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Finite-difference method for the hyperbolic system of equations with nonlocal boundary conditions

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Abstract

In the present paper, the finite-difference method for the initial-boundary value problem for a hyperbolic system of equations with nonlocal boundary conditions is studied. The positivity of the difference analogy of the space operator generated by this problem in the space C with maximum norm is established. The structure of the interpolation spaces generated by this difference operator is investigated. The positivity of this difference operator in Hölder spaces is established. In applications, stability estimates for the solution of the difference scheme for a hyperbolic system of equations with nonlocal boundary conditions are obtained. A numerical example is applied.

MSC: 35L40; 35L45

Keywords: hyperbolic system of equations; nonlocal boundary value problems; difference schemes; interpolation spaces; positivity of the difference operator; stability estimates

1 Introduction

Nonlocal problems are widely used for mathematical modeling of various processes of physics, ecology, chemistry, and industry, when it is impossible to determine the boundary or initial values of the unknown function. The method of operators as a tool for the investigation of the solution of local and nonlocal problems for partial differential equations in Hilbert and Banach spaces has been systematically developed by several authors (see, *e.g.*, [1–27]). It is well known that (see, *e.g.*, [28–32] and the references given therein) many application problems in fluid mechanics, physics, mathematical biology, and chemistry were formulated as nonlocal mathematical models. Note that such problems were not well studied in general.

In the paper [33], the initial-boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} + \delta(u(t,x) - v(t,x)) = f_1(t,x), & 0 < x < l, 0 < t < T, \\ \frac{\partial v(t,x)}{\partial t} - a(x)\frac{\partial v(t,x)}{\partial x} + \delta v(t,x) = f_2(t,x), & 0 < x < l, 0 < t < T, \\ u(t,0) = \gamma u(t,l), & 0 \le \gamma \le 1, \quad \beta v(t,0) = v(t,l), \quad 0 \le \beta \le 1, 0 \le t \le T, \\ u(0,x) = u_0(x), & v(0,x) = v_0(x), \quad 0 \le x \le l \end{cases} \tag{1}$$

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for the hyperbolic system of equations with nonlocal boundary conditions was considered. Here

$$a(x) \geq a > 0, \tag{2}$$

$u_0(x), v_0(x) \ (x \in [0, l]), f_1(t, x), f_2(t, x) \ ((t, x) \in [0, T] \times [0, l])$ are given smooth functions and they satisfy all compatibility conditions which guarantee the problem (1) has a smooth solution $u(t, x)$ and $v(t, x)$. As noted in the paper [33], the problem of sound waves [34] and the problem of the expansion of electricity oscillations [35] can be replaced by the problem (1). Note that, we have the nonclassical initial-boundary value problem (1) with boundary conditions $u(t, 0) = \gamma u(t, l)$, $0 \leq \gamma \leq 1$, $\beta v(t, 0) = v(t, l)$, $0 \leq \beta \leq 1$, $0 \leq t \leq T$. These conditions are given on two boundary points. It is clear that it is impossible to determine the boundary values of the unknown function. So, these conditions are not local.

Let E be a Banach space and $A : D(A) \subset E \to E$ be a linear unbounded operator densely defined in E. We call A a positive operator in the Banach space if the operator $(\lambda I + A)$ has a bounded inverse in E for any $\lambda \geq 0$, and the following estimate holds:

$$\left\| (\lambda I + A)^{-1} \right\|_{E \to E} \leq \frac{M}{\lambda + 1}. \tag{3}$$

Throughout the present paper, M is defined as a positive constant. However, we will use $M(\alpha, \beta, \ldots)$ to stress the fact that the constant depends only on $\alpha, \beta, \ldots$.

For a positive operator A in the Banach space E, let us introduce the fractional spaces $E_\alpha = E_\alpha(E, A)$ $(0 < \beta < 1)$ consisting of those $v \in E$ for which the norm

$$\|v\|_{E_\alpha} = \sup_{\lambda > 0} \lambda^\alpha \left\| A(\lambda + A)^{-1} v \right\|_E + \|v\|_E$$

is finite.

Let us introduce the Banach space $\mathbb{C}^\alpha[0, l] = C^\alpha([0, l], R) \times C^\alpha([0, l], R)$ $(0 \leq \alpha \leq 1)$ of all continuous vector functions $u = \left(\begin{smallmatrix} u_1(x) \\ u_2(x) \end{smallmatrix} \right)$ defined on $[0, l]$ and satisfying a Hölder condition for which the following norm is finite:

$$\|u\|_{\mathbb{C}^\alpha[0, l]} = \|u\|_{\mathbb{C}[0, l]}$$

$$+ \sup_{\substack{x, x+\tau \in [0, l] \\ \tau \neq 0}} \frac{|u_1(x + \tau) - u_1(x)|}{|\tau|^\alpha} + \sup_{\substack{x, x+\tau \in [0, l] \\ \tau \neq 0}} \frac{|u_2(x + \tau) - u_2(x)|}{|\tau|^\alpha}.$$

Here $\mathbb{C}[0, l] = C([0, l], R) \times C([0, l], R)$ is the Banach space of all continuous vector functions $u = \left(\begin{smallmatrix} u_1(x) \\ u_2(x) \end{smallmatrix} \right)$ defined on $[0, l]$ with norm

$$\|u\|_{\mathbb{C}[0, l]} = \max_{x \in [0, l]} |u_1(x)| + \max_{x \in [0, l]} |u_2(x)|.$$

We consider the space operator A generated by the problem (1) defined by the formula

$$Au = \begin{pmatrix} a(x)\frac{du_1(x)}{dx} + \delta u_1(x) & -\delta u_2(x) \\ 0 & -a(x)\frac{du_2(x)}{dx} + \delta u_2(x) \end{pmatrix} \tag{4}$$

with domain

$$D(A) = \left\{ \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} : u_m(x), \frac{du_m(x)}{dx} \in C([0,l],R), m = 1,2; \right.$$

$$\left. u_1(0) = \gamma u_1(l), \beta u_2(0) = u_2(l) \right\}.$$

The Green's matrix function of A was constructed. The positivity of the operator A in the Banach space $\mathbb{C}[0,l]$ was established. It was proved that for any $\alpha \in (0,1)$ the norms in spaces $E_\alpha(\mathbb{C}[0,l],A)$ and $\overset{\circ}{\mathbb{C}}{}^\alpha[0,l]$ are equivalent. The positivity of A in the Hölder spaces of $\overset{\circ}{\mathbb{C}}{}^\alpha[0,l]$, $\alpha \in (0,1)$ was proved. In applications, stability estimates for the solution of the problem (1) for the hyperbolic system of equations with nonlocal boundary conditions were obtained.

In the present paper, the finite-difference method for the initial value problem for the hyperbolic system of equations with nonlocal boundary conditions is applied. The positivity of the difference analogy of the space operator A defined by equation (1) in the difference analogy of $\mathbb{C}[0,l]$ spaces is established. The structure interpolation spaces generated by this difference operator is studied. The positivity of this difference operator in Hölder spaces is established. In practice, stability estimates for the solution of the difference scheme for the hyperbolic system of equations with nonlocal boundary conditions are obtained. The method is illustrated by numerical example.

The organization of the present paper as follows. Section 1 is an introduction where we provide the history and formulation of the problem. In Section 2, the Green's matrix function of the difference space operator is presented and positivity of this operator in the difference analogy of $\mathbb{C}[0,l]$ spaces is proved. In Section 3, the structure of fractional spaces generated by this difference operator is investigated and positivity of this difference operator in Hölder spaces is established. In Section 4, stable difference schemes for the approximate solution of the problem (1) are constructed. A theorem on the stability for the first order of accuracy in the t difference scheme is proved. In Section 5, a numerical application is given. Finally, Section 6 is for our conclusion.

