) \to \mathcal{C}(I)$ and $\mathcal{S}_D : [0,1] \times \mathcal{C}^1(I) \to \mathbb{R} \times \mathcal{C}(I)$ by

$$N_{\widehat{f}}(u)(t) = \int_0^t \widehat{f}\left(s, u(s), u'(s)\right) ds, \tag{3.5}$$

and

$$\mathcal{S}_D(\lambda, u) = \left(\mu, \overline{a}_{\nu-\mu}\left(\lambda N_{\widehat{f}}(u)\right) + \lambda N_{\widehat{f}}(u)\right), \tag{3.6}$$

where $\overline{a}_{\nu-\mu}$ is obtained in Lemma 2.5. Now, we define $\mathcal{D} : [0,1] \times \mathcal{C}^1(I) \to \mathcal{C}^1(I)$ by

$$\mathcal{D} = \Phi \circ \mathcal{S}_D, \tag{3.7}$$

where Φ is defined in Lemma 2.6. We deduce that $\mathcal{S}_D$ is continuous and completely continuous from Lemma 2.5 and from Lemma 2.4 applied with

$$\sigma(t, x, y) = \begin{cases} (\beta(t), \beta'(t)) & \text{if } x > \beta(t), \\ (x, y) & \text{if } \alpha(t) \leq x \leq \beta(t), \\ (\alpha(t), \alpha'(t)) & \text{if } x < \alpha(t). \end{cases}$$

This combined with Lemma 2.6 implies that $\mathcal{D}$ is continuous and completely continuous.

Now, we study the fixed points of $\mathcal{D}$. Let $u \in \mathcal{C}^1(I)$ and $\lambda \in [0,1]$ be such that $u = \mathcal{D}(\lambda, u)$. One has, by Lemma 2.5,

$$u(0) = \mu, \qquad u(T) = \mu + \mathcal{G}\left(\overline{a}_{\nu-\mu}\left(\lambda N_{\widehat{f}}(u)\right), \lambda N_{\widehat{f}}(u)\right) = \mu + (\nu - \mu) = \nu.$$

Also, $\phi(u') = \overline{a}_{\nu-\mu}(\lambda N_{\widehat{f}}(u)) + \lambda N_{\widehat{f}}(u)$. Hence, by Lemma 2.4, it is absolutely continuous and

$$\left(\phi\left(u'(t)\right)\right)' = \lambda \widehat{f}\left(t, u(t), u'(t)\right) \quad \text{a.e. } t \in I.$$

So, fixed points of $\mathcal{D}$ are solutions of (3.1_λ).

Let $M > \max\{\|\alpha\|_0, \|\beta\|_0\} \geq \max\{\mu, \nu\}$ be the constant obtained in Proposition 3.1 and set

$$\mathcal{U} = \big\{u \in \mathcal{C}^1(I) : \|u\|_1 < M\big\}. \tag{3.8}$$

Proposition 3.1 implies that $u \neq \mathcal{D}(\lambda, u)$ for all $(\lambda, u) \in [0,1] \times \partial\mathcal{U}$. Observe that $\mathcal{D}(0, u) = \Phi(\mu, \overline{a}_{\nu-\mu}(0)) = u_0$ with $u_0(t) = \mu + (\nu - \mu)t/T$. One has $u_0 \in \mathcal{U}$. By the properties of the fixed point index (see [18] for more details),

$$\mathrm{index}\big(\mathcal{D}(\lambda, \cdot), \mathcal{U}\big) = \mathrm{index}\big(\mathcal{D}(0, \cdot), \mathcal{U}\big) = \mathrm{index}(u_0, \mathcal{U}) = 1 \quad \forall\lambda \in [0,1]. \tag{3.9}$$

Therefore, for every $\lambda \in [0,1]$, $\mathcal{D}(\lambda, \cdot)$ has a fixed point, and hence (3.1_λ) has a solution. □

Now, we can establish the existence of a solution to (1.1), (1.2).

Theorem 3.3 *Assume that* (H_ϕ), (H_f), (H_D) *and* (WN) *hold. Then Dirichlet problem* (1.1), (1.2) *has a solution* $u \in W(I)$ *such that* $\alpha(t) \leq u(t) \leq \beta(t)$ *for every* $t \in I$.

Proof Proposition 3.2 insures the existence of $u \in W(I)$, a solution of (3.1_λ) for $\lambda = 1$. To conclude, we have to show that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in I$ since $\widehat{f}(t, x, y) = f(t, x, y)$ for $x \in [\alpha(t), \beta(t)]$.

By (H_D),

$$\big(\phi(u'(t))\big)' = f\big(t, \alpha(t), \alpha'(t)\big) + \widehat{g}(t)\big(x(t) - \alpha(t)\big) \leq f\big(t, \alpha(t), \alpha'(t)\big)$$

$$\leq \big(\phi(\alpha'(t))\big)' \quad \text{a.e. } t \in \big\{t \in I : u(t) < \alpha(t)\big\},$$

$$u(0) = \mu \geq \alpha(0) \quad \text{and} \quad u(T) = \nu \geq \alpha(T).$$

It follows from Lemma 2.7 that $u(t) \geq \alpha(t)$ for all $t \in I$.

