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Infinitely many periodic solutions for subquadratic second-order Hamiltonian systems

Hua Gu* and Tianqing An

*Correspondence:
guhuasy@hhu.edu.cn
College of Science, Hohai University,
Nanjing, 210098, China

Abstract
In this paper, we investigate the existence of infinitely many periodic solutions for a class of subquadratic nonautonomous second-order Hamiltonian systems by using the variant fountain theorem.

1 Introduction

Consider the second-order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) + \nabla_u W(t,u) = 0, & \forall t \in \mathbb{R}, \\ u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T), & T > 0, \end{cases} \tag{1.1}$$

where $W(t,u)$ is also T-periodic and satisfies the following assumption (A):

(A) $W(t,u)$ is measurable in t for all $u \in \mathbb{R}^N$, continuously differentiable in u for a.e. $t \in [0,T]$ and there exist $a \in C(R^+, R^+)$ and $b \in L^1([0,T], R^+)$ such that

$$|W(t,u)| \le a(|u|)b(t), \qquad |\nabla_u W(t,u)| \le a(|u|)b(t)$$

for all $u \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Here and in the sequel, $\langle \cdot, \cdot \rangle$ and $|\cdot|$ always denote the standard inner product and the norm in $\mathbb{R}^N$ respectively.

There have been many investigations on the existence and multiplicity of periodic solutions for Hamiltonian systems via the variational methods (see [1–7] and the references therein). In [6], Zhang and Liu studied the asymptotically quadratic case of $W(t,u) = \frac{1}{2}\langle U(t)u, u\rangle + W_1(t,u)$ under the following assumptions:

(AQ_1) $W_1(t,u) \ge 0$ for all $(t,u) \in [0,T] \times \mathbb{R}^N$, and there exist constants $\mu \in (0,2)$ and $R_1 > 0$ such that

$$\langle \nabla_u W_1(t,u), u\rangle \le \mu W_1(t,u), \quad \forall t \in [0,T] \text{ and } |u| \ge R_1;$$

(AQ_2) $\lim_{|u|\to 0} \frac{W_1(t,u)}{|u|^2} = \infty$ uniformly for $t \in [0,T]$, and there exist constants $c_2, R_2 > 0$ such that

$$W_1(t,u) \le c_2|u|, \quad \forall t \in [0,T] \text{ and } |u| \le R_2;$$

(AQ_3) $\liminf_{|u|\to\infty} \frac{W_1(t,u)}{|u|} \geq d > 0$ uniformly for $t \in [0,T]$.

They obtained the existence of infinitely many periodic solutions of (1.1) provided $W_1(t,u)$ *is even in* u (see Theorem 1.1 of [6]).

The subquadratic condition (AQ_1) is widely used in the investigation of nonlinear differential equations. This condition was weakened by some researchers; see, for example, [4] of Jiang and Tang. This paper considers the case of $U(t) \equiv 0$, then $W(t,u) = W_1(t,u)$. Motivated by [4] and [6], we replace (AQ_1) with the following condition:

(AQ_1') $W(t,u) \geq 0$ for all $(t,u) \in [0,T] \times \mathbb{R}^N$, and

$$\lim_{|u|\to\infty} \big(\langle \nabla_u W(t,u), u\rangle - 2W(t,u)\big) = -\infty \quad \text{and}$$

$$\lim_{|u|\to\infty} \frac{W(t,u)}{|u|^2} = 0 \quad \text{uniformly for } t \in [0,T].$$

The condition (AQ_1') implies that for some constant $R_1' > 0$,

$$\langle \nabla_u W(t,u), u\rangle \leq 2W(t,u), \quad \forall t \in [0,T] \text{ and } |u| \geq R_1'. \tag{1.2}$$

By the assumption (A) and the condition (AQ_1'), for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$W(t,u) \leq \epsilon |u|^2 + \max_{s\in[0,\delta]} a(s)b(t), \tag{1.3}$$

for $\forall u \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Meanwhile, we weaken the condition (AQ_3) to (AQ_3') as follows:

(AQ_3') There exists a constant $\varrho \in (0,1]$ such that

$$\liminf_{|u|\to\infty} \frac{W(t,u)}{|u|^{\varrho}} \geq d > 0 \quad \text{uniformly for } t \in [0,T].$$

Then our main result is the following theorem.