2 The Green's matrix function of difference space operator and positivity

Let us introduce the Banach spaces $\mathbb{C}_h^\alpha = C_h^\alpha \times C_h^\alpha$ $(0 \le \alpha \le 1)$ and $\mathbb{C}_h = C_h \times C_h$ of all mesh vector functions $u^h = \left\{ \begin{pmatrix} u_{1,n} \\ u_{2,n-1} \end{pmatrix} \right\}_{n=1}^M$ defined on

$$[0,l]_h = \{x_n = nh, 0 \le n \le M, Mh = l\}$$

with the following norms:

$$\|u^h\|_{\mathbb{C}_h^\alpha} = \|u^h\|_{\mathbb{C}_h}$$

$$+ \sup_{1 \le n < n+m \le M} \frac{|u_{1,n+m} - u_{1,n}|}{(mh)^\alpha} + \sup_{1 \le k < k+m \le M-1} \frac{|u_{2,n+m} - u_{2,n}|}{(mh)^\alpha},$$

$$\|u^h\|_{\mathbb{C}_h} = \max_{1 \le n \le M} |u_{1,n}| + \max_{0 \le n \le M-1} |u_{2,n}|.$$

We consider the difference space operator A_h^x generated by the problem (1) defined by the formula

$$A_h^x u^h = \begin{pmatrix} a(x_n)\frac{u_{1,n}-u_{1,n-1}}{h} + \delta u_{1,n} & -\delta u_{2,n} \\ 0 & -a(x_n)\frac{u_{2,n+1}-u_{2,n}}{h} + \delta u_{2,n} \end{pmatrix} \tag{5}$$

acting on the space of mesh vector functions $u^h = \left\{ \begin{pmatrix} u_{1,n} \\ u_{2,n-1} \end{pmatrix} \right\}_{n=1}^M$ defined on $[0,l]_h$, satisfying the conditions

$$u_{1,0} = \gamma u_{1,M}, \qquad \beta u_{2,0} = u_{2,M}.$$

Here $a_n = a(x_n)$. We will study the resolvent of the difference space operator $-A_h^x$, i.e.

$$A_h^x \begin{pmatrix} u \\ v \end{pmatrix}^h + \lambda \begin{pmatrix} u \\ v \end{pmatrix}^h = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \tag{6}$$

or

$$\begin{cases} a_n\frac{u_n-u_{n-1}}{h} + (\delta+\lambda)u_n - \delta v_n = \varphi_n, & 1 \le n \le M, \\ -a_{n+1}\frac{v_{n+1}-v_n}{h} + (\delta+\lambda)v_n = \psi_n, & 0 \le n \le M-1, \\ u_0 = \gamma u_M, \qquad \beta v_0 = v_M. \end{cases} \tag{7}$$

Lemma 2.1 *For any $\lambda \ge 0$, equation (7) is uniquely solvable and the following formula holds*:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \left(A_h^x + \lambda\right)^{-1} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} = \sum_{s=0}^M G(n,s;\lambda) \begin{pmatrix} \varphi_s \\ \psi_s \end{pmatrix} h$$

$$= \begin{pmatrix} \sum_{n=1}^M G_{11}(n,s;\lambda)\varphi_s h + \sum_{n=0}^{M-1} G_{12}(n,s;\lambda)\psi_s h \\ \sum_{n=0}^{M-1} G_{22}(n,s;\lambda)\psi_s h, \end{pmatrix}, \quad 0 \le n \le M, \tag{8}$$

where

$$G(n,s;\lambda) = \begin{pmatrix} G_{11}(n,s;\lambda) & G_{12}(n,s;\lambda) \\ 0 & G_{22}(n,s;\lambda) \end{pmatrix}.$$

Here

$$G_{11}(n,s;\lambda) = Q \begin{cases} U(n,s-1,\lambda)\frac{1}{a_s}, & 1 \le s \le n, \\ \gamma U(n,0,\lambda)U(M,s-1,\lambda)\frac{1}{a_s}, & n+1 \le s \le M, \end{cases} \tag{9}$$

$$G_{22}(n,s;\lambda) = P \begin{cases} \frac{\beta}{a_{s+1}}U(M,n,\lambda)U(s+1,0,\lambda), & 0 \le s \le n-1, \\ \frac{1}{a_{s+1}}U(s+1,n,\lambda), & n \le s \le M-1, \end{cases} \tag{10}$$

$$G_{12}(n,s;\lambda) = \delta \sum_{k=1}^M G_{11}(n,k-1;\lambda)G_{22}(k,s;\lambda)h, \tag{11}$$

$$P = \left(1 - \beta U(M,0;\lambda)\right)^{-1}, \qquad Q = \left(1 - \gamma U(M,0;\lambda)\right)^{-1},$$

$$U(n,k;\lambda) = \begin{cases} R_n, \ldots, R_{k+1}, & n > k, \\ 1, & n = k, \end{cases}$$

$$R_n = \left(1 + \frac{(\delta + \lambda)h}{a_n}\right)^{-1}, \quad 1 \le n \le M.$$

Proof Using the resolvent equation (7), we get

$$-a_{n+1}\frac{v_{n+1} - v_n}{h} + (\delta + \lambda)v_n = \psi_n, \quad 0 \le n \le M - 1, \qquad \beta v_0 = v_M.$$

From that follows the following recursive formula:

$$v_n = R_{n+1}v_{n+1} + \frac{h}{a_{n+1}}R_{n+1}\psi_n, \quad 0 \le n \le M - 1.$$

Hence

$$v_n = U(M,n;\lambda)v_M + \sum_{s=n}^{M-1} U(s+1,n;\lambda)\frac{h}{a_{s+1}}\psi_s, \quad 0 \le n \le M - 1.$$

From this formula and the nonlocal boundary condition $\beta v_0 = v_M$ it follows that

$$v_M = \beta P \sum_{s=0}^{M-1} U(s+1,0;\lambda)\frac{h}{a_{s+1}}\psi_s.$$

Then,

$$v_n = U(M,n;\lambda)\beta P \sum_{s=0}^{M-1} U(s+1,0;\lambda)\frac{h}{a_{s+1}}\psi_s$$

$$+ \sum_{s=n}^{M-1} U(s+1,n;\lambda)\frac{h}{a_{s+1}}\psi_s = \beta P \sum_{s=0}^{n-1} U(M,n;\lambda)U(s+1,0;\lambda)\frac{h}{a_{s+1}}\psi_s$$

$$+ P \sum_{s=n}^{M-1} U(s+1,n;\lambda)\frac{h}{a_{s+1}}\psi_s = \sum_{s=0}^{M-1} G_{22}(n,s;\lambda)\psi_s h. \tag{12}$$

Using the resolvent equation (7), we get

$$a_n\frac{u_n - u_{n-1}}{h} + (\delta + \lambda)u_n - \delta v_n = \varphi_n, \quad 1 \le n \le M, \qquad u_0 = \gamma u_M.$$

From that follows the system of recursion formulas

$$u_n = R_n u_{n-1} + \frac{h}{a_n}R_n(\delta v_n + \varphi_n), \quad 1 \le n \le M.$$

Hence

$$u_n = U(n,0;\lambda)u_0 + \sum_{k=1}^{n} U(n,k-1;\lambda)\frac{h}{a_k}(\delta v_k + \varphi_k), \quad 1 \le n \le M.$$

From this formula and the nonlocal boundary condition $u_0 = \gamma u_M$ it follows that

$$u_0 = \gamma Q \sum_{k=1}^{M} U(M, k-1; \lambda) \frac{h}{a_k} (\delta v_k + \varphi_k).$$

Therefore,

$$
u_n = U(n, 0; \lambda) \left[\gamma Q \sum_{k=1}^{M} U(M, k-1; \lambda) \frac{h}{a_k} (\delta v_k + \varphi_k) \right] + \sum_{k=1}^{n} U(n, k-1; \lambda) \frac{h}{a_k} (\delta v_k + \varphi_k)
$$

$$
= U(n, 0; \lambda) \gamma Q \sum_{s=n+1}^{M} U(M, s-1; \lambda) \frac{h}{a_s} (\delta v_s + \varphi_s) + \sum_{s=1}^{n} U(n, s-1; \lambda) \frac{h}{a_s} (\delta v_s + \varphi_s)
$$

$$
= \sum_{s=1}^{M} G_{11}(n, s-1; \lambda) \varphi_s h + \delta \sum_{s=1}^{M} G_{11}(n, s-1; \lambda) v_s h.
$$

