Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On the convergence of $x_n = f(x_{n-2}, x_{n-1})$ when $f(x,y) < x$

Gabor Nyerges*

*Correspondence: nyergeg@lsbu.ac.uk
Deptartment of Engineering and Design, London South Bank University, 103 Borough Road, London, SE1 0AA, UK

Abstract

Given the second-order difference equation $x_n = f(x_{n-2}, x_{n-1})$, if $f \in C[(0,\infty)^2, (0,\infty)]$ and $f(x,y) < x$, then (x_{2n}, x_{2n+1}) tends either to $(L,0)$ or to $(0,L)$ for some $L \geq 0$. In this paper we show that if $f(x,y)$ decreases in y, then for any x_0 there is an x_1 such that x_n monotonically decreases to 0. We also prove that if $x - y \geq f(x,y) - f(y, f(x,y))$, then for any $L \geq 0$ and $x_0 > L$ there is an x_1 such that $(x_{2n}, x_{2n+1}) \to (L,0)$ and similarly, for any x_0 there is an x_1 such that $(x_{2n}, x_{2n+1}) \to (0,L)$. The class of functions satisfying the latter condition includes any function of the form $f(x,y) = \frac{x}{1+h(x,y)y}$, where h is symmetric and increases in y.

MSC: 39A11; 39A23

Keywords: second-order difference equation; convergence to zero; convergence to prime period two sequences; approximating initial conditions for convergence

1 Introduction

Let $\mathcal{F}$ be the set of functions $f \in C[(0,\infty)^2, (0,\infty)]$ satisfying the condition

(A) $f(x,y) < x$.

In this paper, we investigate the convergence of the solutions of the second-order difference equation

$$x_n = f(x_{n-2}, x_{n-1}), \quad n = 2, 3, \ldots \tag{1}$$

for the following two subsets of $\mathcal{F}$:

$$\mathcal{F}_1 = \{f \in \mathcal{F} : f(x,y) \text{ decreases in } y\},$$
$$\mathcal{F}_2 = \{f \in \mathcal{F} : x - y \geq f(x,y) - f(y, f(x,y))\}.$$

We note that by 'decreasing' and 'increasing' we mean non-increasing and non-decreasing, respectively.

Continuity and (A) imply that if $f \in \mathcal{F}$, then (x_{2n}, x_{2n+1}) tends either to $(L,0)$ or to $(0,L)$ for some $L \geq 0$. The convergence of the positive solutions of equation (1) has been investigated before for functions whose restriction to $(0,\infty)^2$ is in $\mathcal{F}$ and which are strictly monotonic in both of their variables, typically assuming that

(a) $f \in C[[0,\infty)^2, [0,\infty)]$ and its restriction $\bar{f}$ to $(0,\infty)^2$ is in $\mathcal{F}$,

(b) $f(x,0) = x$,

(c) $\bar{f}$ strictly increases in x and strictly decreases in y.

Let f be of this type. Then, since $\bar{f} \in \mathcal{F}$, every positive solution tends either to $(L, 0, L, 0, \ldots)$ or to $(0, L, 0, L, \ldots)$ for some $L \geq 0$, the only periodic solutions of equation (1), with $(0, 0, 0, 0, \ldots)$ being the unique equilibrium solution (see, *e.g.* [1]). The question of the existence of positive solutions converging to the equilibrium solution in the special case $f(x,y) = \frac{x}{1+\alpha y}$ $(x, y \geq 0)$ was raised by Kulenovic and Ladas in [2]. Kent gave an affirmative answer in [1] by showing that, in general, if f satisfies (a), (b), (c) and the condition that

(d) f is differentiable and there is a differentiable function g such that $x_{n-2} = g(x_{n-1}, x_n)$, with some further properties detailed in [1],

then for any $x_0 > 0$ there is an $x_1 > 0$ such that x_n monotonically decreases to 0. We note that Janssen and Tjaden [3] had previously proved this for $f(x,y) = \frac{x}{1+y}$ and $x_0 = 1$. As we shall see, Kent's conclusion follows from much weaker assumptions, namely that (a) holds and $\bar{f}(x,y)$ decreases in y.

To be more precise, we show (*cf.* Theorem 2.1) that if $f \in \mathcal{F}_1$, then for any x_0 there is an x_1 such that x_n monotonically decreases to 0. Also, for any x_1, if $x_1 \leq f(x, x_1)$ for some x, then there is an x_0 such that x_n monotonically decreases to 0. Theorem 2.1 also applies to $f(x,y) = x\frac{\alpha y+\beta x}{Ay+\beta x}$, investigated by Chan *et al.* in [4]. In their paper, they prove, *inter alia*, that if $0 < \alpha < A$ and $0 < \beta$, then there are positive initial values for which x_n in equation (1) monotonically decreases to 0. Since the restriction of f to $(0, \infty)^2$ is in $\mathcal{F}_1$, their result also follows from Theorem 2.1 in a stronger form.

