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Chebyshev wavelets method for solving Bratu's problem

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Abstract

A numerical method for one-dimensional Bratu's problem is presented in this work. The method is based on Chebyshev wavelets approximates. The operational matrix of derivative of Chebyshev wavelets is introduced. The matrix together with the collocation method are then utilized to transform the differential equation into a system of algebraic equations. Numerical examples are presented to verify the efficiency and accuracy of the proposed algorithm. The results reveal that the method is accurate and easy to implement.

1 Introduction

In this paper, we consider the boundary-value problem and initial value problem of Bratu's problem. It is well known that Bratu's boundary value problem in one-dimensional planar coordinates is of the form

$$u'' + \lambda e^{u} = 0, \quad 0 < x < 1, \tag{1}$$

with the boundary conditions $u(0) = u(1) = 0$. For $\lambda > 0$ is a constant, the exact solution of equation (1) is given by [1]

$$u(x) = -2\ln\left[\frac{\cosh(0.5\theta(x-0.5))}{\cosh(0.25\theta)}\right], \tag{2}$$

where θ satisfies

$$\theta = \sqrt{2\lambda}\sinh(0.25\theta). \tag{3}$$

The problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$, respectively, where the critical value λ_c satisfies the equation

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\cosh\left(\frac{1}{4}\theta\right).$$

It was evaluated in [1–3] that the critical value λ_c is given by $\lambda_c = 3.513830719$.

In addition, an initial value problem of Bratu's problem

$$u'' + \lambda e^{u} = 0, \quad 0 < x < 1, \tag{4}$$

with the initial conditions $u(0) = u'(0) = 0$ will be investigated.

Bratu's problem is also used in a large variety of applications such as the fuel ignition model of the thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [4–11].

A substantial amount of research work has been done for the study of Bratu's problem. Boyd [2, 12] employed Chebyshev polynomial expansions and the Gegenbauer as base functions. Syam and Hamdan [8] presented the Laplace decomposition method for solving Bratu's problem. Also, Aksoy and Pakdemirli [13] developed a perturbation solution to Bratu-type equations. Wazwaz [10] presented the Adomian decomposition method for solving Bratu's problem. In addition, the applications of spline method, wavelet method and Sinc-Galerkin method for solution of Bratu's problem have been used by [14–17].

In recent years, the wavelet applications in dealing with dynamic system problems, especially in solving differential equations with two-point boundary value constraints have been discussed in many papers [4, 16, 18]. By transforming differential equations into algebraic equations, the solution may be found by determining the corresponding coefficients that satisfy the algebraic equations. Some efforts have been made to solve Bratu's problem by using the wavelet collocation method [16].

In the present article, we apply the Chebyshev wavelets method to find the approximate solution of Bratu's problem. The method is based on expanding the solution by Chebyshev wavelets with unknown coefficients. The properties of Chebyshev wavelets together with the collocation method are utilized to evaluate the unknown coefficients and then an approximate solution to (1) is identified.

2 Chebyshev wavelets and their properties

2.1 Wavelets and Chebyshev wavelets

In recent years, wavelets have been very successful in many science and engineering fields. They constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet $\psi(x)$. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [19]:

$$\psi_{a,b}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R}, a \neq 0.$$

Chebyshev wavelets $\psi_{n,m} = \psi(k, n, m, x)$ have four arguments, $n = 1, 2, \ldots, 2^{k-1}$, k can assume any positive integer, m is the degree of Chebyshev polynomials of first kind and x denotes the time.

$$\psi_{n,m}(x) = \begin{cases} \frac{\alpha_m 2^{(k-1)/2}}{\sqrt{\pi}} T_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}; \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where

$$\alpha_m = \begin{cases} \sqrt{2}, & m = 0; \\ 2, & m = 1, 2, \ldots \end{cases}$$

and $m = 0, 1, 2, \dots, M-1$, $n = 1, 2, \dots, 2^{k-1}$. Here $T_m(x)$ are the well-known Chebyshev polynomials of order m, which are orthogonal with respect to the weight function $\omega(x) = 1/\sqrt{1-x^2}$ and satisfy the following recursive formula:

$$T_0(x) = 1,$$
$$T_1(x) = x,$$
$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x).$$

We should note that the set of Chebyshev wavelets is orthogonal with respect to the weight function $\omega_n(x) = \omega(2^k x - 2n + 1)$.