A similar argument yields $u(t) \leq \beta(t)$ for all $t \in I$. □

Remark 3.4 The hypothesis (WN) can be generalized by

$$\min\left\{\int_{\phi(d_0)}^{\infty} \frac{ds}{\psi(|\phi^{-1}(s)|)}, \int_{-\infty}^{\phi(-d_0)} \frac{ds}{\psi(|\phi^{-1}(s)|)}\right\} > c,$$

with $d_0 = (\nu - \mu)/T$ and c, a suitable constant which can be deduced from the proof of Lemma 2.9.

Example 3.5 Let us consider the following problem:

$$\big(\phi(u'(t))\big)' = \big((u'(t))^3 + 1\big)\big(h(t)(u(t) + a(t)) + g(u(t))u'(t)\big) \quad \text{a.e. } t \in [0, T],$$

$$u(0) = 0, \qquad u(T) = 0, \tag{3.10}$$

where $\phi(s) = s^3$, $a \in C[0, T]$, $g \in C(\mathbb{R})$ and $h \in L^1[0, T]$ with $h(t) \geq 0$ a.e. $t \in [0, T]$. Let $m > 0$ be such that $|a(t)| \leq m$ for all $t \in T$. Then $\alpha = -m$ and $\beta = m$ are respectively lower

and upper solutions of (3.10). Let $\psi(s) = (|s|^3 + 1)$, $c \geq \max\{|g(x)| : |x| \leq m\}$ and $l = 2mh$. One has

$$|f(t,x,y)| \leq \psi(|y|)(l(t) + c|y|) \quad \forall x \in [-m,m], y \in \mathbb{R}, \text{ and a.e. } t \in [0,T],$$

and

$$\int_{-\infty}^{\infty} \frac{ds}{\psi(|\phi^{-1}(s)|)} = \int_{-\infty}^{\infty} \frac{ds}{|s| + 1} = \infty.$$

By Theorem 3.3, (3.10) has a solution u such that $|u(t)| \leq m$ for all $t \in [0,T]$. Notice that f does not satisfy a growth condition of Nagumo type.

4 The periodic problem

In this section, we consider problem (1.1) with the periodic boundary condition. In order to establish the existence of a solution to (1.1), (1.3), we consider the following family of problems defined for $\lambda \in [0,1]$:

$$\left(\phi\left(u'(t)\right)\right)' = \lambda \widehat{f}\left(t,u(t),u'(t)\right) + \frac{(1-\lambda)}{T} \int_0^T \widehat{f}\left(t,u(t),u'(t)\right) dt \quad \text{a.e. } t \in I,$$

$$u(0) = u(T), \qquad \phi\left(u'(0)\right) = \phi\left(u'(T)\right),$$

$$(4.1_\lambda)$$

where $\widehat{f}$ is defined in (3.2).

We show that the solutions to these problems are *a priori* bounded.

Proposition 4.1 *Assume that* (H_ϕ), (H_f), (H_P) *and* (WN) *hold. Then there exists* $M > \max\{\|\alpha\|_0, \|\beta\|_0\}$ *such that any solution u of* (4.1_λ) *satisfies* $\|u\|_1 < M$.

Proof Fix $M_1 > 0$ such that

$$-M_1 \leq \min_{t \in I} \alpha(t) - 1 \quad \text{and} \quad M_1 \geq \max_{t \in I} \beta(t) + 1.$$

We claim that any solution u of (4.1_λ) is such that $\|u\|_0 \leq M_1$. Observe that a solution of (4.1_λ) satisfies

$$0 = \phi\left(u'(T)\right) - \phi\left(u'(0)\right) = \lambda \int_0^T \widehat{f}\left(t,u(t),u'(t)\right) dt + (1-\lambda) \int_0^T \widehat{f}\left(t,u(t),u'(t)\right) dt$$

$$= \int_0^T \widehat{f}\left(t,u(t),u'(t)\right) dt.$$

So, it satisfies

$$\left(\phi\left(u'(t)\right)\right)' = \lambda \widehat{f}\left(t,u(t),u'(t)\right) \quad \text{a.e. } t \in I,$$

with periodic boundary condition (1.3). Arguing as in the proof of Proposition 3.1, we obtain that

$$\left(\phi\left(u'(t)\right)\right)' \leq 0 \quad \text{a.e. on } \left\{t \in I : u(t) < -M_1\right\},$$

$$\left(\phi\left(u'(t)\right)\right)' \geq 0 \quad \text{a.e. on } \left\{t \in I : u(t) > M_1\right\}.$$

It follows from Lemma 2.7 that $\|u\|_0 \leq M_1$, or there exists $c > 0$ such that $|u(t)| = M_1 + c$ for all $t \in I$. If $u \equiv M_1 + c$,

$$0 = \int_0^T \widehat{f}(t, u(t), u'(t))\,dt = \int_0^T f(t, \beta(t), \beta'(t)) - \widehat{h}(t)(M_1 + c - \beta(t))\,dt > 0,$$

a contradiction. Similarly, one cannot have $u \equiv -(M_1 + c)$. Hence, $\|u\|_0 \leq M_1$.