For functions in $\mathcal{F}_2$ we show (*cf.* Theorem 2.2) that if $L \geq 0$, then for any $x_0 > L$ there is an x_1 such that $(x_{2n}, x_{2n+1}) \to (L, 0)$ and for any x_1, if $x_1 + L \leq f(x, x_1)$ for some x, then there is an x_0 such that $(x_{2n}, x_{2n+1}) \to (L, 0)$. Similar results hold for convergence to $(0, L)$.

The class $\mathcal{F}_2$ includes several types of functions of interest. For instance, as we shall see later on, $\mathcal{F}_2 \cap \mathcal{F}_1$ contains any function of the form $f(x,y) = \frac{x}{1+h(x,y)y}$, where $h \in C[(0,\infty)^2, (0,\infty)]$ is symmetric, *i.e.* $h(x,y) = h(y,x)$, and increases in y. In particular, $\mathcal{F}_2 \cap \mathcal{F}_1$ includes $f(x,y) = \frac{x}{1+\alpha y}$ for any $\alpha > 0$. Actually, functions of this form, defined on $[0,\infty)^2$, belong to another subset of the set of functions satisfying (a), (b) and (c) above, for which a result similar to, but stronger than Theorem 2.2 was proved in [5]. Given any function in this set, for any $L \geq 0$, the set of positive initial values (x_0, x_1) for which $(x_{2n}, x_{2n+1}) \to (L, 0)$ is a surjective, strictly increasing (and hence continuous) function from (L, ∞) to $(0, \infty)$.

2 Main results

In order to prove our main results, we shall think of the difference equation (1) as a recursively defined sequence of functions of the initial conditions, namely

$$
\begin{aligned}
f_0(x,y) &= x, \\
f_1(x,y) &= y, \\
f_n(x,y) &= f\bigl(f_{n-2}(x,y), f_{n-1}(x,y)\bigr) \quad \text{for } n \geq 2.
\end{aligned}
$$

Putting $r_n = f_{2n}$, $s_n = f_{2n+1}$, we have $r_0(x,y) = x$, $s_0(x,y) = y$ and, for $n \geq 1$,

$$
\begin{aligned}
r_n(x,y) &= f\bigl(r_{n-1}(x,y), s_{n-1}(x,y)\bigr), \\
s_n(x,y) &= f\bigl(s_{n-1}(x,y), r_n(x,y)\bigr).
\end{aligned}
$$

Let $f \in \mathcal{F}$. Then the recursive relationships above imply that $r_n, s_n \in C[(0,\infty)^2, (0,\infty)]$ for any n, and by (A)

$$x = r_0(x,y) > r_1(x,y) > r_2(x,y) > \cdots,$$
$$y = s_0(x,y) > s_1(x,y) > s_2(x,y) > \cdots.$$

In particular, for $n \geq 1$, r_n satisfies (A), hence $r_n \in \mathcal{F}$, and s_n satisfies the condition

(A*) $f(x,y) < y$.

We shall write R and S for the pointwise limits of r_n and s_n, respectively. R and S map $(0,\infty)^2$ to $[0,\infty)$ and, since $r_n(x,y) > R(x,y)$, $s_n(x,y) > S(x,y)$, R and S satisfy (A) and (A*), respectively. However, as we shall see in Example 3.4, R and S are not necessarily continuous. If $L = R(x,y)$ and $M = S(x,y)$ were both positive, then, by the recursive relationship between r_n and s_n and the continuity of f, $L = f(L,M)$, contradicting (A). Therefore, either $R(x,y) = 0$ or $S(x,y) = 0$, *i.e.* $R(x,y) \cdot S(x,y) = 0$ for any x and y. We also observe that R and S have the same range, since $R(x,y) = S(y, f(x,y))$ and $S(x,y) = R(y, f(x,y))$. In particular, 0 is in the range of both functions.

Our main results will characterise, in terms of the functions R and S, the sets of initial conditions under which (x_{2n}, x_{2n+1}) in equation (1) tends to $(L,0)$ or to $(0,L)$. For any $L \geq 0$, let us put R_L and S_L, respectively, for these sets, so that

$$R_L = \{(x,y) : R(x,y) = L \text{ and } S(x,y) = 0\},$$
$$S_L = \{(x,y) : R(x,y) = 0 \text{ and } S(x,y) = L\}.$$

We note that for $L > 0$, since $R(x,y) \cdot S \cdot S(x,y) = 0$, R_L and S_L are simply the L-level sets of R and S, respectively. Also, R_0 is the intersection of the 0-level sets of R and S and $R_0 = S_0$.