Applying equation (12), we get

$$
\delta \sum_{k=1}^{M} G_{11}(n, k-1; \lambda) v_k h = \delta \sum_{k=1}^{M} G_{11}(n, k-1; \lambda) \left[\sum_{s=0}^{M-1} G_{22}(k, s; \lambda) \psi_s h \right] h
$$

$$
= \sum_{s=0}^{M-1} \left[\delta \sum_{k=1}^{M} G_{11}(n, k-1; \lambda) G_{22}(k, s; \lambda) h \right] \psi_s h
$$

$$
= \sum_{s=0}^{M-1} G_{12}(n, s; \lambda) \psi_s h.
$$

From the last two formulas it follows that

$$
u_n = \sum_{s=1}^{M} G_{11}(n, s-1; \lambda) \varphi_s h + \sum_{s=0}^{M-1} G_{12}(n, s; \lambda) \psi_s h. \tag{13}
$$

Lemma 2.1 is proved. $\qquad\square$

Lemma 2.2 *The following pointwise estimates hold; see equation* (7):

$$
|P|, |Q| \leq \frac{1}{1 - r^M}, \tag{14}
$$

$$
|G_{11}(n, s; \lambda)| \leq \frac{1}{a(1 - r^M)} \begin{cases} r^{n-s+1}, & 1 \leq s \leq n, \\ r^{M+n-s+1}, & n+1 \leq s \leq n \leq M, \end{cases} \tag{15}
$$

$$
|G_{22}(n, s; \lambda)| \leq \frac{1}{a(1 - r^M)} \begin{cases} r^{M+s+1-n}, & 0 \leq s \leq n-1, \\ r^{s+1-n}, & n \leq s \leq M-1, \end{cases} \tag{16}
$$

$$
|G_{12}(n, s; \lambda)| \leq \frac{1}{a(1 - r^M)} \begin{cases} r^{n-s+1}, & 0 \leq s \leq n, \\ r^{s-n+1}, & n+1 \leq s \leq n \leq M-1. \end{cases} \tag{17}
$$

Here $r = \dfrac{1}{1 + \frac{(\delta+\lambda)h}{a}}$.

Proof It is easy to see that the estimates of equations (14), (15), and (16) follow from the triangle inequality. Applying the triangle inequality, we get

$$
\left|G_{12}(n,s;\lambda)\right| \leq \delta \sum_{p=1}^{s} \left|G_{11}(n,p-1;\lambda)\right|\left|G_{22}(p,s;\lambda)\right| h. \tag{18}
$$

If $1 \leq s \leq n-1$. Then, using the estimates of equations (14), (15), (16), and inequality (18), we get

$$
\begin{aligned}
\left|G_{12}(n,s;\lambda)\right| &\leq \delta \sum_{p=1}^{s} \left|G_{11}(n,p-1;\lambda)\right|\left|G_{22}(p,s;\lambda)\right| h \\
&\quad + \delta \sum_{p=s+1}^{n} \left|G_{11}(n,p-1;\lambda)\right|\left|G_{22}(p,s;\lambda)\right| h \\
&\quad + \delta \sum_{p=n+1}^{M} \left|G_{11}(n,p-1;\lambda)\right|\left|G_{22}(p,s;\lambda)\right| h \\
&\leq \frac{\delta}{a^2(1-r^M)^2}\left[\sum_{p=1}^{s} r^{n-p+1} r^{s+1-p} h + \sum_{p=s+1}^{n} r^{n-p+1} r^{M+s+1-p} h \right. \\
&\quad \left. + \sum_{p=n+1}^{M} r^{M+n-p+1} r^{M+s+1-p} h \right] \\
&= \frac{\delta r^{n-s+2}}{a^2(1-r^M)^2}\left[\sum_{p=1}^{s} r^{2s-2p} h + \sum_{p=s+1}^{n} r^{M+2s-2p} h + \sum_{p=n+1}^{M} r^{2M+2s-2p} h \right] \\
&= \frac{\delta h r^{n-s+2}}{a^2(1-r^M)^2(1-r^2)^2}\left[1 - r^{2s} + \left(1-r^{2(n-s)}\right) r^{M+2s-2n} + r^{2s}\left(1-r^{2(M-n)}\right) \right] \\
&= \frac{\delta h r^{n-s+2}(1-r^M)}{a^2(1-r^M)^2(1-r^2)}\left[1 + r^{M+2s-2n} \right] \\
&= \frac{\delta h r^{n-s+2}(1-r^M)}{a^2(1-r^M)(1-r^2)}\left[1 + r^{M+2s-2n} \right] \\
&= \frac{\delta h r^{n-s+2}(1-r^M)}{a^2(1-r^M) r^{\frac{\delta+\lambda}{a}} h(1+r)}\left[1 + r^{M+2s-2n} \right] \leq \frac{r^{n-s+1}}{a(1-r^M)}.
\end{aligned}
$$

If $s = n$. Then, using the estimates of equations (14), (15), (16), and inequality (18), we get

$$
\begin{aligned}
\left|G_{12}(n,s;\lambda)\right| &\leq \delta \sum_{p=1}^{n} \left|G_{11}(n,p-1;\lambda)\right|\left|G_{22}(p,s;\lambda)\right| h \\
&\quad + \delta \sum_{p=n+1}^{M} \left|G_{11}(n,p-1;\lambda)\right|\left|G_{22}(p,s;\lambda)\right| h \\
&\leq \frac{\delta}{a^2(1-r^M)^2}\left[\sum_{p=1}^{n} r^{n-p+1} r^{s+1-p} h + \sum_{p=n+1}^{M} r^{M+n-p+1} r^{M+s+1-p} h \right] \\
&= \frac{\delta r^2}{a^2(1-r^M)^2}\left[\sum_{p=1}^{n} r^{2s-2p} h + \sum_{p=n+1}^{M} r^{2M+2s-2p} h \right]
\end{aligned}
$$

$$= \frac{\delta h r^2}{a^2 (1 - r^M)^2 (1 - r^2)} \left[1 - r^{2n} + \left(1 - r^{2(M-s)} \right) r^{2s} \right]$$

$$= \frac{\delta h r^2 (1 - r^M)}{a^2 (1 - r^M)^2 (1 - r^2)} = \frac{\delta h r^2 (1 - r^M)}{a^2 (1 - r^M) r^{\frac{\delta + \lambda}{a}} h (1 + r)} \le \frac{r}{a(1 - r^M)}.$$

Here $n \le s \le M$. Then, using the estimates of equations (14), (15), (16), and inequality (18), we get

$$\left| G_{12}(n, s; \lambda) \right| \le \delta \sum_{p=1}^{n} \left| G_{11}(n, p - 1; \lambda) \right| \left| G_{22}(p, s; \lambda) \right| h$$

$$+ \delta \sum_{p=n+1}^{s} \left| G_{11}(n, p - 1; \lambda) \right| \left| G_{22}(p, s; \lambda) \right| h$$

$$+ \delta \sum_{p=s+1}^{M} \left| G_{11}(n, p - 1; \lambda) \right| \left| G_{22}(p, s; \lambda) \right| h$$

$$\le \frac{\delta}{a^2 (1 - r^M)^2} \left[\sum_{p=1}^{n} r^{n-p+1} r^{s+1-p} h + \sum_{p=n+1}^{s} r^{s+2-p} r^{M+n-p} h \right.$$

$$\left. + \sum_{p=s+1}^{M} r^{s+2-p} r^{M+n-p} h \right]$$

$$= \frac{\delta r^{s-n+2}}{a^2 (1 - r^M)^2} \left[\sum_{p=1}^{n} r^{2n-2p} h + \sum_{p=n+1}^{s} r^{M+2n-2p} h + \sum_{p=s+1}^{M} r^{2M+2n-2p} h \right]$$

$$= \frac{\delta h r^{s-n+2}}{a^2 (1 - r^M)^2 (1 - r^2)} \left[1 - r^{2n} + \left(1 - r^{2(n-s)} \right) r^{M+2s-2n} + r^{2n} \left(1 - r^{2(M-s)} \right) \right]$$

$$= \frac{\delta h r^{s-n+2} (1 - r^M)}{a^2 (1 - r^M)^2 (1 - r^2)} \left[1 + r^{M+2s-2n} \right]$$

$$= \frac{\delta h r^{s-n+1} (1 - r^M)}{a^2 (1 - r^M)(1 - r^2)} \left[1 + r^{M+2s-2n} \right]$$

$$= \frac{\delta h r^{s-n+2}}{a^2 (1 - r^M) r^{\frac{\delta + \lambda}{a}} h (1 + r)} \left[1 + r^{M+2s-2n} \right] \le \frac{r^{s-n+1}}{a(1 - r^M)}.$$