The derivative of Chebyshev polynomials is a linear combination of lower-order Chebyshev polynomials, in fact [20],

$$\begin{cases} T'_m(x) = 2m\sum_{k=1}^{m-1} T_k(x), & m \text{ even;} \\ T'_m(x) = 2m\sum_{k=1}^{m-1} T_k(x) + mT_0(x), & m \text{ odd.} \end{cases} \tag{6}$$

2.2 Function approximation

A function $u(x)$ defined over $[0, 1)$ may be expanded as

$$u(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{7}$$

where $c_{nm} = (u(x), \psi_{nm}(x))$, in which $(\cdot, \cdot)$ denotes the inner product with the weight function $\omega_n(x)$. If $u(x)$ in (7) is truncated, then (7) can be written as

$$u(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = C^T\Psi(x), \tag{8}$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_1, c_2, \dots, c_{2^{k-1}}]^T,$$
$$\Psi(x) = [\psi_1, \psi_2, \dots, \psi_{2^{k-1}}]^T$$

and

$$c_i = [c_{i0}, c_{i1}, \dots, c_{i,M-1}],$$
$$\psi_i(x) = \bigl[\psi_{i0}(x), \psi_{i1}(x), \dots, \psi_{i,M-1}(x)\bigr], \quad i = 1, 2, 3, \dots, 2^{k-1}.$$

3 Chebyshev wavelets operational matrix of derivative

In this section we first derive the operational matrix D of derivative which plays a great role in dealing with Bratu's problem.

In the interval $[(n-1)/2^{k-1}, n/2^{k-1})$,

$$\psi_{n,m}(x) = \frac{\alpha_m 2^{(k-1)/2}}{\sqrt{\pi}} T_m\bigl(2^k x - 2n + 1\bigr).$$

Applying (6) the derivative of $\psi_{n,m}(x)$ is

$$\psi'_{n,m}(x) = \begin{cases} \frac{\alpha_m 2^{(k-1)/2}}{\sqrt{\pi}} \cdot 2^k \cdot 2m \sum_{k=1}^{m-1} T_k(2^k x - 2n + 1), & m \text{ even}; \\ \frac{\alpha_m 2^{(k-1)/2}}{\sqrt{\pi}} \cdot 2^k \cdot [2m \sum_{k=1}^{m-1} T_k(2^k x - 2n + 1) + mT_0(2^k x - 2n + 1)], & m \text{ odd}. \end{cases}$$

The function $\psi_i(x)$ is zero outside the interval $[(i-1)/2^{k-1}, i/2^{k-1})$, so

$$\psi'_i(x) = \psi_i(x)M, \quad i = 1, 2, \ldots, 2^{k-1}, \tag{9}$$

where

$$M = 2^k \cdot \begin{pmatrix} 0 & \sqrt{2} & 0 & 3\sqrt{2} & 0 & 5\sqrt{2} & \cdots & (M-1)\sqrt{2} \\ 0 & 0 & 4 & 0 & 8 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & \cdots & 2(M-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2(M-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{M \times M} \quad \text{for even } M,$$

$$M = 2^k \cdot \begin{pmatrix} 0 & \sqrt{2} & 0 & 3\sqrt{2} & 0 & 5\sqrt{2} & \cdots & 0 \\ 0 & 0 & 4 & 0 & 8 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & \cdots & 2(M-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2(M-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{M \times M} \quad \text{for odd } M.$$

In fact we have shown that

$$\Psi'(x) = D\Psi(x), \tag{10}$$

where

$$D = \operatorname{diag}\left(M^T, M^T, \ldots, M^T\right).$$

From (10), it can be generalized for any $n \in N$ as

$$\frac{d^n \Psi(x)}{dx^n} = D^n \Psi(x), \quad n = 1, 2, 3, \ldots. \tag{11}$$

4 Solution of Bratu's problem

Consider Bratu's problem given in (1). In order to use Chebyshev wavelets, we first approximate $u(x)$ as

$$u(x) = C^T \Psi(x).$$

Applying (11) we can get

$$u''(x) = C^T D^2 \Psi(x).$$

Thus we have

$$C^T D^2 \Psi(x) + \lambda e^{C^T \Psi(x)} = 0. \tag{12}$$

We now collocate (12) at $2^{k-1}M - 2$ points at x_i as

$$C^T D^2 \Psi(x_i) + \lambda e^{C^T \Psi(x_i)} = 0. \tag{13}$$

Suitable collocation points are

$$x_i = \frac{1}{2}\left[1 + \cos\left(\frac{(i-1)\pi}{2^{k-1}M - 1}\right)\right], \quad i = 2, 3, \ldots, 2^{k-1}M - 1.$$

Thus with the boundary conditions $u(0) = u(1) = 0$, we have

$$C^T \Psi(0) = 0, \tag{14}$$

$$C^T \Psi(1) = 0. \tag{15}$$

Equations (13), (14) and (15) generate $2^{k-1}M$ set of nonlinear equations. The approximate solution of the vector C is obtained by solving the nonlinear system using the Gauss-Newton method.