As in the proof of Proposition 3.1, one has that any solution u of (4.1_λ) satisfies

$$\left|\left(\phi(u'(t))\right)'\right| \leq \psi\left(|u'(t)|\right)\left(l_0(t) + c(t)|u'(t)|^{(p-1)/p}\right) \quad \text{a.e. } t \in I,$$

where l_0 is defined in (3.4). It follows from Lemma 2.9 applied with $d_0 = 0$ that there exists M_0 such that any solution u of (4.1_λ) satisfies $\|u'\|_0 < M_0$.

Finally, set $M = \max\{M_0, 1 + M_1\}$. We have that $\|u\|_1 < M$ for any solution u of (4.1_λ).
$\square$

Proposition 4.2 *Assume that* (H_ϕ), (H_f), (H_P) *and* (WN) *hold. Then, for every* $\lambda \in [0,1]$, *problem* (4.1_λ) *has at least one solution.*

Proof Let $N_{\widehat{f}} : \mathcal{C}^1(I) \to \mathcal{C}(I)$ be defined in (3.5). Let us consider the operators $\widehat{L} : \mathcal{C}^1(I) \to \mathcal{C}(I)$ and $\mathcal{S}_P : [0,1] \times \mathcal{C}^1(I) \to \mathbb{R} \times \mathcal{C}(I)$ defined by

$$\widehat{L}(u)(t) = \frac{t}{T}N_{\widehat{f}}(u)(T), \tag{4.2}$$

and

$$\mathcal{S}_P(\lambda, u) = \left(u(0) + \frac{1}{T}N_{\widehat{f}}(u)(T), \overline{a}_0\left(\lambda\left(N_{\widehat{f}}(u) - \widehat{L}(u)\right)\right) + \lambda\left(N_{\widehat{f}}(u) - \widehat{L}(u)\right)\right), \tag{4.3}$$

where $\overline{a}_0$ is obtained in Lemma 2.5. Now, we define $\mathcal{P} : [0,1] \times \mathcal{C}^1(I) \to \mathcal{C}^1(I)$ by

$$\mathcal{P} = \Phi \circ \mathcal{S}_P. \tag{4.4}$$

Again, we deduce from Lemmas 2.4, 2.5 and 2.6 that $\mathcal{S}_P$ and $\mathcal{P}$ are continuous and completely continuous.

Now, we study the fixed points of $\mathcal{P}$. Let $u \in \mathcal{C}^1(I)$ and $\lambda \in [0,1]$ be such that $u = \mathcal{P}(\lambda, u)$. One has, by Lemma 2.5,

$$u(0) = u(0) + \frac{1}{T}N_{\widehat{f}}(u)(T),$$

$$u(T) = u(0) + \frac{1}{T}N_{\widehat{f}}(u)(T) + \mathcal{G}\left(\overline{a}_0\left(\lambda\left(N_{\widehat{f}}(u) - \widehat{L}(u)\right)\right), \lambda\left(N_{\widehat{f}}(u) - \widehat{L}(u)\right)\right)$$

$$= u(0).$$

Notice that $(N_{\widehat{f}}(u)(T))/T = 0$. Also,

$$\phi(u') = \overline{a}_0\left(\lambda\left(N_{\widehat{f}}(u) - \widehat{L}(u)\right)\right) + \lambda\left(N_{\widehat{f}}(u) - \widehat{L}(u)\right).$$

Hence, by Lemma 2.4, it is absolutely continuous and

$$\left(\phi\left(u'(t)\right)\right)' = \lambda\left(\widehat{f}\left(t, u(t), u'(t)\right) - \frac{1}{T} N_{\widehat{f}}(u)(T)\right)$$

$$= \lambda\widehat{f}\left(t, u(t), u'(t)\right) + \frac{(1-\lambda)}{T} \int_0^T \widehat{f}\left(t, u(t), u'(t)\right) dt \quad \text{a.e. } t \in I.$$

Moreover,

$$\phi\left(u'(0)\right) = \overline{a}_0\left(\lambda\left(N_{\widehat{f}}(u) - \widehat{L}(u)\right)\right) + \lambda\left(N_{\widehat{f}}(u)(0) - \widehat{L}(u)(0)\right),$$

$$\phi\left(u'(T)\right) = \overline{a}_0\left(\lambda\left(N_{\widehat{f}}(u) - \widehat{L}(u)\right)\right) + \lambda\left(N_{\widehat{f}}(u)(T) - \widehat{L}(u)(T)\right)$$

$$= \phi\left(u'(0)\right).$$

So, fixed points of $\mathcal{P}$ are solutions of (4.1_λ).