Regarding R_L and S_L as relations, the theorems below characterise their domain and their range, Theorem 2.1 for $L = 0$ and any $f \in \mathcal{F}_1$ and Theorem 2.2 for any $L \geq 0$ and any $f \in \mathcal{F}_2$.

Theorem 2.1 *Let $f \in \mathcal{F}_1$. Then*

(i) *for any a there is a b in the interval $[f(a,a), a]$ for which $R(a,b) = S(a,b) = 0$,*
(ii) *for any b, if there is a b' such that $b \leq f(b',b)$, then there is an a in the interval $[b,b']$ for which $R(a,b) = S(a,b) = 0$.*

Theorem 2.2 *Let $f \in \mathcal{F}_2$ and $L \geq 0$. Then*

(i) *for any $a > L$ there is a b in the interval $[f(a-L,a), a-L]$ for which $R(a,b) = L$, $S(a,b) = 0$,*
(ii) *for any b, if there is a b' such that $b + L \leq f(b',b)$, then there is an a in the interval $[b+L, b']$ for which $R(a,b) = L$, $S(a,b) = 0$,*
(iii) *for any a there is a b in the interval $[f(a+L,a), a+L]$ for which $R(a,b) = 0$, $S(a,b) = L$,*
(iv) *for any $b > L$, if there is a b' such that $b - L \leq f(b',b)$, then there is an a in the interval $[b-L, b']$ for which $R(a,b) = 0$, $S(a,b) = L$.*

Remark 2.1 If $\lim_{x\to\infty} f(x,b) = \infty$, then of course there is a b' satisfying the condition in (ii) of Theorem 2.1 and in (ii), (iv) of Theorem 2.2. The condition is not necessary, as shown later on by Example 3.1.

The proofs of Theorem 2.1 and Theorem 2.2 will follow immediately from the two lemmas below. In order to motivate these lemmas, we observe that given $f \in \mathcal{F}$ and $L \geq 0$, a sufficient condition for $R(a,b) = L$, $S(a,b) = 0$ is that for all n

$$r_n(a,b) - L \geq s_n(a,b) \geq r_{n+1}(a,b) - L,$$

i.e. $r_n(a,b) - s_n(a,b) - L \geq 0$ and $s_n(a,b) - r_{n+1}(a,b) + L \geq 0$. This is sufficient, since either $R(a,b) = 0$ or $S(a,b) = 0$. Similarly, a sufficient condition for $R(a,b) = 0$, $S(a,b) = L$ is that for all n

$$r_n(a,b) \geq s_n(a,b) - L \geq r_{n+1}(a,b),$$

i.e. $r_n(a,b) - s_n(a,b) + L \geq 0$ and $s_n(a,b) - r_{n+1}(a,b) - L \geq 0$. In particular, if $L = 0$, then $r_n(a,b) \geq s_n(a,b) \geq r_{n+1}(a,b)$ and so $R(a,b) = S(a,b) = 0$. We also observe that if $f \in \mathcal{F}_2$, then the first condition is also necessary for $R(a,b) = L$, $S(a,b) = 0$ and the second one is also necessary for $R(a,b) = 0$, $S(a,b) = L$. This is because if $f \in \mathcal{F}_2$, then for any x, y and n, $r_n(x,y) - s_n(x,y) \geq r_{n+1}(x,y) - s_{n+1}(x,y)$ and $s_n(x,y) - r_{n+1}(x,y) \geq s_{n+1}(x,y) - r_{n+2}(x,y)$.

According to the discussion above, in order to prove that given a there is a b such that $R(a,b) = L$ and $S(a,b) = 0$, it is sufficient to show that there is a b for which the functions $p_n(y) = r_n(a,y) - s_n(a,y) - L$ and $q_n(y) = s_n(a,y) - r_{n+1}(a,y) + L$ are both non-negative for all n. Similarly, given a and $p_n(y) = r_n(a,y) - s_n(a,y) + L$, $q_n(y) = s_n(a,y) - r_{n+1}(a,y) - L$, if both functions are non-negative for all n, then $R(a,b) = 0$ and $S(a,b) = L$.

Lemma 2.1 below provides sufficient conditions ensuring that, in general, the functions in two sequences of continuous real functions are all non-negative for some argument. Lemma 2.2 uses Lemma 2.1 to show that if $L \geq 0$ and $f \in \mathcal{F}$ has the property that $f(x,y) - f(y,f(x,y)) \leq \pm L$ whenever $x - y = \pm L$, then statements (i), (ii), (iii) and (iv) in Theorem 2.2 are true for L and f. Since any function in $\mathcal{F}_1$ has this property for $L = 0$, Theorem 2.1 follows from Lemma 2.1, and since any function in $\mathcal{F}_2$ has this property for any $L \geq 0$, so does Theorem 2.2.