5 Error analysis

Theorem 5.1 *A function $u(x) \in L^2_\omega([0,1])$, with bounded second derivative, say $|u''(x)| \leq N$, can be expanded as an infinite sum of Chebyshev wavelets, and the series converges uniformly to $u(x)$, that is* [21],

$$u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}.$$

Since the truncated Chebyshev wavelets series is an approximate solution of Bratu's problem, so one has an error function $E(x)$ for $u(x)$ as follows:

$$E(x) = \left| u(x) - C^T \Psi(x) \right|.$$

The error bound of the approximate solution by using Chebyshev wavelets series is given by the following theorem.

Theorem 5.2 *Suppose that $u(x) \in C^m[0,1]$ and $C^T \Psi(x)$ is the approximate solution using the Chebyshev wavelets method. Then the error bound would be obtained as follows*:

$$E(x) \leq \left\| \frac{2}{m! 4^m 2^{m(k-1)}} \max_{x \in [0,1]} \left| u^m(x) \right| \right\|^2.$$

Proof Applying the definition of norm in the inner product space, we have

$$\|E(x)\|^2 = \int_0^1 \left[u(x) - C^T\Psi(x)\right]^2 dx.$$

Because the interval $[0,1]$ is divided into 2^{k-1} subintervals $I_n = [(n-1)/2^{k-1}, n/2^{k-1}]$, $n = 1,2,\ldots,2^{k-1}$, then we can obtain

$$\begin{aligned}
\|E(x)\|^2 &= \int_0^1 \left[u(x) - C^T\Psi(x)\right]^2 dx \\
&= \sum_{k=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[u(x) - C^T\Psi(x)\right]^2 dt \\
&\leq \sum_{k=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[u(x) - P_m(x)\right]^2 dt,
\end{aligned}$$

where $P_m(x)$ is the interpolating polynomial of degree m which agrees with $u(x)$ at the Chebyshev nodes on I_n with the following error bound for interpolating [22, 23]:

$$\left|u(x) - P_m(x)\right| \leq \frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in I_n} \left|u^m(x)\right|.$$

Therefore, using the above equation, we would get

$$\begin{aligned}
\|E(x)\|^2 &\leq \sum_{k=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[u(x) - P_m(x)\right]^2 dt \\
&\leq \sum_{k=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in I_n} \left|u^m(x)\right|\right]^2 dt \\
&\leq \sum_{k=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in [0,1]} \left|u^m(x)\right|\right]^2 dt \\
&= \int_0^1 \left[\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in [0,1]} \left|u^m(x)\right|\right]^2 dt \\
&= \left\|\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in [0,1]} \left|u^m(x)\right|\right\|^2.
\end{aligned}$$

□

6 Numerical examples

To illustrate the ability and reliability of the method for Bratu's problem, some examples are provided. The results reveal that the method is very effective and simple.

Example 6.1 Consider the first case for Bratu's equation as follows, when $\lambda = 2$ [14, 15]:

$$\begin{aligned}
&u'' + 2e^u = 0, \quad 0 < x < 1, \\
&u(0) = u(1) = 0.
\end{aligned} \tag{16}$$

We solve the equation by using the Chebyshev wavelets method with $k = 2$, $M = 4, 6, 8$. The numerical results obtained are presented in Table 1. Table 1 shows the comparison

Table 1 Computed absolute errors for Example 6.1

x	$k = 2, M = 4$	$k = 2, M = 6$	$k = 2, M = 8$
0.1	1.33462×10^{-3}	1.18825×10^{-5}	5.01033×10^{-7}
0.2	2.57841×10^{-3}	2.62388×10^{-5}	1.16521×10^{-6}
0.3	5.88781×10^{-3}	2.26840×10^{-5}	2.33696×10^{-6}
0.4	3.36183×10^{-3}	5.20260×10^{-5}	5.48713×10^{-6}
0.5	5.98507×10^{-3}	4.44306×10^{-5}	1.33512×10^{-6}
0.6	3.59903×10^{-3}	2.53383×10^{-5}	2.40800×10^{-6}
0.7	5.14675×10^{-3}	4.17100×10^{-5}	3.73327×10^{-6}
0.8	1.43561×10^{-3}	3.25335×10^{-5}	7.18825×10^{-6}
0.9	1.25962×10^{-3}	1.71598×10^{-5}	1.46832×10^{-6}

between the absolute error of exact and approximate solutions for various values of M (with $k = 2$). Moreover, higher accuracy can be achieved by taking higher order approximations.