Let $M > \max\{\|\alpha\|_0, \|\beta\|_0\}$ be the constant obtained in Proposition 4.1 and set

$$\mathcal{U} = \left\{u \in \mathcal{C}^1(I) : \|u\|_1 < M\right\}. \tag{4.5}$$

Proposition 4.1 implies that $u \neq \mathcal{P}(\lambda, u)$ for all $(\lambda, u) \in [0,1] \times \partial\mathcal{U}$. By the homotopy property of the fixed point index,

$$\text{index}\left(\mathcal{P}(\lambda, \cdot), \mathcal{U}\right) = \text{index}\left(\mathcal{P}(0, \cdot), \mathcal{U}\right) \quad \forall\lambda \in [0,1].$$

Observe that

$$\mathcal{P}(0, u) = \Phi\left(u(0) + \frac{1}{T} N_{\widehat{f}}(u)(T), \overline{a}_0(0)\right) = u(0) + \frac{1}{T} N_{\widehat{f}}(u)(T) \in \mathbb{R}.$$

One has $\partial(\mathcal{U} \cap \mathbb{R}) = \{-M, M\}$, and

$$\mathcal{P}(0, M) = M + \frac{1}{T} \int_0^T \widehat{f}(t, M, 0) \, dt$$

$$= M + \frac{1}{T} \int_0^T f\left(t, \beta(t), \beta'(t)\right) - \widehat{h}(t)\left(M - \beta(t)\right) dt$$

$$> M,$$

since $M > \|\beta\|_0$ and $\widehat{h}(t) < \min\{0, f(t, \beta(t), \beta'(t))\}$. Similarly,

$$\mathcal{P}(0, -M) < -M.$$

By the contraction property of the fixed point index (see [18, Chapter 4, Section 12, Theorem 6.2]),

$$\text{index}\left(\mathcal{P}(0, \cdot), \mathcal{U}\right) = \text{index}\left(\mathcal{P}(0, \cdot), (\mathcal{U} \cap \mathbb{R})\right) = -1.$$

Therefore, for every $\lambda \in [0,1]$,

$$\text{index}\big(\mathcal{P}(\lambda, \cdot), \mathcal{U}\big) = -1. \tag{4.6}$$

Thus, $\mathcal{P}(\lambda, \cdot)$ has a fixed point, and hence (4.1_λ) has a solution. $\qquad\square$

Now, we can establish the existence of a solution to (1.1), (1.3).

Theorem 4.3 *Assume that* (H_ϕ), (H_f), (H_P) *and* (WN) *hold. Then periodic problem* (1.1), (1.3) *has a solution* $u \in W(I)$ *such that* $\alpha(t) \le u(t) \le \beta(t)$ *for every* $t \in I$.

Proof Proposition 4.2 insures the existence of $u \in W(I)$, a solution of (4.1_λ) for $\lambda = 1$. To conclude, we have to show that $\alpha(t) \le u(t) \le \beta(t)$ for every $t \in I$, since $\widehat{f}(t,x,y) = f(t,x,y)$ for $x \in [\alpha(t), \beta(t)]$.

Using (H_P), we obtain that

$$\big(\phi\big(u'(t)\big)\big)' < \big(\phi\big(\alpha'(t)\big)\big)' \quad \text{a.e. } t \in \big\{t \in I : u(t) < \alpha(t)\big\},$$

$$\big(\phi\big(u'(t)\big)\big)' > \big(\phi\big(\beta'(t)\big)\big)' \quad \text{a.e. } t \in \big\{t \in I : u(t) > \beta(t)\big\}.$$

It follows from Lemma 2.7 that $\alpha(t) \le u(t) \le \beta(t)$ for all $t \in I$. $\qquad\square$

5 The Neumann problem

In this section, we consider problem (1.1) with the Neumann boundary condition. In order to establish the existence of a solution to (1.1), (1.3), we consider the following family of problems defined for $\lambda \in [0,1]$:

$$\big(\phi\big(u'(t)\big)\big)' = \lambda \widetilde{f}\big(t, u(t), u'(t)\big) + \frac{(1-\lambda)}{T} \int_0^T \widetilde{f}\big(t, u(t), u'(t)\big)\,dt \quad \text{a.e. } t \in I,$$

$$\phi\big(u'(0)\big) = \mu, \qquad \phi\big(u'(T)\big) = \nu, \tag{5.1_\lambda}$$

where $\widetilde{f} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\widetilde{f}(t,x,y) = \begin{cases} f(t, \beta(t), \beta'(t)) - \widetilde{h}(t)(x - \beta(t)) & \text{if } x > \beta(t), \\ f(t,x,y) & \text{if } \alpha(t) \le x \le \beta(t), \\ f(t, \alpha(t), \alpha'(t)) + \widetilde{g}(t)(x - \alpha(t)) & \text{if } x < \alpha(t); \end{cases} \tag{5.2}$$

with $\widetilde{g}, \widetilde{h} \in L^1(I)$ chosen such that

$$\widetilde{g}(t) > \max\left\{0, f\big(t, \alpha(t), \alpha'(t)\big) - \frac{1}{T}(\nu - \mu)\right\},$$

$$\widetilde{h}(t) < \min\left\{0, f\big(t, \beta(t), \beta'(t)\big) - \frac{1}{T}(\nu - \mu)\right\}. \tag{5.3}$$