Lemma 2.2 is proved. $\qquad\square$

Theorem 2.1 *The operator* $(\lambda I + A_h^x)$ *has a bounded inverse in* $\mathbb{C}_h$ *for any* $\lambda \ge 0$ *and the following estimate holds*:

$$\left\| \left(\lambda + A_h^x \right)^{-1} \right\|_{\mathbb{C}_h \to \mathbb{C}_h} \le \frac{M_1}{1 + \lambda}. \tag{19}$$

Proof Using the formula equation (13) and the triangle inequality, we get

$$|u_n| \le \sum_{k=1}^{M} h \left| G_{11}(n, k - 1; \lambda) \right| \max_{1 \le k \le M} |\varphi_k| + \sum_{k=0}^{M-1} h \left| G_{12}(n, k; \lambda) \right| \max_{0 \le k \le M-1} |\psi_k|$$

$$\le \left[\sum_{k=1}^{M} h \left| G_{11}(n, k - 1; \lambda) \right| + \sum_{k=0}^{M-1} h \left| G_{12}(n, k; \lambda) \right| \right] \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{C_h[0,l]},$$

$$|v_n| \leq \sum_{k=0}^{M-1} h |G_{22}(n,k;\lambda)| \max_{0 \leq k \leq M-1} |\psi_k|$$

$$\leq \sum_{k=0}^{M-1} h |G_{22}(n,k;\lambda)| \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{C_h[0,l]},$$

for any $n = 0, 1, \ldots, M$. Using the estimate of equation (15), we get

$$\sum_{k=1}^{M} h |G_{11}(n,k-1;\lambda)| \leq \frac{r}{a(1-r^M)} \left[\sum_{k=1}^{n} r^{n-k} h + \sum_{k=n+1}^{M} r^{M+n-k} h \right]$$

$$= \frac{hr}{a(1-r^M)r^{\frac{\delta+\lambda}{a}} h} \left[1 - r^n + r^n \left(1 - r^{M-n} \right) \right] = \frac{1}{\delta + \lambda}.$$

Using the estimate of equation (16), we get

$$\sum_{k=0}^{M-1} h |G_{22}(n,k;\lambda)| \leq \frac{1}{a(1-r^M)} \left[\sum_{k=0}^{n-1} r^{M+k+1-n} h + \sum_{k=n}^{M-1} r^{k+1-n} h \right]$$

$$= \frac{hr}{a(1-r^M)r^{\frac{\delta+\lambda}{a}} h} \left[1 - r^{M-n} + r^{M-n} \left(1 - r^n \right) \right] = \frac{1}{\delta + \lambda}.$$

Using the estimate of equation (17), we get

$$\sum_{k=0}^{M-1} h |G_{12}(n,k;\lambda)| \leq \frac{r}{a(1-r^M)} \left[\sum_{k=0}^{n} r^{n-k} h + \sum_{k=n+1}^{M-1} r^{k-n} h \right]$$

$$= \frac{hr}{a(1-r^M)r^{\frac{\delta+\lambda}{a}} h} \left[1 - r^{n+1} + r \left(1 - r^{M-n-1} \right) \right] \leq \frac{2}{\delta + \lambda}.$$

Therefore,

$$\max_{1 \leq n \leq M} |u_n| \leq \left[\sum_{k=1}^{M} h |G_{11}(n,k-1;\lambda)| + \sum_{k=0}^{M-1} h |G_{12}(n,k;\lambda)| \right] \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{C_h[0,l]}$$

$$\leq \frac{3}{\delta + \lambda} \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{C_h[0,l]},$$

$$\max_{0 \leq n \leq M-1} |v_n| \leq \sum_{k=0}^{M-1} h |G_{22}(n,k;\lambda)| \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{C_h[0,l]} \leq \frac{1}{\delta + \lambda} \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{C_h[0,l]}.$$

From this it follows that

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix}^h \right\|_{C_h[0,l]} \leq \frac{3}{\delta + \lambda} \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{C_h[0,l]}.$$

Theorem 2.1 is proved. $\square$

3 The structure of fractional spaces $E_\alpha(\mathbb{C}_h, A_h^x)$ and positivity of A_h^x in Hölder spaces

Clearly, the operator A_h^x and its resolvent $(A_h^x + \lambda)^{-1}$ commute. By the definition of the norm in the fractional space $E_\alpha = E_\alpha(\mathbb{C}_h, A_h^x)$, we get

$$\left\| \left(A_h^x + \lambda\right)^{-1} \right\|_{E_\alpha \to E_\alpha} \leq \left\| \left(A_h^x + \lambda\right)^{-1} \right\|_{\mathbb{C}_h \to \mathbb{C}_h}.$$

Thus, from Theorem 2.1 it follows that A_h^x is a positive operator in the fractional spaces $E_\alpha(\mathbb{C}_h, A_h^x)$. Moreover, we have the following result.

Theorem 3.1 *For $\alpha \in (0,1)$, the norms of the spaces $E_\alpha(\mathbb{C}_h, A_h^x)$ and the Hölder space $\mathring{\mathbb{C}}_h^\alpha$ are equivalent uniformly with respect to h. Here*

$$\mathring{\mathbb{C}}_h^\alpha = \left\{ \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \in \mathbb{C}_h^\alpha : \right.$$

$$\left. \varphi_0 = \gamma \varphi_M, 0 \leq \gamma \leq 1, \beta \psi_0 = \psi_M, 0 \leq \beta \leq 1 \right\}. \tag{20}$$

Proof For any $\lambda \geq 0$ we have the obvious equality

$$A_h^x \left(A_h^x + \lambda\right)^{-1} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} - \lambda \left(A_h^x + \lambda\right)^{-1} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}.$$

By equation (8), we can write

$$A_h^x \left(A_h^x + \lambda\right)^{-1} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}$$

$$= \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} - \lambda \sum_{k=0}^M G(n,k;\lambda) h \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}$$

$$= \begin{pmatrix} [1 - \lambda \sum_{k=1}^n G_{11}(n,k;\lambda) h] \varphi_n - \lambda \sum_{k=n+1}^M G_{11}(n,k;\lambda) h \varphi_M \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} -\lambda \sum_{k=0}^{M-1} G_{12}(n,k;\lambda) h \psi_n \\ -\lambda \sum_{k=0}^{n-1} G_{22}(n,k;\lambda) h \psi_0 + [1 - \lambda \sum_{k=n}^{M-1} G_{22}(n,k;\lambda) h] \psi_n \end{pmatrix}$$

$$+ \begin{pmatrix} \lambda \sum_{k=1}^n G_{11}(n,k;\lambda)(\varphi_n - \varphi_k) h + \lambda \sum_{k=n+1}^M G_{11}(n,k;\lambda)(\varphi_M - \varphi_k) h \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \lambda \sum_{k=0}^{M-1} G_{12}(n,k;\lambda)(\psi_n - \psi_k) h \\ \lambda \sum_{k=0}^{n-1} G_{12}(n,k;\lambda)(\psi_0 - \psi_k) h + \lambda \sum_{k=n}^{M-1} G_{22}(n,k;\lambda)(\psi_n - \psi_k) h \end{pmatrix}. \tag{21}$$