Theorem 1.1 *Assume that* (AQ_1'), (AQ_2), (AQ_3') *hold and* $W(t,u)$ *is even in* u. *Then* (1.1) *possesses infinitely many solutions*.

Remark The conditions (AQ_1) and (AQ_3) are stronger than (AQ_1') and (AQ_3'). Then Theorem 1.1 above is different from Theorem 1.1 of [6].

2 Preliminaries

In this section, we establish the variational setting for our problem and give the variant fountain theorem. Let $E = H_T^1$ be the usual Sobolev space with the inner product

$$\langle u, v\rangle_E = \int_0^T \big\langle u(t), v(t)\big\rangle\,dt + \int_0^T \big\langle \dot{u}(t), \dot{v}(t)\big\rangle\,dt.$$

We define the functional on E by

$$\Phi(u) = \frac{1}{2}\int_0^T |\dot{u}|^2\,dt - \Psi(u),$$

where $\Psi(u) = \int_0^T W(t,u(t))\,dt$. Then Φ and Ψ are continuously differentiable and

$$\langle \Phi'(u), v\rangle = \int_0^T \langle \dot{u}, \dot{v}\rangle\,dt - \int_0^T \langle \nabla_u W(t,u), v\rangle\,dt.$$

Define a self-adjoint linear operator $\mathcal{B}: L^2([0,T];\mathbb{R}^N) \to L^2([0,T];\mathbb{R}^N)$ by

$$\int_0^T \langle \mathcal{B}u, v\rangle\,dt = \int_0^T \langle \dot{u}(t), \dot{v}(t)\rangle\,dt$$

with the domain $D(\mathcal{B}) = E$. Then $\mathcal{B}$ has a sequence of eigenvalues $\sigma_k = \frac{4k^2\pi^2}{T^2}$ $(k = 0,1,2,\ldots)$. Let $\{e_j\}_{j=0}^{+\infty}$ be the system of eigenfunctions corresponding to $\{\sigma_j\}_{j=0}^{+\infty}$, it forms an orthogonal basis in L^2. Denote by $E^+ = \{u \in E \mid \int_0^T u(t)\,dt = 0\}$, $E^0 = \mathbb{R}^N$, it is well known that

$$E^0 = \ker \mathcal{B} = \operatorname{span}\{e_0\},$$
$$E^+ = \operatorname{span}\{e_j \mid j = 1,2,\ldots\},$$

and E possesses orthogonal decomposition $E = E^0 \oplus E^+$. For $u \in E$, we have

$$u = u^0 + u^+ \in E^0 \oplus E^+.$$

We can define on E a new inner product and the associated norm by

$$\langle u, v\rangle_0 = \langle \mathcal{B}u^+, v^+\rangle_{L^2} + \langle u^0, v^0\rangle_{L^2},$$

and

$$\|u\| = \langle u, u\rangle_0^{\frac{1}{2}}.$$

Therefore, Φ can be written as

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \Psi(u). \tag{2.1}$$

Direct computation shows that

$$\begin{aligned} \langle \Psi'(u), v\rangle &= \int_0^T \langle \nabla_u W(t,u), v\rangle\,dt, \\ \langle \Phi'(u), v\rangle &= \langle u^+, v^+\rangle_0 - \langle \Psi'(u), v\rangle \end{aligned} \tag{2.2}$$

for all $u, v \in E$ with $u = u^0 + u^+$ and $v = v^0 + v^+$ respectively. It is known that $\Psi' : E \to E$ is compact.