We show that the solutions to these problems are *a priori* bounded. Let $w \in \mathcal{C}^1(I)$ and $v \in W(I)$ be defined by

$$w(t) = \mu + \frac{t}{T}(\nu - \mu) \quad \text{and} \quad v = \Phi(0, w). \tag{5.4}$$

Observe that

$$\phi(v') = w.$$

Proposition 5.1 *Assume that* (H_ϕ), (H_f), (H_N) *and* (WN) *hold. Then there exists* $M > \|v'\|_0$ *such that any solution* u *of* (5.1_λ) *satisfies* $\|u - v\|_0 < M$ *and* $\|u'\|_0 < M$. *Moreover,*

$$\left\{ u \in \mathcal{C}^1(I) : \alpha(t) \le u(t) \le \beta(t) \; \forall t \in I \right\} \subset \left\{ u \in \mathcal{C}^1(I) : \|u - v\|_0 < M \right\}.$$

Proof Fix $M_1 > 0$ such that

$$-M_1 + v(t) \le \min_{t \in I} \alpha(t) - 1 \quad \text{and} \quad M_1 + v(t) \ge \max_{t \in I} \beta(t) + 1 \quad \forall t \in I.$$

We claim that any solution u of (5.1_λ) is such that $\|u - v\|_0 \le M_1$. For every $u \in W(I)$, one has almost everywhere on $\{t \in I : u(t) < -M_1 + v(t)\}$,

$$\begin{aligned}
\widetilde{f}\big(t, u(t), u'(t)\big) &= f\big(t, \alpha(t), \alpha'(t)\big) + \widetilde{g}(t)\big(u(t) - \alpha(t)\big) \\
&< f\big(t, \alpha(t), \alpha'(t)\big) - \widetilde{g}(t) \\
&\le \frac{1}{T}(v - \mu).
\end{aligned} \tag{5.5}$$

Similarly,

$$\widetilde{f}\big(t, u(t), u'(t)\big) > \frac{1}{T}(v - \mu) \quad \text{a.e. on } \left\{ t \in I : u(t) > M_1 + v(t) \right\}. \tag{5.6}$$

Let u be a solution of (5.1_λ). One has

$$v - \mu = \int_0^T \big(\phi(u'(t))\big)' \, dt = \int_0^T \widetilde{f}\big(t, u(t), u'(t)\big) \, dt. \tag{5.7}$$

Combining (5.5), (5.6) and (5.7), we deduce that almost everywhere on $\{t \in I : u(t) < -M_1 + v(t)\}$,

$$\begin{aligned}
\big(\phi(u'(t))\big)' &= \lambda \widetilde{f}\big(t, u(t), u'(t)\big) + \frac{(1 - \lambda)}{T}(v - \mu) \\
&\le \frac{1}{T}(v - \mu) \\
&= \big(\phi(v'(t))\big)'.
\end{aligned}$$

Similarly,

$$\big(\phi(u'(t))\big)' \ge \big(\phi(v'(t))\big)' \quad \text{a.e. on } \left\{ t \in I : u(t) > M_1 + v(t) \right\}.$$

Moreover,

$$\phi(u'(0)) = \mu = \phi(v'(0)) \quad \text{and} \quad \phi(u'(T)) = v = \phi(v'(T)).$$

It follows from Lemma 2.7 that $\|u - v\|_0 \le M_1$, or there exists $c > 0$ such that $|u(t) - v(t)| = M_1 + c$ for all $t \in I$. If $u(t) = v(t) + M_1 + c$ for all $t \in I$, then

$$
\begin{aligned}
\nu - \mu &= \int_0^T \widetilde{f}\big(t, u(t), u'(t)\big)\, dt \\
&= \int_0^T f\big(t, \beta(t), \beta'(t)\big) - \widetilde{h}(t)\big(v(t) + M_1 + c - \beta(t)\big)\, dt \\
&> \int_0^T \big(\phi\big(\beta'(t)\big)\big)'\, dt \\
&= \phi\big(\beta'(T)\big) - \phi\big(\beta'(0)\big) \\
&\ge \nu - \mu,
\end{aligned}
$$

a contradiction. Similarly, one cannot have $u(t) = v(t) - M_1 - c$ for all $t \in I$. Hence, $\|u - v\|_0 \le M_1$.