Applying equation (21) and the following obvious equalities:

$$1 - \lambda \sum_{k=1}^n G_{11}(n,k;\lambda) h = 1 - Q\lambda \sum_{k=1}^n U(n,s-1;\lambda) \frac{h}{a_s}$$

$$= 1 - Q\frac{\lambda}{\delta + \lambda} \sum_{k=1}^n \left(U(n,s-1;\lambda) - U(n,s;\lambda)\right)$$

$$= 1 - Q\frac{\lambda}{\delta + \lambda}\big(1 - U(n, 0; \lambda)\big)$$

$$= \frac{\delta}{\delta + \lambda}Q - \gamma QU(M, 0; \lambda) + \frac{\lambda}{\delta + \lambda}QU(n, 0; \lambda),$$

$$-\lambda \sum_{k=n+1}^{M} G_{11}(n, k; \lambda) = -Q\lambda \sum_{k=n+1}^{M} \frac{\gamma}{a_s}U(n, s-1; \lambda)U(M, 0; \lambda)$$

$$= -\frac{\lambda}{\delta + \lambda}Q \sum_{k=n+1}^{M} \gamma U(M, 0; \lambda)\big(U(n, s-1; \lambda) - U(n, s; \lambda)\big)$$

$$= -\frac{\lambda}{\delta + \lambda}\gamma QU(M, 0; \lambda)\big(U(n, 1; \lambda) - U(n, M; \lambda)\big)$$

$$= -\frac{\lambda}{\delta + \lambda}\gamma Q\big[U(M, 0; \lambda)U(n, 1; \lambda) - U(n, 0; \lambda)\big],$$

$$1 - \lambda \sum_{s=n}^{M-1} G_{22}(n, k; \lambda)h = 1 - P\lambda \sum_{s=n}^{M-1} \frac{1}{a_{s+1}}U(s+1, n; \lambda)h$$

$$= 1 - P\frac{\lambda}{\delta + \lambda}\sum_{s=n}^{M-1}\big[U(s, n; \lambda) - U(s+1, n; \lambda)\big]h$$

$$= 1 - P\frac{\lambda}{\delta + \lambda}\big[1 - U(M, n; \lambda)\big]h$$

$$= P\frac{\delta}{\delta + \lambda} - P\beta U(M, 0; \lambda) + P\frac{\lambda}{\delta + \lambda}U(M, n; \lambda),$$

$$-\lambda \sum_{s=0}^{n-1} G_{22}(n, k; \lambda)h = -P\lambda \sum_{s=0}^{n-1} \frac{\beta}{a_{s+1}}U(M, n; \lambda)U(s+1, 0; \lambda)h$$

$$- \lambda\beta \sum_{s=0}^{n-1} PU(M, n; \lambda)\frac{1}{\delta + \lambda}\big[U(s, n; \lambda) - U(s+1, n; \lambda)\big]h$$

$$- \lambda\beta P\frac{1}{\delta + \lambda}U(M, n; \lambda)\big[U(0, n; \lambda) - 1\big]h,$$

$$-\lambda \sum_{s=0}^{M-1} G_{12}(n, s; \lambda)h = -\lambda \sum_{s=0}^{M-1}\delta h\sum_{k=1}^{M} G_{11}(n, k-1; \lambda)G_{22}(k, s; \lambda)h$$

$$= -\lambda \sum_{k=1}^{M}\Bigg[\sum_{s=0}^{k-1}\delta hG_{11}(n, k-1; \lambda)G_{22}(k, s; \lambda)h$$

$$+ \sum_{s=k}^{M-1}\delta hG_{11}(n, k-1; \lambda)G_{22}(k, s; \lambda)h\Bigg]$$

$$= -\lambda \sum_{k=1}^{n+1}\sum_{s=0}^{k-1}\delta hG_{11}(n, k-1; \lambda)G_{22}(k, s; \lambda)h$$

$$- \lambda \sum_{k=n+2}^{M}\sum_{s=0}^{k-1}\delta hG_{11}(n, k-1; \lambda)G_{22}(k, s; \lambda)h$$

$$= -\lambda \sum_{k=1}^{n+1}\sum_{s=k}^{M-1}\delta hG_{11}(n, k-1; \lambda)G_{22}(k, s; \lambda)h$$

$$- \lambda \sum_{k=n+2}^{M} \sum_{s=k}^{M-1} \delta h G_{11}(n, k-1; \lambda) G_{22}(k, s; \lambda) h$$

$$= -\lambda \sum_{k=1}^{n+1} \sum_{s=0}^{k-1} \delta h U(n, k-2; \lambda) \frac{1}{a_{k-1}} \frac{\beta}{a_{s+1}}$$

$$\times U(M, k; \lambda) U(s+1, 0; \lambda) h$$

$$= -\lambda \sum_{k=n+2}^{M} \sum_{s=0}^{k-1} \delta h \gamma U(M, 0; \lambda) U(n, k-2; \lambda) \frac{1}{a_{k-1}} \frac{\beta}{a_{s+1}}$$

$$\times U(M, k; \lambda) U(s+1, 0; \lambda) h$$

$$= -\lambda \sum_{k=1}^{n+1} \sum_{s=k}^{M-1} \delta h U(n, k-2; \lambda) \frac{1}{a_{k-1}} \frac{1}{a_{s+1}} U(s+1, k; \lambda) h$$

$$= -\lambda \sum_{k=n+2}^{M} \sum_{s=k}^{M-1} \delta h \gamma U(M, 0; \lambda) U(n, k-2; \lambda) \frac{1}{a_{k-1}} \frac{1}{a_{s+1}} U(s+1, k; \lambda) h$$

$$= -\lambda \sum_{k=1}^{n+1} \sum_{s=0}^{k-1} \delta h U(n, k-2; \lambda) \frac{1}{a_{k-1}} U(M, k; \lambda) \beta \frac{1}{\delta + \lambda}$$

$$\times \left[U(s, 0; \lambda) - U(s+1, 0; \lambda) \right]$$

$$- \lambda \sum_{k=n+2}^{M} \sum_{s=0}^{k-1} \delta h \gamma U(M, 0; \lambda) U(n, k-2; \lambda) \frac{1}{a_{k-1}} \beta \frac{1}{\delta + \lambda}$$

$$\times \left[U(s, 0; \lambda) - U(s+1, 0; \lambda) \right]$$

$$- \lambda \sum_{k=1}^{n+1} \sum_{s=k}^{M-1} \delta h U(n, k-2; \lambda) \frac{1}{a_{k-1}} \frac{1}{\delta + \lambda}$$

$$\times \left[U(s, 0; \lambda) - U(s+1, 0; \lambda) \right]$$

$$- \lambda \sum_{k=n+2}^{M} \sum_{s=k}^{M-1} \delta h \gamma U(M, 0; \lambda) U(n, k-2; \lambda) \frac{1}{a_{k-1}} \beta \frac{1}{\delta + \lambda}$$

$$\times \left[U(s, 0; \lambda) - U(s+1, 0; \lambda) \right]$$

$$= -\lambda \sum_{k=1}^{n+1} \delta h U(n, k-2; \lambda) \frac{1}{a_{k-1}} U(M, k; \lambda) \beta \frac{1}{\delta + \lambda} \left[1 - U(k, 0; \lambda) \right]$$

$$- \lambda \sum_{k=n+2}^{M} \delta h \gamma U(M, 0; \lambda) U(n, k-2; \lambda) \frac{1}{a_{k-1}} \beta \frac{1}{\delta + \lambda} \left[1 - U(k, 0; \lambda) \right]$$

$$- \lambda \sum_{k=1}^{n+1} \delta h U(n, k-2; \lambda) \frac{1}{a_{k-1}} \frac{1}{\delta + \lambda} \left[U(k, 0; \lambda) - U(M, 0; \lambda) \right]$$