Denote by $|\cdot|_p$ the usual norm of L^P, then there exists a $\tau_p > 0$ such that

$$|u|_p \le \tau_p \|u\|, \quad \forall u \in E. \tag{2.3}$$

We state an abstract critical point theorem founded in [8]. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=1}^k X_j$ and $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$. Consider the following C^1-functional $\Phi_\lambda : E \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) := A(u) - \lambda B(u), \quad \lambda \in [1,2].$$

Theorem 2.1 [8, Theorem 2.2] *Assume that the functional Φ_λ defined above satisfies the following*:

- (T_1) *Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$*;
- (T_2) *$B(u) \geq 0$ for all $u \in E$, and $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace of E*;
- (T_3) *There exist $\rho_k > r_k > 0$ such that*

$$\alpha_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \geq 0 > \beta_k(\lambda) := \max_{u\in Y_k,\|u\|=r_k} \Phi_\lambda(u), \quad \forall\lambda \in [1,2]$$

and

$$\xi_k(\lambda) := \inf_{u\in Z_k,\|u\|\leq\rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

Then there exist $\lambda_n \to 1$, $u_{\lambda_n} \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \qquad \Phi_{\lambda_n}(u_{\lambda_n}) \to \eta_k \in \left[\xi_k(2), \beta_k(1)\right] \quad \text{as } n \to \infty.$$

Particularly, if $\{u_{\lambda_n}\}$ has a convergent subsequence for every k, then Φ_1 has infinitely many nontrivial critical points $\{u_k\} \subset E \setminus \{0\}$ satisfying $\Phi_1(u_k) \to 0^-$ as $k \to \infty$.

In order to apply this theorem to prove our main result, we define the functionals A, B and Φ_λ on our working space E by

$$A(u) = \frac{1}{2}\left\|u^+\right\|^2, \qquad B(u) = \int_0^T W(t,u)\,dt \tag{2.4}$$

and

$$\Phi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2}\left\|u^+\right\|^2 - \lambda\int_0^T W(t,u)\,dt \tag{2.5}$$

for all $u = u^0 + u^+ \in E = E^0 + E^+$ and $\lambda \in [1,2]$. Then $\Phi_\lambda \in C^1(E,\mathbb{R})$ for all $\lambda \in [1,2]$. Let $X_j = \operatorname{span}\{e_j\}$, $j = 0,1,2,\ldots$. Note that $\Phi_1 = \Phi$, where Φ is the functional defined in (2.1).

3 Proof of Theorem 1.1

We firstly establish the following lemmas.

Lemma 3.1 *Assume that* (AQ'_1) *and* (AQ'_3) *hold. Then $B(u) \geq 0$ for all $u \in E$ and $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace of E.*

Proof Since $W(t,u) \geq 0$, by (2.4), it is obvious that $B(u) \geq 0$ for all $u \in E$.

By the proof of Lemma 2.6 of [6], for any finite-dimensional subspace $Y \subset E$, there exists a constant $\epsilon > 0$ such that

$$m\bigl(\bigl\{t \in [0,T] : |u| \geq \epsilon\|u\|\bigr\}\bigr) \geq \epsilon, \quad \forall u \in Y \setminus \{0\}, \tag{3.1}$$

where $m(\cdot)$ is the Lebesgue measure.

For the ϵ given in (3.1), let

$$\Lambda_u = \bigl\{t \in [0,T] : |u| \geq \epsilon\|u\|\bigr\}, \quad \forall u \in Y \setminus \{0\}.$$

Then $m(\Lambda_u) \geq \epsilon$. By (AQ_3'), there exists a constant $R_3 > R_1'$ such that

$$W(t,u) \geq d|u|^{\varrho}/2, \quad \forall t \in [0,T] \text{ and } |u| \geq R_3, \tag{3.2}$$

where R_1' is the constant given in (1.2). Note that

$$\bigl|u(t)\bigr| \geq R_3, \quad \forall t \in \Lambda_u \tag{3.3}$$

for any $u \in Y$ with $\|u\| \geq R_3/\epsilon$. Thus,

$$\begin{aligned} B(u) &= \int_0^T W(t,u)\,dt \geq \int_{\Lambda_u} W(t,u)\,dt \geq \int_{\Lambda_u} d|u|^{\varrho}/2\,dt \\ &\geq d\epsilon^{\varrho}\|u\|^{\varrho} \cdot m(\Lambda_u)/2 \geq d\epsilon^{\varrho+1}\|u\|^{\varrho}/2 \end{aligned}$$

for any $u \in Y$ with $\|u\| \geq R_3/\epsilon$. This implies $B(u) \to \infty$ as $\|u\| \to \infty$ on Y. □