Let

$$
\widetilde{l}_1(t) = \max\big\{ \big| f\big(t, \alpha(t), \alpha'(t)\big)\big| + 2M_1\big|\widetilde{g}(t)\big|,\ \big|f\big(t, \beta(t), \beta'(t)\big)\big| + 2M_1\big|\widetilde{h}(t)\big|\big\}.
$$

Observe that, by (WN), one has that for any u solution of (5.1_λ),

$$
\begin{aligned}
\big|\big(\phi\big(u'(t)\big)\big)'\big| &\le \max\big\{\widetilde{l}_1(t), \psi\big(\big|u'(t)\big|\big)\big(l(t) + c(t)\big|u'(t)\big|^{(p-1)/p}\big)\big\} \\
&\le \psi\big(\big|u'(t)\big|\big)\big(\widetilde{l}_0(t) + c(t)\big|u'(t)\big|^{(p-1)/p}\big) \quad \text{a.e. } t \in I,
\end{aligned}
$$

with

$$
\widetilde{l}_0(t) = \max\left\{ l(t), \frac{\widetilde{l}_1(t)}{k} \right\}. \tag{5.8}
$$

It follows from Lemma 2.9 applied with $d_0 = \phi^{-1}(\mu)$ that there exists M_0 such that any solution u of (5.1_λ) satisfies $\|u'\|_0 < M_0$.

Finally, set $M = \max\{M_0, 1 + M_1, 1 + \|v'\|_0\}$, we get the conclusion. □

Proposition 5.2 *Assume that* (H_ϕ), (H_f), (H_N) *and* (WN) *hold. Then, for every* $\lambda \in [0,1]$, *problem* (5.1_λ) *has at least one solution.*

Proof Let $N_{\widetilde{f}} : \mathcal{C}^1(I) \to \mathcal{C}(I)$, $\widetilde{L} : \mathcal{C}^1(I) \to \mathcal{C}(I)$ and $\mathcal{S}_N : [0,1] \times \mathcal{C}^1(I) \to \mathbb{R} \times \mathcal{C}(I)$ be defined respectively by

$$
N_{\widetilde{f}}(u)(t) = \int_0^t \widetilde{f}\big(s, u(s), u'(s)\big)\, ds,
$$

$$
\widetilde{L}(u)(t) = \frac{t}{T} N_{\widetilde{f}}(u)(T),
$$

$$
\mathcal{S}_N(\lambda, u) = \left(u(0) + \frac{1}{T}\big(\mu - \nu + N_{\widetilde{f}}(u)(T)\big), w + \lambda\big(N_{\widetilde{f}}(u) - \widetilde{L}(u)\big)\right),
$$

with $w \in \mathcal{C}^1(I)$ defined in (5.4). Now, we define $\mathcal{N} : [0,1] \times \mathcal{C}^1(I) \to \mathcal{C}^1(I)$ by

$$\mathcal{N} = \Phi \circ \mathcal{S}_N. \tag{5.9}$$

Again, we deduce from Lemmas 2.4, 2.5 and 2.6 that $\mathcal{S}_N$ and $\mathcal{N}$ are continuous and completely continuous.

Now, we study the fixed points of $\mathcal{N}$. Let $u \in \mathcal{C}^1(I)$ and $\lambda \in [0,1]$ be such that $u = \mathcal{N}(\lambda, u)$. One has

$$u(0) = u(0) + \frac{1}{T}\big(\mu - \nu + N_{\tilde{f}}(u)(T)\big).$$

So,

$$\nu - \mu = N_{\tilde{f}}(u)(T).$$

Also, $\phi(u') = w + \lambda(N_{\tilde{f}}(u) - \widetilde{L}(u))$. Hence, by Lemma 2.4, it is absolutely continuous and

$$\big(\phi\big(u'(t)\big)\big)' = \lambda\left(\widetilde{f}\big(t, u(t), u'(t)\big) - \frac{1}{T}N_{\tilde{f}}(u)(T)\right) + \frac{1}{T}(\nu - \mu)$$

$$= \lambda\widetilde{f}\big(t, u(t), u'(t)\big) + \frac{(1-\lambda)}{T}\int_0^T \widetilde{f}\big(t, u(t), u'(t)\big)\, dt \quad \text{a.e. } t \in I.$$

Moreover,

$$\phi\big(u'(0)\big) = w(0) + \lambda\big(N_{\tilde{f}}(u)(0) - \widetilde{L}(u)(0)\big) = \mu,$$
$$\phi\big(u'(T)\big) = w(T) + \lambda\big(N_{\tilde{f}}(u)(T) - \widetilde{L}(u)(T)\big) = \nu.$$

So, fixed points of $\mathcal{N}$ are solutions of (5.1_λ).