$$- \lambda \sum_{k=n+2}^{M} \delta h \gamma U(M, 0; \lambda) U(n, k-2; \lambda) \frac{1}{a_{k-1}} \frac{1}{\delta + \lambda}$$

$$\times \left[U(k, 0; \lambda) - U(M, 0; \lambda) \right],$$

and using the nonlocal boundary conditions

$$\varphi_0 = \gamma\varphi_M, \qquad \beta\psi_0 = \psi_M,$$

we get

$$A_h^x\left(\lambda + A_h^x\right)^{-1}\begin{pmatrix}\varphi_n \\ \psi_n\end{pmatrix}$$

$$= \begin{pmatrix}[\frac{\delta}{\delta+\lambda}Q - \gamma QU(M,0;\lambda)]\varphi_n + Q\frac{\lambda}{\delta+\lambda}U(n,0;\lambda)(\varphi_n - \varphi_0) \\ 0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{\lambda}{\delta+\lambda}\gamma QU(M,0;\lambda)U(n,1;\lambda)\varphi_M \\ -\lambda\beta PU(M,n;\lambda)\frac{1}{\delta+\lambda}U(0,n;\lambda)\psi_0\end{pmatrix}$$

$$+ \begin{pmatrix}\{-\lambda\sum_{k=1}^{n+1}\delta hU(n,k-2;\lambda)\frac{1}{a_{k-1}}U(M,k;\lambda)\beta\frac{1}{\delta+\lambda}[1 - U(k,0;\lambda)]\}\psi_n \\ [P\frac{\delta}{\delta+\lambda} - P\beta U(M,0;\lambda)]\psi_n + P\frac{\lambda}{\delta+\lambda}U(M,n;\lambda)(\psi_n - \psi_M)\end{pmatrix}$$

$$+ \begin{pmatrix}-\lambda\sum_{k=1}^{n+1}\delta hU(n,k-2;\lambda)\frac{1}{a_{k-1}}U(M,k;\lambda)\beta\frac{1}{\delta+\lambda}[1 - U(k,0;\lambda)]\psi_n \\ 0\end{pmatrix}$$

$$+ \begin{pmatrix}-\lambda\sum_{k=n+2}^{M}\delta h\gamma U(M,0;\lambda)U(n,k-2;\lambda)\frac{1}{a_{k-1}}\beta\frac{1}{\delta+\lambda}[1 - U(k,0;\lambda)]\psi_n \\ 0\end{pmatrix}$$

$$+ \begin{pmatrix}-\lambda\sum_{k=1}^{n+1}\delta hU(n,k-2;\lambda)\frac{1}{a_{k-1}}\frac{1}{\delta+\lambda}[U(k,0;\lambda) - U(M,0;\lambda)]\psi_n \\ 0\end{pmatrix}$$

$$+ \begin{pmatrix}-\lambda\sum_{k=n+2}^{M}\delta h\gamma U(M,0;\lambda)U(n,k-2;\lambda)\frac{1}{a_{k-1}}\beta\frac{1}{\delta+\lambda}[U(k,0;\lambda) - U(M,0;\lambda)]\psi_n \\ 0\end{pmatrix}$$

$$+ \begin{pmatrix}\lambda\sum_{k=1}^{n+1}G_{11}(n,k;\lambda)(\varphi_n - \varphi_k)h + \lambda\sum_{k=n+1}^{M}G_{11}(n,k;\lambda)(\varphi_M - \varphi_k)h \\ 0\end{pmatrix}$$

$$+ \begin{pmatrix}\lambda\sum_{k=0}^{M-1}G_{12}(n,k;\lambda)(\psi_n - \psi_k)h \\ \lambda\sum_{k=0}^{n-1}G_{22}(n,k;\lambda)(\psi_0 - \psi_k)h + \lambda\sum_{k=n}^{M-1}G_{22}(n,k;\lambda)(\psi_n - \psi_k)h\end{pmatrix}.$$

Using this formula, the triangle inequality and the definition of spaces $E_\alpha(\mathbf{C}[0,l]_h, A_h^x)$ and $\mathbf{C}^\alpha[0,l]_h$, we get

$$\left\|\lambda^\alpha A_h^x\left(A_h^x + \lambda\right)^{-1}\begin{pmatrix}\varphi \\ \psi\end{pmatrix}^h\right\|_{\mathbb{C}[0,l]_h}$$

$$\leq \max_{1\leq n\leq M}\left[\frac{\lambda^\alpha\delta}{\delta+\lambda}|Q| + \gamma|Q|\lambda^\alpha\left|U(M,0;\lambda)\right| + \frac{\lambda^{1+\alpha}}{\delta+\lambda}\gamma|Q|\left|U(M,0;\lambda)\right|\left|U(n,1;\lambda)\right|\right]$$

$$\times \max_{0\leq n\leq M}|\psi_n|$$

$$+ \max_{1\leq n\leq M}\frac{\lambda^{1+\alpha}}{\delta+\lambda}|Q|(nh)^\alpha\left|U(n,0;\lambda)\right|\sup_{1\leq n\leq M}\frac{|\varphi_n - \varphi_0|}{(nh)^\alpha}$$

$$+ \max_{1\leq n\leq M}\left[\frac{\lambda^\alpha\delta}{\delta+\lambda}|P| + \beta|P|\lambda^\alpha\left|U(M,0;\lambda)\right| + \frac{\lambda^{1+\alpha}}{\delta+\lambda}\beta|P|\left|U(M,n;\lambda)\right|\left|U(0,n;\lambda)\right|\right]$$

$$\times \max_{0 \le n \le M} |\psi_n| + \max_{1 \le n \le M} \frac{\lambda^{1+\alpha}}{\delta + \lambda} |P| \big((M-n)h\big)^{\alpha} \big|U(M,n;\lambda)\big|$$

$$\times \sup_{0 \le n \le M-1} \frac{|\psi_n - \psi_M|}{((M-n)h)^{\alpha}}$$

$$+ \max_{1 \le n \le M} \lambda^{1+\alpha} \sum_{k=1}^{n+1} \delta h \big|U(n,k-2;\lambda)\big| \frac{1}{a_{k-1}} \big|U(M,k;\lambda)\big|$$

$$\times \beta \frac{1}{\delta + \lambda} \big|1 - U(k,0;\lambda)\big| \max_{0 \le n \le M} |\psi_n|$$

$$+ \max_{1 \le n \le M} \lambda^{1+\alpha} \sum_{k=n+2}^{M} \delta h \gamma \big|U(M,0;\lambda)\big| \big|U(n,k-2;\lambda)\big| \frac{1}{a_{k-1}} \beta \frac{1}{\delta + \lambda} \big|1 - U(k,0;\lambda)\big|$$

$$\times \max_{0 \le n \le M} |\psi_n|$$

$$+ \max_{1 \le n \le M} \lambda^{1+\alpha} \sum_{k=1}^{n+1} \delta h \big|U(n,k-2;\lambda)\big| \frac{1}{a_{k-1}} \frac{1}{\delta + \lambda} \big|U(k,0;\lambda) - U(M,k;\lambda)\big| \max_{0 \le n \le M} |\psi_n|$$

$$+ \max_{1 \le n \le M} \lambda^{1+\alpha} \sum_{k=n+2}^{M} \delta h \gamma \big|U(M,0;\lambda)\big| \big|U(n,k-2;\lambda)\big| \frac{1}{a_{k-1}}$$

$$\times \frac{1}{\delta + \lambda} \big|U(k,0;\lambda) - U(M,0;\lambda)\big| \max_{0 \le n \le M} |\psi_n|$$

$$+ \max_{1 \le n \le M} \lambda^{1+\alpha} \sum_{k=1}^{n+1} \big((n-k)h\big)^{\alpha} \big|G_{11}(n,k;\lambda)\big| h \sup_{1 \le k \le n \le M} \frac{|\varphi_n - \varphi_k|}{((n-k)h)^{\alpha}}$$

$$+ \max_{1 \le n \le M} \lambda^{1+\alpha} \sum_{k=n+1}^{M} \big((M-n)h\big)^{\alpha} \big|G_{11}(n,k;\lambda)\big| h \sup_{1 \le n \le M-1} \frac{|\varphi_M - \varphi_k|}{((M-n)h)^{\alpha}}$$