Lemma 2.1 *Let $p_n, q_n \in C[[a,b],(-\infty,\infty)]$, $n = 0,1,\ldots$, and put*

$$S_{ab} = \{x : p_n(x) \geq 0 \text{ and } q_n(x) \geq 0 \text{ for all } n\}.$$

(i) *If*

(a) *for all n and x, either $p_n(x) > 0$ or $q_n(x) > 0$,*
(b) *for all n and x, if $p_n(x) = 0$ then $p_{n+1}(x) \leq 0$ and if $q_n(x) = 0$ then $q_{n+1}(x) \leq 0$,*
(c) $p_0(b) \leq 0$ *and* $q_0(a) \leq 0$,

then $S_{ab} \neq \emptyset$.

(ii) *If* (a) *holds and* (b), (c) *are replaced by*

(b′) *for all n and x, $p_n(x) \geq p_{n+1}(x)$ and $q_n(x) \geq q_{n+1}(x)$,*
(c′) $p_N(b) \leq 0$ *and* $q_M(a) \leq 0$ *for some $N, M \geq 0$,*

then $S_{ab} \neq \emptyset$.

Proof (i) Let $p, q \in C[[a,b], (-\infty, \infty)]$ be any two functions such that for any x either $p(x) > 0$ or $q(x) > 0$ and $p(b) \leq 0$, $q(a) \leq 0$. Then $q(a) \leq 0$ implies that $p(a) > 0$ and so, since $p(b) \leq 0$, p has a smallest root b' in $(a,b]$. Therefore, $p > 0$ on $[a, b')$ and $p(b') = 0$. But then $q(b') > 0$ and so, since $q(a) \leq 0$, q has a greatest root a' in $[a, b')$. Hence, $q > 0$ on $(a', b']$ and $q(a') = 0$. Thus, there is an interval $[a', b'] \subseteq [a,b]$ such that $p > 0$ on $[a', b')$ and $q > 0$ on $(a', b']$ and $p(b') = q(a') = 0$.

Let us now assume that the hypothesis of (i) holds. Then, by the above, there is an interval $[a_0, b_0] \subseteq [a,b]$ such that $p_0 > 0$ on $[a_0, b_0)$ and $q_0 > 0$ on $(a_0, b_0]$ and $p_0(b_0) = q_0(a_0) = 0$. Now suppose that there is an interval $[a_n, b_n]$ such that $p_n > 0$ on $[a_n, b_n)$, $q_n > 0$ on $(a_n, b_n]$ and $p_n(b_n) = q_n(a_n) = 0$. Then, by (b), $p_{n+1}(b_n) \leq p_n(b_n) = 0$ and $q_{n+1}(a_n) \leq q_n(a_n) = 0$. Therefore, by the above, there is an interval $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ such that $p_{n+1} > 0$ on $[a_{n+1}, b_{n+1})$, $q_{n+1} > 0$ on $(a_{n+1}, b_{n+1}]$ and $p_{n+1}(b_{n+1}) = q_{n+1}(a_{n+1}) = 0$. Hence, by induction, there is a descending sequence $[a_0, b_0] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$ of intervals such that $p_n > 0$ on $[a_n, b_n)$, $q_n > 0$ on $(a_n, b_n]$ and $p_n(b_n) = q_n(a_n) = 0$.

Finally, let $I = \bigcap_n [a_n, b_n]$. Then I is a non-empty closed interval and $I \subseteq S_{ab}$. We note that if $\operatorname{int}(I) \neq \emptyset$, then p_n and q_n are both positive on $\operatorname{int}(I)$.

(ii) If, say, $M \leq N$, then $p_N(a) \leq 0$ by (b′). Therefore, by (i)

$$\{x : p_n(x) \geq 0 \text{ and } q_n(x) \geq 0 \text{ for } n \geq N\} \neq \emptyset$$

and by (b′), $\{x : p_n(x) \geq 0 \text{ and } q_n(x) \geq 0 \text{ for } n \geq N\} = S_{ab}$. □

Remark 2.2 (Approximating an element of S_{ab}) Suppose that p_n, q_n satisfy the conditions of Lemma 2.1(ii) on an interval $[a,b]$, so that $S_{ab} \neq \emptyset$. Choose a $c \in (a,b)$, say $c = (a+b)/2$. Suppose that either $p_N(c) \leq 0$ or $q_N(c) \leq 0$ for some $N \geq 0$. Then, by Lemma 2.1(ii), p_n and q_n satisfy the conditions of Lemma 2.1(ii) either on $[a,c]$ if $p_N(c) \leq 0$, or on $[c,b]$ if $q_N(c) \leq 0$. Therefore, if $p_N(c) \leq 0$, then $S_{ac} \neq \emptyset$ and if $q_N(c) \leq 0$, then $S_{cb} \neq \emptyset$. Since $S_{ac}, S_{cb} \subseteq S_{ab}$, in either case we have a better approximation of an element of S_{ab}. The method will fail if $p_n(c) > 0$ and $q_n(c) > 0$ for all $n \geq 0$ (in which case $c \in S_{ab}$).