Let $M > 0$ be the constant obtained in Proposition 5.1 and $v = \Phi(0, w)$. We set

$$\mathcal{U} = \big\{u \in \mathcal{C}^1(I) : \|u - v\|_0 < M, \|u'\|_0 < M\big\}. \tag{5.10}$$

Proposition 5.1 implies that $u \neq \mathcal{N}(\lambda, u)$ for all $(\lambda, u) \in [0,1] \times \partial\mathcal{U}$. By the homotopy property of the fixed point index,

$$\text{index}\big(\mathcal{N}(\lambda, \cdot), \mathcal{U}\big) = \text{index}\big(\mathcal{N}(0, \cdot), \mathcal{U}\big) \quad \forall\lambda \in [0,1]. \tag{5.11}$$

Observe that

$$\mathcal{N}(0, u) = u(0) + \frac{1}{T}\big(\mu - \nu + N_{\tilde{f}}(u)(T)\big) + \Phi(0, w) \in \mathbb{R} + v.$$

Let $X = \{u = r + v : r \in \mathbb{R}\} \subset \mathcal{C}^1(I)$. Notice that X is not a normed vectorial space if $v \neq 0$. Nevertheless, if $v \neq 0$, X is an affine space and hence it is an ANR. By the contraction property of the fixed point index (see [18, Chapter 4, Section 12, Theorem 6.2]),

$$\text{index}\big(\mathcal{N}(0, \cdot), \mathcal{U}\big) = \text{index}\big(\mathcal{N}(0, \cdot), \mathcal{U} \cap X\big). \tag{5.12}$$

One has $\partial(\mathcal{U} \cap X) = \{-M + \nu, M + \nu\}$. Consider $H : [-M, M] \to \mathbb{R}$ defined by

$$H(r) = \mathcal{N}(0, r + \nu) - \nu.$$

By (5.6),

$$H(M) = \mathcal{N}(0, M + \nu) - \nu = M + \frac{1}{T}\left(\mu - \nu + \int_0^T \widetilde{f}(t, M + \nu(t), \nu'(t))\, dt\right) > M,$$

since $M + \nu(t) > \beta(t)$ for every $t \in I$. Similarly,

$$H(-M) < -M.$$

By the commutativity property of the fixed point index (see [18, Chapter 4, Section 12, Theorem 6.2]),

$$\text{index}\big(\mathcal{N}(0, \cdot), \mathcal{U} \cap X\big) = \text{index}\big(H, \,]-M, M[\big) = -1. \tag{5.13}$$

Combining (5.11), (5.12) and (5.13), we deduce that for every $\lambda \in [0, 1]$,

$$\text{index}\big(\mathcal{N}(\lambda, \cdot), \mathcal{U}\big) = -1. \tag{5.14}$$

Thus, $\mathcal{N}(\lambda, \cdot)$ has a fixed point, and hence (5.1_λ) has a solution. □

Now, we can establish the existence of a solution to (1.1), (1.4).

Theorem 5.3 *Assume that* (H_ϕ), (H_f), (H_N) *and* (WN) *hold. Then Neumann problem* (1.1), (1.4) *has a solution* $u \in W(I)$ *such that* $\alpha(t) \le u(t) \le \beta(t)$ *for every* $t \in I$.

Proof Proposition 5.2 insures the existence of $u \in W(I)$, a solution of (5.1_λ) for $\lambda = 1$. To conclude, we have to show that $\alpha(t) \le u(t) \le \beta(t)$ for every $t \in I$, since $\widetilde{f}(t, x, y) = f(t, x, y)$ for $x \in [\alpha(t), \beta(t)]$.

Using (H_N), we obtain that

$$\big(\phi\big(u'(t)\big)\big)' < \big(\phi\big(\alpha'(t)\big)\big)' \quad \text{a.e. } t \in \big\{t \in I : u(t) < \alpha(t)\big\},$$
$$\big(\phi\big(u'(t)\big)\big)' > \big(\phi\big(\beta'(t)\big)\big)' \quad \text{a.e. } t \in \big\{t \in I : u(t) > \beta(t)\big\}.$$

It follows from Lemma 2.7 that $\alpha(t) \le u(t) \le \beta(t)$ for all $t \in I$. □

6 Multiplicity result

In this section, we establish the existence of at least three solutions to problem (1.1).

Theorem 6.1 *Assume that* (H_ϕ), (H_f) *and the following conditions are satisfied*:

$(H_\mathcal{B})'$ *For* $i = 1, 2$, *there exist* $\alpha_i, \beta_i \in W(I)$, *respectively strict lower and upper solutions of* (1.1), *such that* $\alpha_i(t) < \beta_i(t)$, $\alpha_1(t) \le \alpha_2(t)$, $\beta_1(t) \le \beta_2(t)$ *for all* $t \in I$, *and* $\{t \in I : \alpha_2(t) > \beta_1(t)\} \ne \emptyset$.