Lemma 3.2 *Assume that* (AQ_1'), (AQ_2) *and* (AQ_3') *hold. Then there exist a positive integer* k_1 *and two sequences* $0 < r_k < \rho_k \to 0$ *as* $k \to \infty$ *such that*

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) > 0, \quad \forall k \geq k_1, \tag{3.4}$$

$$\xi_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \to 0 \quad \textit{as } k \to \infty \textit{ uniformly for } \lambda \in [1,2], \tag{3.5}$$

and

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N}, \tag{3.6}$$

where $Y_k = \bigoplus_{j=0}^k X_j = \operatorname{span}\{e_0, e_1, \ldots, e_k\}$ *and* $Z_k = \overline{\bigoplus_{j=k}^\infty X_j} = \overline{\operatorname{span}\{e_k, e_{k+1}, \ldots\}}$ *for all* $k \in \mathbb{N}$.

Proof Comparing this lemma with Lemma 2.7 of [6], we find that these two lemmas have the same condition (AQ_2) which is the key in the proof of Lemma 2.7 of [6]. We can prove our lemma by using the same method of [6], so the details are omitted. □

Now it is the time to prove our main result Theorem 1.1.

Proof of Theorem 1.1 By virtue of (1.3), (2.3) and (2.5), Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Obviously, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$ since $W(t,u)$ is even in u. Consequently, the condition (T_1) of Theorem 2.1 holds. Lemma 3.1 shows that the condition (T_2) holds, whereas Lemma 3.2 implies that the condition (T_3) holds for all $k \geq k_1$, where k_1 is given there. Therefore, by Theorem 2.1, for each $k \geq k_1$, there exist $\lambda_n \to 1$ and $u_{\lambda_n} \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \qquad \Phi_{\lambda_n}(u_{\lambda_n}) \to \eta_k \in \left[\xi_k(2), \beta_k(1)\right] \quad \text{as } n \to \infty. \tag{3.7}$$

For the sake of notational simplicity, in the following we always set $u_n = u_{\lambda_n}$ for all $n \in \mathbb{N}$.

Step 1. We firstly prove that $\{u_n\}$ is bounded in E.

Since $\{u_n\}$ satisfies (3.7), one has

$$\lim_{n\to\infty} \left(\left\langle \Phi'_{\lambda_n}|_{Y_n}(u_n), u_n \right\rangle - 2\Phi_{\lambda_n}(u_n) \right) = -2\eta_k.$$

More precisely,

$$\lim_{n\to\infty} \int_0^T \left(\left\langle \nabla_u W(t,u_n), u_n \right\rangle - 2W(t,u_n) \right) dt = 2\eta_k. \tag{3.8}$$

Now, we prove that $\{u_n\}$ is bounded. Otherwise, without loss of generality, we may assume that

$$\|u_n\| \to \infty \quad \text{as } n \to \infty.$$

Put $z_n = \frac{u_n}{\|u_n\|}$, we have $\|z_n\| = 1$. Going to a subsequence if necessary, we may assume that

$$z_n \rightharpoonup z \quad \text{in } E, \qquad z_n \to z \quad \text{in } L^2 \quad \text{and} \quad z_n(t) \to z(t) \quad \text{for a.e. } t \in [0,T].$$

By (1.3), we have

$$\begin{aligned}
\Phi_{\lambda_n}(u_n) &= \frac{1}{2}\left\|u_n^+\right\|^2 - \lambda_n \int_0^T W(t,u_n)\,dt \\
&\geq \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\left\|u_n^0\right\|^2 - \lambda_n \left(\epsilon \int_0^T |u_n|^2\,dt + \max_{s\in[0,\delta]} a(s) \int_0^T b(t)\,dt \right) \\
&\geq \frac{1}{2}\|u_n\|^2 - \left(\frac{1}{2} + \lambda_n \epsilon \right) \int_0^T |u_n|^2\,dt - \lambda_n c_1,
\end{aligned}$$

where $c_1 = \max_{s\in[0,\delta]} a(s) \int_0^T b(t)\,dt$. Therefore, one obtains