Lemma 2.2 *Let $f \in \mathcal{F}$. Suppose that for an $L \geq 0$ and any x and y,*

$$\text{if } x - y = \pm L \text{ then } f(x,y) - f\bigl(y, f(x,y)\bigr) \leq \pm L. \tag{2}$$

Then statements (i), (ii), (iii) *and* (iv) *in Theorem* 2.2 *are true for f and L.*

Proof First we observe that

$$\begin{aligned} &\bigl(r_n(x,y) - s_n(x,y) \pm L\bigr) + \bigl(s_n(x,y) - r_{n+1}(x,y) \mp L\bigr) \\ &\quad = r_n(x,y) - r_{n+1}(x,y) \\ &\quad > 0. \end{aligned} \tag{3}$$

Also, it follows from (2) that

$$\begin{aligned} &\text{if } r_n(x,y) - s_n(x,y) \pm L = 0 \text{ then } r_{n+1}(x,y) - s_{n+1}(x,y) \pm L \leq 0, \\ &\text{if } s_n(x,y) - r_{n+1}(x,y) \pm L = 0 \text{ then } s_{n+1}(x,y) - r_{n+2}(x,y) \pm L \leq 0. \end{aligned} \tag{4}$$

Then:

(i) Let $a > L$. Put $p_n(y) = r_n(a,y) - s_n(a,y) - L$ and $q_n(y) = s_n(a,y) - r_{n+1}(a,y) + L$. Then $p_0(a-L) = a-(a-L)-L = 0$. By (2), $f(a-L,a)-f(a,f(a-L,a)) \leq -L$ and so $q_0(f(a-L,a)) = f(a-L,a) - f(a,f(a-L,a)) + L \leq 0$. Together with (3) and (4) this means that p_n and q_n satisfy the hypothesis of Lemma 2.1(i) on the interval $[f(a-L,a), a-L]$. Therefore, there is a b in the interval $[f(a-L,a), a-L]$ such that $p_n(b) \geq 0$ and $q_n(b) \geq 0$ for all $n \geq 0$ and so $R(a,b) = L$, $S(a,b) = 0$.

(ii) Given b, suppose that there is a b' such that $b + L \leq f(b',b)$. Put $p_n(x) = s_n(x,b) - r_{n+1}(x,b) + L$ and $q_n(x) = r_n(x,b) - s_n(x,b) - L$. Then $p_0(b') = b - f(b',b) + L \leq 0$ and $q_0(b + L) = (b + L) - b - L = 0$. Together with (3) and (4) this means that p_n and q_n satisfy the hypothesis of Lemma 2.1(i) on the interval $[b+L, b']$. Therefore, there is an a in the interval $[b + L, b']$ such that $p_n(b) \geq 0$ and $q_n(b) \geq 0$ for all $n \geq 0$ and so $R(a,b) = L$, $S(a,b) = 0$.

(iii) Given a, put $p_n(y) = r_n(a,y) - s_n(a,y) + L$ and $q_n(y) = s_n(a,y) - r_{n+1}(a,y) - L$. Then $p_0(a+L) = a-(a+L)+L = 0$. By (2), $f(a+L,a)-f(a,f(a+L,a)) \leq L$ and so $q_0(f(a+L,a)) = f(a+L,a) - f(a,f(a+L,a)) - L \leq 0$. Together with (3) and (4) this means that p_n and q_n satisfy the hypothesis of Lemma 2.1(i) on the interval $[f(a+L), a+L]$. Therefore, there is a b in the interval $[f(a+L,a), a+L]$ such that $p_n(b) \geq 0$ and $q_n(b) \geq 0$ for all $n \geq 0$ and so $R(a,b) = 0$, $S(a,b) = L$.

(iv) Let $b > L$ and suppose that there is a b' such that $b - L \leq f(b',b)$. Put $p_n(x) = s_n(x,b) - r_{n+1}(x,b) - L$ and $q_n(x) = r_n(x,b) - s_n(x,b) + L$. Then $p_0(b') = b - f(b',b) - L \leq 0$ and $q_0(b - L) = (b - L) - b + L = 0$. Together with (3) and (4) this means that p_n and q_n satisfy the hypothesis of Lemma 2.1(i) on the interval $[b - L, b']$. Therefore there is an a in the interval $[b - L, b']$ such that $p_n(b) \geq 0$ and $q_n(b) \geq 0$ for all $n \geq 0$ and so $R(a,b) = 0$, $S(a,b) = L$. □

We can now prove Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1 *and Theorem* 2.2 If $f \in \mathcal{F}_1$, then $f(x,x) - f(x,f(x,x)) \leq 0$, since $f(x,x) < x$. Thus f satisfies the hypothesis of Lemma 2.2 for $L = 0$ and so Theorem 2.1 holds.