(WN)′ *There exist* $k > 0$, $p \in]1, \infty]$, $c \in L^p(I, [0, \infty[)$, $l \in L^1(I, [0, \infty[)$ *and* $\psi : [0, \infty[\to [k, \infty[$ *such that*

$$\frac{1}{\psi \circ \phi^{-1}} \in L^1_{loc}(\mathbb{R}), \qquad \int_0^{\pm\infty} \frac{ds}{\psi(|\phi^{-1}(s)|)} = \pm\infty,$$

and

$$\left| f(t, x, y) \right| \leq \psi(|y|)\left(l(t) + c(t)|y|^{(p-1)/p} \right)$$

$$\forall x \in [\alpha_1(t), \beta_2(t)], y \in \mathbb{R}, \text{ and a.e. } t \in I,$$

with $(p - 1)/p = 1$ *if* $p = \infty$.

Then problem (1.1) *has at least three solutions* u_1, u_2, u_3 *such that*

$$\alpha_1(t) < u_3(t) < \beta_2(t), \qquad \alpha_i(t) < u_i(t) < \beta_i(t) \quad \forall t \in I, \text{ and } i = 1, 2,$$

and $\{t \in I : \beta_1(t) \leq u_3(t) \leq \alpha_2(t)\} \neq \emptyset$.

Proof Let $\mathcal{B}$ denote Dirichlet boundary condition (1.2). Consider $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$, the operators defined as in (3.7) and associated to the pairs of lower and upper solutions (α_1, β_1), (α_2, β_2) and (α_1, β_2), respectively. Let $\mathcal{U}_i \subset \mathcal{C}^1(I)$ be the open set defined as in (3.8) and associated to the operator $\mathcal{D}_i$ for $i = 1, 2, 3$. From equation (3.9), one has

$$\text{index}(\mathcal{D}_i(1, \cdot), \mathcal{U}_i) = 1 \quad \text{for } i = 1, 2, 3.$$

From the proof of Theorem 3.3, we deduce that any fixed point u of $\mathcal{D}_i(1, \cdot)$ is a solution of (1.1), (1.2) and is such that $\alpha_i(t) \leq u(t) \leq \beta_i(t)$ for all $t \in I$ if $i = 1, 2$, and such that $\alpha_1(t) \leq u(t) \leq \beta_2(t)$ for all $t \in I$ if $i = 3$. Using the fact that α_i and β_i are respectively strict lower and upper solutions of (1.1), it follows from Lemma 2.8 that $\mathcal{D}_i(1, \cdot)$ has no fixed points in $\mathcal{U}_i \backslash \overline{\mathcal{V}_i}$, with

$$\mathcal{V}_i = \left\{ u \in \mathcal{C}^1(I) : \alpha_i(t) < u(t) < \beta_i(t) \ \forall t \in I \right\} \quad \text{for } i = 1, 2,$$
$$\mathcal{V}_3 = \left\{ u \in \mathcal{C}^1(I) : \alpha_1(t) < u(t) < \beta_2(t) \ \forall t \in I \right\}.$$

Hence, by the excision property of the fixed point index,

$$\text{index}(\mathcal{D}_i(1, \cdot), \mathcal{V}_i) = 1 \quad \text{for } i = 1, 2, 3.$$

Since $\alpha_1(t) \leq \beta_1(t) \leq \beta_2(t)$, $\alpha_1(t) \leq \alpha_2(t) \leq \beta_2(t)$ for all $t \in I$ and $\{t \in I : \alpha_2(t) > \beta_1(t)\} \neq \emptyset$, one has

$$\mathcal{V}_1 \cup \mathcal{V}_2 \subset \mathcal{V}_3, \quad \text{and} \quad \mathcal{V}_3 \backslash (\overline{\mathcal{V}_1 \cup \mathcal{V}_2}) \neq \emptyset,$$

and

$$\mathcal{D}_i(1, u) = \mathcal{D}_3(1, u) \quad \forall u \in \mathcal{V}_i, i = 1, 2.$$

This combined with the additivity of the fixed point index implies that

$$\text{index}\big(\mathcal{D}_3(1,\cdot),\mathcal{V}_3\backslash(\overline{\mathcal{V}_1\cup\mathcal{V}_2})\big) = \text{index}\big(\mathcal{D}_3(1,\cdot),\mathcal{V}_3\big)$$
$$- \text{index}\big(\mathcal{D}_3(1,\cdot),\mathcal{V}_1\big) - \text{index}\big(\mathcal{D}_3(1,\cdot),\mathcal{V}_2\big)$$
$$= -1.$$

Therefore, problem (1.1) has at least three solutions $u_1 \in \mathcal{V}_1$, $u_2 \in \mathcal{V}_2$ and $u_3 \in \mathcal{V}_3\backslash(\overline{\mathcal{V}_1\cup\mathcal{V}_2})$.

We argue similarly if $\mathcal{B}$ denotes periodic boundary condition (1.3) (resp. Neumann boundary condition (1.4)) by using the results of Section 4 (resp. Section 5). □

Authors' contributions
Authors contributed equally. They read and approved the final manuscript.

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