Example 6.2 Consider the initial value problem [10, 14–16, 24]

$$\begin{aligned} &u'' - 2e^u = 0, \quad 0 < x < 1, \\ &u(0) = 0, \qquad u'(0) = 0. \end{aligned} \tag{17}$$

The exact solution is $u(x) = -2\ln(\cos(x))$. Here we solve it using Chebyshev wavelets, with $k = 1$, $M = 6$. First we assume that the unknown function $u(x)$ is given by

$$u(x) = C^T \Psi(x).$$

Applying (12) we get

$$C^T D^2 \Psi(x_i) - 2e^{C^T \Psi(x_i)} = 0. \tag{18}$$

Using the initial condition, we obtain

$$\begin{aligned} &C^T \Psi(0) = 0, \\ &C^T D\Psi(0) = 0. \end{aligned} \tag{19}$$

Equations (18) and (19) generate a system of nonlinear equations. These equations can be solved for unknown coefficients of the vector C. A comparison between the exact and the approximate solutions is demonstrated in Figure 1. From Figure 1, it can be found that the obtained approximate solutions are very close to the exact solution. In addition, Table 2 shows the exact and approximate solutions using the method presented in Section 3 and compares the results with the method presented in [16]. Also, by comparing the results of the table, we see that the results of the proposed method are more accurate.

7 Conclusions

The aim of present work is to develop an efficient and accurate method for solving Bratu's problems. The Chebyshev wavelet operational matrix of derivative together with the collocation method are used to reduce the problem to the solution of nonlinear algebraic

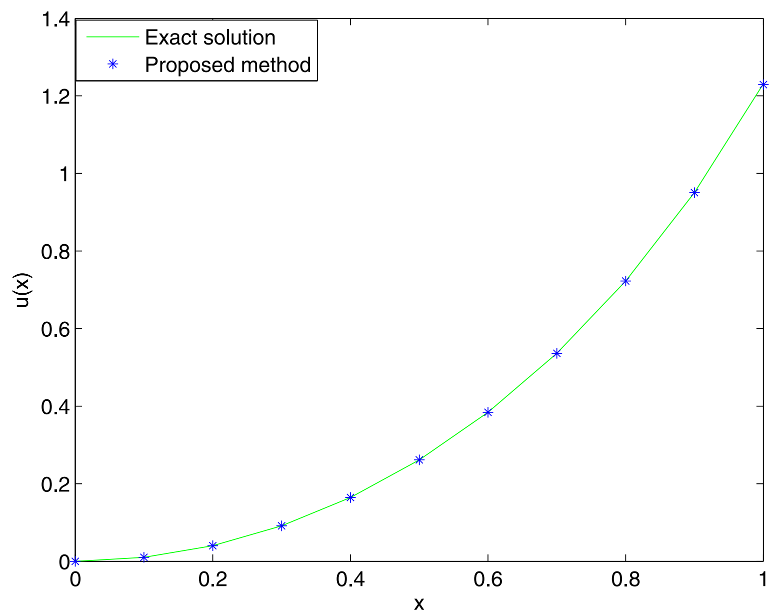

Figure 1 Comparison of solutions for Example 6.2.

Table 2 Comparison of the results of the Chebyshev and Legendre wavelets method for Example 6.2

x	Chebyshev wavelets	Legendre wavelets	Exact solutions
0.1	1.0016711×10^{-2}	1.0016801×10^{-2}	1.0016711×10^{-2}
0.2	4.0269541×10^{-2}	4.0269696×10^{-2}	4.0269546×10^{-2}
0.3	9.1383326×10^{-2}	9.1382697×10^{-2}	9.1383311×10^{-2}
0.4	1.6445871×10^{-1}	1.6444915×10^{-1}	1.6445803×10^{-1}
0.5	2.6116111×10^{-1}	2.6111176×10^{-1}	2.6116848×10^{-1}
0.6	3.8339360×10^{-1}	3.8367456×10^{-1}	3.8393033×10^{-1}
0.7	5.3617551×10^{-1}	5.3524690×10^{-1}	5.3617151×10^{-1}
0.8	7.2271751×10^{-1}	7.1991951×10^{-1}	7.2278149×10^{-1}
0.9	9.5086960×10^{-1}	9.4297240×10^{-1}	9.5088488×10^{-1}

equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CY completed the main study, carried out the results of this article and drafted the manuscript. JH checked the proofs and verified the calculation. All the authors read and approved the final manuscript.

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