If $f \in \mathcal{F}_2$ then $x - y \geq f(x,y) - f(y,f(x,y))$ and so f satisfies the hypothesis of Lemma 2.2 for any $L \geq 0$ and Theorem 2.2 follows. □

Remark 2.3 (Approximating initial conditions for convergence) Let $f \in \mathcal{F}_2$ and $L \geq 0$. Given any $a > L$, define p_n and q_n as in (i) of the proof of Lemma 2.2 above. Then, using the bisection method described in Remark 2.2, we can (usually) approximate a b such that $R(a,b) = L$ and $S(a,b) = 0$ to any degree of accuracy. Similarly, if we define p_n and q_n as in (iii) of the proof of Lemma 2.2, then, given any a, we can (usually) approximate a b such that $R(a,b) = 0$ and $S(a,b) = L$.

3 Applications

In this section, we give a few examples of applying the results of Section 2. We also show that R (and hence S) is not necessarily continuous for functions in $\mathcal{F}$. We note that, in general, the elements of $\mathcal{F}$ can be written as $f(x,y) = \frac{x}{1+g(x,y)}$, with $g \in C[(0,\infty)^2, (0,\infty)]$ and *vice versa*.

We shall use the following proposition in the examples.

Proposition 3.1

(i) *Let $\mathcal{F}_{21}$ be the set of functions of the form*

$$f(x,y)=\frac{x}{1+h(x,y)y},\quad x,y>0,$$

where $h\in C[(0,\infty)^2,(0,\infty)]$ is symmetric and increases in y (and hence also in x). Then $\mathcal{F}_{21}\subseteq\mathcal{F}_2\cap\mathcal{F}_1$.

(ii) *Let $\mathcal{F}_{22}$ be the set of functions of the form*

$$f(x,y)=\frac{x}{1+g(y)},\quad x,y>0,$$

where $g:(0,\infty)\to(0,\infty)$ is differentiable, $g(y)\to 0$ as $y\to 0$, $y\le g(y)$ and $g'(y)\le(1+g(y))^2$. Then $\mathcal{F}_{22}\subseteq\mathcal{F}_2$.

Proof (i) First of all, functions in $\mathcal{F}_{21}$ (strictly) decrease in y and so they are in $\mathcal{F}_1$. Next we note that if $f\in\mathcal{F}$ is written as $f(x,y)=\frac{x}{1+g(x,y)}$, then $x-f(x,y)=f(x,y)g(x,y)$. Therefore, if $f\in\mathcal{F}_{21}$, then

$$\begin{aligned}x-f(x,y)&=f(x,y)h(x,y)y=f(x,y)h(y,x)y\\&>f(x,y)h\big(y,f(x,y)\big)f\big(y,f(x,y)\big)\\&=f\big(y,f(x,y)\big)h\big(y,f(x,y)\big)f(x,y)\\&=y-f\big(y,f(x,y)\big).\end{aligned}$$

Therefore, $\mathcal{F}_{21}\subseteq\mathcal{F}_2$.

(ii) Let us put $e(x,y)=(x-y)-(f(x,y)-f(y,f(x,y)))$. Then, for any y, $e(x,y)\to 0$ as $x\to 0$, and so in order to show that $e(x,y)\ge 0$, *i.e.* that $\mathcal{F}_{22}\subseteq\mathcal{F}_2$, it will be sufficient to demonstrate that for any y, $e(x,y)$ increases in x.

The partial derivative e_x can be written as

$$\begin{aligned}e_x(x,y)&=1-\big(f_x(x,y)-f_y\big(y,f(x,y)\big)f_x(x,y)\big)\\&=1-f_x(x,y)\big(1-f_y\big(y,f(x,y)\big)\big)\\&=1-\frac{1}{1+g(y)}\left(1+y\frac{g'(f(x,y))}{(1+g(f(x,y)))^2}\right).\end{aligned}$$

If $g'(y)\le(1+g(y))^2$ and $y\le g(y)$, then $1+y\frac{g'(f(x,y))}{(1+g(f(x,y)))^2}\le 1+y\le 1+g(y)$ and so $e_x(x,y)\ge 0$, *i.e.* $e(x,y)$ increases in x for any y, as claimed. □

Functions in $\mathcal{F}_{21}$ do not have to be monotonic in x. For instance, $f(x,y)=\frac{x}{1+x^2y^3}$ is not monotonic in x (for any y). Also, functions in $\mathcal{F}_{22}$ strictly increase in x, but they are not necessarily monotonic in y (*cf.* Example 3.2). Finally, we note that $f(x,y)=\frac{x}{1+y}$ is in $\mathcal{F}_{21}\cap\mathcal{F}_{22}$.