$$\begin{aligned}
\frac{\Phi_{\lambda_n}(u_n)}{\|u_n\|^2} &\geq \frac{1}{2} - \left(\frac{1}{2} + \lambda_n \epsilon \right) \int_0^T \left(\frac{|u_n|}{\|u_n\|} \right)^2 dt - \frac{\lambda_n c_1}{\|u_n\|^2} \\
&= \frac{1}{2} - \left(\frac{1}{2} + \lambda_n \epsilon \right) \|z_n\|_2^2 - \frac{\lambda_n c_1}{\|u_n\|^2}.
\end{aligned}$$

Passing to the limit in the inequality, by using $\Phi_{\lambda_n}(u_n) \to \eta_k$ and $\lambda_n \to 1$ as $n \to \infty$, we obtain

$$\frac{1}{2} - \left(\frac{1}{2} + \epsilon\right) \|z\|_2^2 \le 0.$$

Thus, $z \neq 0$ on a subset Ω of $[0, T]$ with positive measure.

By (1.2), we have

$$\langle \nabla_u W(t,u), u\rangle - 2W(t,u) \le 0, \quad \forall t \in [0,T] \text{ and } |u| \ge R_1',$$

and by the assumption (A), we obtain

$$\langle \nabla_u W(t,u), u\rangle - 2W(t,u) \le c_3 b(t), \quad \text{for all } |u| \le R_1' \text{ and a.e. } t \in [0,T],$$

where $c_3 = (2 + R_1') \max_{[0,R_1']} a(s)$. So, we get

$$\langle \nabla_u W(t,u), u\rangle - 2W(t,u) \le c_3 b(t)$$

for all $u \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Hence,

$$\begin{aligned}
&\int_0^T \big(\langle \nabla_u W(t,u_n), u_n\rangle - 2W(t,u_n)\big)\,dt \\
&\quad = \int_\Omega \big(\langle \nabla_u W(t,u_n), u_n\rangle - 2W(t,u_n)\big)\,dt + \int_{[0,T]\setminus\Omega} \big(\langle \nabla_u W(t,u_n), u_n\rangle - 2W(t,u_n)\big)\,dt \\
&\quad \le \int_\Omega \big(\langle \nabla_u W(t,u_n), u_n\rangle - 2W(t,u_n)\big)\,dt + \int_{[0,T]\setminus\Omega} c_3 b(t)\,dt.
\end{aligned}$$

An application of Fatou's lemma yields

$$\int_\Omega \big(\langle \nabla_u W(t,u_n), u_n\rangle - 2W(t,u_n)\big)\,dt \to -\infty \quad \text{as } n \to \infty,$$

which is a contradiction to (3.8).

Step 2. We prove that $\{u_n\}$ has a convergent subsequence in E.

Since $\{u_n\}$ is bounded in E, E is reflexible and $\dim E^0 < \infty$, without loss of generality, we assume

$$u_n^0 \to u_0^0, \qquad u_n^+ \rightharpoonup u_0^+ \quad \text{and} \quad u_n \rightharpoonup u_0 \quad \text{as } n \to \infty \tag{3.9}$$

for some $u_0 = u_0^0 + u_0^+ \in E = E^0 \oplus E^+$.

Note that

$$0 = \Phi_{\lambda_n}'|_{Y_n}(u_n) = u_n^+ - \lambda_n P_n \Psi'(u_n), \quad \forall n \in \mathbb{N},$$

where $P_n : E \to Y_n$ is the orthogonal projection for all $n \in \mathbb{N}$, that is,

$$u_n^+ = \lambda_n P_n \Psi'(u_n), \quad \forall n \in \mathbb{N}. \tag{3.10}$$

In view of the compactness of Ψ' and (3.9), the right-hand side of (3.10) converges strongly in E and hence $u_n^+ \to u_0^+$ in E. Together with (3.9), we have $u_n \to u_0$ in E.

Now, from the last assertion of Theorem 2.1, we know that $\Phi = \Phi_1$ has infinitely many nontrivial critical points. The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HG wrote the first draft and TA corrected and improved the final version. All authors read and approved the final draft.

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