$$+ \max_{1 \le n \le M} \lambda^{1+\alpha} \sum_{k=0}^{M-1} (|k-n|h)^{\alpha} \big|G_{12}(n,k;\lambda)\big| h \sup_{\substack{1 \le k,n \le M, \\ k \ne n}} \frac{|\psi_n - \psi_k|}{(|n-k|h)^{\alpha}}$$

$$+ \max_{1 \le n \le M} \lambda^{1+\alpha} \sum_{k=0}^{n-1} (kh)^{\alpha} \big|G_{22}(n,k;\lambda)\big| h \sup_{1 \le k \le M} \frac{|\psi_k - \psi_0|}{(kh)^{\alpha}}$$

$$+ \max_{1 \le n \le M} \lambda^{1+\alpha} \sum_{k=n+1}^{M-1} \big((k-n)h\big)^{\alpha} \big|G_{22}(n,k;\lambda)\big| h \sup_{1 \le n \le k \le M-1} \frac{|\psi_n - \psi_k|}{((k-n)h)^{\alpha}}$$

$$\le J \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^{h} \right\|_{\mathbb{C}_h^{\alpha}}.$$

Here

$$J = \frac{\lambda^{\alpha}\delta}{\delta + \lambda} |Q| + \gamma |Q| \lambda^{\alpha} \big|U(M,0;\lambda)\big|$$

$$+ \max_{1 \le n \le M} \left\{ \frac{\lambda^{1+\alpha}}{\delta + \lambda} \gamma |Q| \big|U(M,0;\lambda)\big| \big|U(n,1;\lambda)\big| + \frac{\lambda^{1+\alpha}}{\delta + \lambda} |Q|(nh)^{\alpha} \big|U(n,0;\lambda)\big| \right.$$

$$\times \frac{\lambda^{\alpha}\delta}{\delta + \lambda} |P| + \beta |P| \lambda^{\alpha} \big|U(M,0;\lambda)\big| + \frac{\lambda^{1+\alpha}}{\delta + \lambda} \beta |P| \big|U(M,n;\lambda)\big| \big|U(0,n;\lambda)\big|$$

$$+ \frac{\lambda^{1+\alpha}}{\delta + \lambda} |P| \big((M-n)\tau\big)^{\alpha} |U(M,n;\lambda)|$$

$$+ \lambda^{1+\alpha} \sum_{k=1}^{n+1} \delta h |U(n,k-2;\lambda)| \frac{1}{a_{k-1}} |U(M,k;\lambda)| \beta \frac{1}{\delta+\lambda} |1 - U(k,0;\lambda)|$$

$$+ \lambda^{1+\alpha} \sum_{k=n+2}^{M} \delta h \gamma |U(M,0;\lambda)| |U(n,k-2;\lambda)| \frac{1}{a_{k-1}} \beta \frac{1}{\delta+\lambda} |1 - U(k,0;\lambda)|$$

$$+ \lambda^{1+\alpha} \sum_{k=1}^{n+1} \delta h |U(n,k-2;\lambda)| \frac{1}{a_{k-1}} \frac{1}{\delta+\lambda} |U(k,0;\lambda) - U(M,k;\lambda)| + \lambda^{1+\alpha}$$

$$\times \sum_{k=n+2}^{M} \delta h \gamma |U(M,0;\lambda)| |U(n,k-2;\lambda)| \frac{1}{a_{k-1}} \frac{1}{\delta+\lambda} |U(k,0;\lambda) - U(M,0;\lambda)|$$

$$+ \lambda^{1+\alpha} \sum_{k=1}^{n+1} \big((n-k)\tau\big)^{\alpha} |G_{11}(n,k;\lambda)| h + \lambda^{1+\alpha} \sum_{k=n+1}^{M} \big((M-n)\tau\big)^{\alpha} |G_{11}(n,k;\lambda)| h$$

$$+ \lambda^{1+\alpha} \sum_{k=0}^{M-1} \big(|k-n|\tau\big)^{\alpha} |G_{12}(n,k;\lambda)| h + \lambda^{1+\alpha} \sum_{k=0}^{n-1} (k\tau)^{\alpha} |G_{22}(n,k;\lambda)| h$$

$$+ \lambda^{1+\alpha} \sum_{k=n+1}^{M-1} \big((k-n)\tau\big)^{\alpha} |G_{22}(n,k;\lambda)| h \Bigg\}.$$

Using the estimates

$$\frac{\lambda^{\alpha} \delta^{1-\alpha}}{\delta + \lambda} \le 1, \qquad \frac{\lambda^{1+\alpha} \delta^{1-\alpha}}{(\delta + \lambda)^2} \le 1,$$

and the estimates of equations (14), (15), (16), and (17), we get

$$J \le \max_{1 \le n \le M} \Bigg\{ \frac{\lambda^{\alpha} \delta}{\delta+\lambda} \frac{1}{1-r^{M}} + \frac{1}{1-r^{M}} \lambda^{\alpha} \left(1 + \frac{(\delta+\lambda)h}{a}\right)^{-M}$$

$$+ \frac{\lambda^{1+\alpha}}{\delta+\lambda} \frac{1}{1-r^{M}} \left(1 + \frac{(\delta+\lambda)h}{a}\right)^{-M-n+1} + \frac{\lambda^{1+\alpha}}{\delta+\lambda} \frac{1}{1-r^{M}} (n\tau)^{\alpha} \left(1 + \frac{(\delta+\lambda)h}{a}\right)^{-n}$$

$$+ \frac{\lambda^{\alpha} \delta}{\delta+\lambda} \frac{1}{1-r^{M}} + \frac{1}{1-r^{M}} \lambda^{\alpha} \left(1 + \frac{(\delta+\lambda)h}{a}\right)^{-M} + \frac{\lambda^{1+\alpha}}{\delta+\lambda} \frac{1}{1-r^{M}} \left(1 + \frac{(\delta+\lambda)h}{a}\right)^{-M}$$

$$+ \frac{\lambda^{1+\alpha}}{\delta+\lambda} \frac{1}{1-r^{M}} \big((M-n)h\big)^{\alpha} \left(1 + \frac{(\delta+\lambda)h}{a}\right)^{-M+n}$$

$$+ \lambda^{1+\alpha} \sum_{k=1}^{n+1} \delta h r^{n-k+2+M-k} \frac{1}{a} \frac{1}{\delta+\lambda} + \lambda^{1+\alpha} \sum_{k=n+2}^{M} \delta h r^{M+n-k+2} \frac{1}{a} \frac{1}{\delta+\lambda}$$

$$+ \lambda^{1+\alpha} \sum_{k=1}^{n+1} \delta h r^{n-k+2} \frac{1}{a} \frac{1}{\delta+\lambda} + \lambda^{1+\alpha} \sum_{k=n+2}^{M} \delta h r^{M+n-k+2} \frac{1}{a} \frac{1}{\delta+\lambda}$$

$$+ \lambda^{1+\alpha} \sum_{k=1}^{n+1} \big((n-k)h\big)^{\alpha} r^{n-k+2} \frac{1}{a} h + \lambda^{1+\alpha} \sum_{k=n+1}^{M} \big((M-k)h\big)^{\alpha} r^{M-k+2} \frac{1}{a} h$$

$$+ \lambda^{1+\alpha} \sum_{k=0}^{M-1} \left(|k-n|h\right)^{\alpha} r^{|n-k|+1} \frac{1}{a} h + \lambda^{1+\alpha} \sum_{k=0}^{n-1} (kh)^{\alpha} r^{k+1} \frac{1}{a} h$$

$$+ \lambda^{1+\alpha} \sum_{k=n+1}^{M-1} \left((k-n)h\right)^{\alpha} r^{k-n+2} \frac{1}{a} h \Bigg\} \le M(a,\delta).$$