Example 3.1 Let $f(x,y)=\frac{x}{1+xy^2}$, $x,y>0$. We note that if we extend f to $[0,\infty)^2$, then it satisfies (a), (b) and (c) of Section 1.

Since f is in $\mathcal{F}_{21}$, Theorem 2.2 holds for f by Proposition 3.1(i). For instance, if $L=3$ and $a=4$, then there is a b in the interval $[f(4-3,4),4-3]=[\frac{1}{17},1]$ such that $R(4,b)=3$. Using the bisection method described in Remark 2.3, we can get a better approximation for such a b. After seven iterations we find that the interval $[0.264706, 0.272059]$ contains a b such that $R(4, b) = 3$.

Similarly, after seven iterations of the bisection method we find that there is a b in the interval $[1.01538, 1.04615]$ such that $R(4, b) = S(4, b) = 0$. Given such a b, $f(x, b) < \frac{1}{b^2} < 1 < b$ for any x, showing that the condition in (ii) and (iv) of Lemma 2.2 - and hence in (ii) of Theorem 2.1 and (ii) and (iv) of Theorem 2.2 - is not necessary.

Since $R(x, y) < r_1(x, y) = f(x, y) < \frac{1}{y^2}$, given any b and L such that $L \geq \frac{1}{b^2}$, there is no a satisfying $R(a, b) = L$. In fact, there is a $B > 1$ such that if $b \geq B$ then $R(x, b) = 0$ and $S(x, b) > 0$ for any x. In order to show this, let us put $h(u, v) = v - f(u, v)$ and let $z = \{(u, v) : h(u, v) = 0\}$ be the level set of h for 0. As $h(u, v)$ (strictly) decreases in u and strictly increases in v, z is a strictly increasing function. Since $h(u, v)$ strictly increases in v,

$$\text{if } v < z(u) \text{ then } v < f(u, v). \tag{5}$$

For any $u \in (0, \infty)$, $\lim_{v \to 0} h(u, v) = -u < 0$ and $h(u, 1) = 1 - \frac{u}{1+u} > 0$. Therefore, $0 < z(u) < 1$, *i.e.* z maps $(0, \infty)$ to $(0, 1)$. Also, for any $v \in (0, 1)$, $h(\frac{v}{1-v^3}, v) = 0$ and so z is surjective. Thus z is a strictly increasing surjective function from $(0, \infty)$ to $(0, 1)$, hence it is also continuous. Therefore, since $\lim_{u \to 0} z(u) = 0$ and $\lim_{u \to \infty} z(u) = 1$, $z(u) = \frac{1}{u^2}$ has a unique solution $B > 1$, and then

$$\frac{1}{u^2} \leq z(u) \quad \text{for any } u \geq B. \tag{6}$$

Since $f(x, b) < \frac{1}{b^2}$ for any x and, by (6), $\frac{1}{b^2} \leq z(b)$ for any $b \geq B$, we see that if $b \geq B$, then $f(x, b) < z(b)$ for any x. Therefore, by (5), if $b \geq B$ then $r_1(x, b) = f(x, b) < f(b, f(x, b)) = s_1(x, b)$ for any x. Since $f \in \mathcal{F}_2$, this means that if $b \geq B$ then $R(x, b) = 0$ and $S(x, b) > 0$ for any x.

Example 3.2 Let $f(x, y) = \frac{x}{1+y+\frac{1}{2}y\sin^2(y)}$, $x, y > 0$. Put $g(y) = y + \frac{1}{2}y\sin^2(y)$. Then $g(y) \to 0$ as $y \to 0$, $y \leq g(y)$ and $g'(y) = 1 + \frac{1}{2}\sin^2(y) + \frac{1}{2}y\sin(2y) \leq 1 + y \leq 1 + g(y) \leq (1 + g(y))^2$. Therefore, $f \in \mathcal{F}_{22}$ and so, by Proposition 3.1(ii), Theorem 2.2 holds for f. We observe that $g(y)$ is not monotonic in y (for any x), so neither is $f(x, y)$.

Example 3.3 Let $f(x, y) = \frac{x}{1+y/x}$, $x, y > 0$. Since

$$(x - y) - \bigl(f(x, y) - f\bigl(y, f(x, y)\bigr)\bigr) = \frac{xy^3}{x^3 + 2x^2y + 2xy^2 + y^3} \geq 0,$$

f is in $\mathcal{F}_2$ and so Theorem 2.2 is true for f. We also note that for any b, $\lim_{x \to \infty} f(x, b) = \infty$ and so the consequent of Theorem 2.2(ii) holds for any $L \geq 0$ and b and that of Theorem 2.2(iv) for any $L \geq 0$ and $b > L$.

Finally, we show that R (and hence S) is not necessarily continuous for functions in $\mathcal{F}$. Our counterexample uses Proposition 3.2 below, which is a kind of comparison test for functions in $\mathcal{F}$. In what follows, we use superscripts (*e.g.* r_n^f) to distinguish between r_n, s_n, R and S for different functions.

Proposition 3.2 *Let $f, g \in \mathcal{F}$, $C > 0$ and assume that*

(i) *$g(x,y)$ increases in x and decreases in y,*

(ii) *if $C < x$ then $g(x,y) \leq f(x,y)$ and if $x \leq C$ then $g(x,y) \geq f(x,y)$.*

Let $C \leq L < a$, where L is a value of R^g. Then there is a $b' \leq C$ such that $R^f(a,b) \geq L$ for all $b \leq b'$.

Proof Let $x' \leq x$, $y' \geq y$. Then, by (i), $g(x',y') \leq g(x,y)$ and so, by (ii), if $C < x$ then $g(x',y') \leq g(x,y) \leq f(x,y)$. Similarly, if $x' \geq x$, $y' \leq y$ and $x \leq C$ then $g(x',y') \geq g(x,y) \geq f(x,y)$.

If $C \leq L < a$, where L is a value of R^g, then $R^g(a',b') = L$ for some a', b' such that $C \leq L < a' \leq a$, $C \geq b'$. Then, by the above, for any b such that $b' \geq b$

$$g(a',b') \leq f(a,b),$$
$$g(b', g(a',b')) \geq f(b, f(a,b))$$

and also, $C \leq L \leq L < g(a',b')$, $C \geq g(b', g(a',b'))$.

Hence, by induction, for all $n \geq 0$, $r_n^g(a',b') \leq r_n^f(a,b)$ (and $s_n^g(a',b') \geq s_n^f(a,b)$). Therefore, $R^f(a,b) \geq R^g(a,b) = L$, as claimed. □

Example 3.4 Let $f \in \mathcal{F}$ be defined as

$$f(x,y) = \begin{cases} \frac{x}{1+(1-x)+y} & \text{if } x \leq 1, \\ \frac{x}{1+y} & \text{if } x > 1, \end{cases}$$

and put $g(x,y) = \frac{x}{1+y}$. We note that f is actually in $\mathcal{F}_1$, in fact it strictly decreases in y and strictly increases in x.

By Theorem 2.2, the range of R^g is $[0,\infty)$. Therefore, by Proposition 3.2, R^f has values ≥ 1, since $f(x,y) \leq g(x,y)$ for $x \leq 1$ and $f(x,y) = g(x,y)$ for $x > 1$. If $R^f(x,y) < 1$, then eventually $r_n^f(x,y) < 1$ and hence $R^f(x,y) = \frac{R^f(x,y)}{1+(1-R^f(x,y))+S^f(x,y)}$. Therefore, $R^f(x,y) = 0$, *i.e.* R^f has no positive values <1. Since 0 is a value of R^f, R^f is not continuous. Since R^f and S^f have the same range, S^f is not continuous either. We also note that if $x, y \leq 1$, then $R^f(x,y) = S^f(x,y) = 0$, since then $R^f(x,y) < 1$ and $S^f(x,y) < 1$.

Competing interests

The author declares that he has no competing interests.

Author's contributions

The sole author had responsibility for all parts of the manuscript.

Acknowledgements

The author would like to thank the referees for their helpful comments and suggestions.

Received: 23 September 2013 Accepted: 11 December 2013 Published: 07 Jan 2014

References

1. Kent, CM: Convergence of solutions in a nonhyperbolic case. Nonlinear Anal. **47**, 4651-4665 (2001)
2. Kulenovic, MRS, Ladas, G: Dynamics of Second Order Rational Difference Equations: With Open Problems and Conjectures. Chapman & Hall/CRC, Boca Raton (2001)
3. Janssen, AJEM, Tjaden, DLA: Solution to problem 86-2. Math. Intell. **9**, 40-43 (1987)
4. Chan, DM, Kent, CM, Ortiz-Robinson, NL: Convergence results on a second-order rational difference equation with quadratic terms. Adv. Differ. Equ. **2009**, 985161 (2009)
5. Kalikow, S, Knopf, PM, Huang, YS, Nyerges, G: Convergence properties in the nonhyperbolic case $x_{n+1} = x_{n-1}/(1 + f(x_n))$. J. Math. Anal. Appl. **326**, 456-467 (2007)

10.1186/1687-1847-2014-8
Cite this article as: Nyerges: On the convergence of $x_n = f(x_{n-2}, x_{n-1})$ when $f(x, y) < x$. *Advances in Difference Equations* 2014, 2014:8