Then

$$\left\| \lambda^{\alpha} A_h^x \left(A_h^x + \lambda\right)^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{\mathbb{C}_h} \le M(a,\delta) \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{\mathbb{C}_h^{\alpha}}$$

for any $\lambda \ge 0$. This means that

$$\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{E_{\alpha}(C_h, A_h^x)} \le M(a,\delta) \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{\mathbb{C}_h^{\alpha}}.$$

Let us prove the opposite inequality. For any positive operator A_h^x we can write

$$\begin{pmatrix} f \\ g \end{pmatrix} = \int_0^{\infty} A_h^x \left(\lambda + A_h^x\right)^{-2} \begin{pmatrix} f \\ g \end{pmatrix} d\lambda.$$

From the relation and formula (21) it follows that

$$\begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} = \int_0^{\infty} \left(\lambda + A_h^x\right)^{-1} A_h^x \left(\lambda + A_h^x\right)^{-1} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} d\lambda$$

$$= \int_0^{\infty} \sum_{k=0}^{M} G(n,k;\lambda) h A_h^x \left(\lambda + A_h^x\right)^{-1} \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} d\lambda$$

$$= \int_0^{\infty} \sum_{s=1}^{M} G_{11}(n,s;\lambda) h A_h^x \left(\lambda + A_h^x\right)^{-1} \begin{pmatrix} \varphi_s \\ 0 \end{pmatrix} d\lambda$$

$$+ \int_0^{\infty} \sum_{s=0}^{M-1} G_{12}(n,s;\lambda) h A_h^x \left(\lambda + A_h^x\right)^{-1} \begin{pmatrix} 0 \\ \psi_s \end{pmatrix} d\lambda$$

$$+ \int_0^{\infty} \sum_{s=0}^{M-1} G_{22}(n,s;\lambda) h A_h^x \left(\lambda + A_h^x\right)^{-1} \begin{pmatrix} 0 \\ \psi_s \end{pmatrix} d\lambda.$$

Consequently,

$$\begin{pmatrix} \varphi_{n+m} \\ \psi_{n+m} \end{pmatrix} = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}$$

$$= \int_0^{\infty} \sum_{k=0}^{M-1} \left(G(n+m,k;\lambda) - G(n,k;\lambda)\right) h A_h^x \left(\lambda + A_h^x\right)^{-1} \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} d\lambda,$$

whence

$$
|\varphi_{n+m} - \varphi_n| \leq \int_0^\infty \sum_{s=1}^M \lambda^{-\alpha} |G_{11}(n+m,s;\lambda) - G_{11}(n,s;\lambda)| h\,d\lambda \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{E_\alpha(C_h,A_h^x)}
$$

$$
+ \int_0^\infty \sum_{s=0}^{M-1} \lambda^{-\alpha} |G_{12}(n+m,s;\lambda) - G_{12}(n,s;\lambda)| h\,d\lambda \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{E_\alpha(C_h,A_h^x)},
$$

$$
|\psi_{n+m} - \psi_n| \leq \int_0^\infty \sum_{s=0}^{M-1} \lambda^{-\alpha} |G_{22}(n+m,s;\lambda) - G_{22}(n,s;\lambda)| h\,d\lambda \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{E_\alpha(C_h,A_h^x)}.
$$

Let

$$
P = (mh)^{-\alpha} \int_0^\infty \sum_{s=1}^M \lambda^{-\alpha} |G_{11}(n+m,s;\lambda) - G_{11}(n,s;\lambda)| h\,d\lambda
$$

$$
+ (mh)^{-\alpha} \int_0^\infty \sum_{s=0}^{M-1} \lambda^{-\alpha} |G_{12}(n+m,s;\lambda) - G_{12}(n,s;\lambda)| h\,d\lambda
$$

$$
+ (mh)^{-\alpha} \int_0^\infty \sum_{s=0}^{M-1} \lambda^{-\alpha} |G_{22}(n+m,s;\lambda) - G_{22}(n,s;\lambda)| h\,d\lambda.
$$

Then for any $n+m, n \in \{0, 1, \ldots, M\}$, we have

$$
(mh)^{-\alpha} |\varphi_{n+m} - \varphi_n| + (mh)^{-\alpha} |\psi_{n+m} - \psi_n| \leq P \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{E_\alpha(C[0,l]_h,A_h^x)}.
$$

Now let us estimate $P = P_1 + P_2 + P_3$, where

$$
P_1 = (mh)^{-\alpha} \int_0^\infty \sum_{s=1}^M \lambda^{-\alpha} |G_{11}(n+m,s;\lambda) - G_{11}(n,s;\lambda)| h\,d\lambda,
$$

$$
P_2 = (mh)^{-\alpha} \int_0^\infty \sum_{s=0}^{M-1} \lambda^{-\alpha} |G_{12}(n+m,s;\lambda) - G_{12}(n,s;\lambda)| h\,d\lambda,
$$

$$
P_3 = (mh)^{-\alpha} \int_0^\infty \sum_{s=0}^{M-1} \lambda^{-\alpha} |G_{22}(n+m,s;\lambda) - G_{22}(n,s;\lambda)| h\,d\lambda.
$$

Note that it suffices to consider the case when $0 \leq mh \leq \frac{1}{2}$. Applying the scheme of the paper [33] and using equations (8), (9), (10), (11), and the estimates of equations (14), (15), (16), and (17), we can establish the following estimate:

$$
P_r \leq \frac{M_6(a,\delta)}{\alpha(1-\alpha)} \tag{22}
$$

for $r = 1, 2, 3$. Applying the triangle inequality and the estimate of equation (22), we get

$$
P \leq \frac{M_7(a,\delta)}{\alpha(1-\alpha)}.
$$

Thus for any $n + m, n \in \{0, 1, \dots, M\}$ we have

$$|mh|^{-\alpha}|\varphi_{n+m} - \varphi_n| + |mh|^{-\alpha}|\psi_{n+m} - \psi_n| \leq \frac{M_8(a, \delta)}{\alpha(1 - \alpha)} \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{E_\alpha(C_h, A_h)}.$$

This means that the following inequality holds:

$$\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{\mathring{C}_h^\alpha} \leq \frac{M_8(a, \delta)}{\alpha(1 - \alpha)} \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix}^h \right\|_{E_\alpha(C_h, A_h)}.$$

Theorem 3.1 is proved. □

Since the A_h^x is a positive operator in the fractional spaces $E_\alpha(\mathbb{C}_h, A_h^x)$, from the result of Theorem 3.1 it follows that it is also a positive operator in the Hölder space $\mathring{\mathbb{C}}_h^\alpha$. Namely, we have the following.

Theorem 3.2 *The operator $(\lambda I + A_h^x)$ has a bounded inverse in $\mathring{\mathbb{C}}_h^\alpha$ uniformly with respect to h for any $\lambda \geq 0$ and the following estimate holds*:

$$\left\| (\lambda I + A_h^x)^{-1} \right\|_{\mathring{\mathbb{C}}_h^\alpha \to \mathring{\mathbb{C}}_h^\alpha} \leq \frac{M_8(a, \delta)}{\alpha(1 - \alpha)} \frac{M_1}{1 + \lambda}.$$

4 Applications

In this section we consider the application of results of Sections 2 and 3. For a positive operator A in E the following result was established in papers [36, 37].

Theorem 4.1 *Let A be a positive operator in E. Then it obeys the following estimate*:

$$\left\| \mathbf{R}_{q, q-1}^k(\tau A) \right\|_{E \to E} \leq M_1, \quad 1 \leq k \leq N, N\tau = T, \tag{23}$$

where M_1 does not depend on τ and k. Here $\mathbf{R}_{q, q-1}(z)$ is the Padé approximation of $\exp(-z)$ near $z = 0$.

For a numerical solution of the initial-boundary value problem (1) the following difference scheme is presented:

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} + a(x_n)\frac{u_n^k - u_{n-1}^k}{h} + \delta(u_n^k - v_n^k) = f_{1,n}^k, & f_{1,n}^k = f_1(t_k, x_n), \\ t_k = k\tau, x_n = nh, 1 \leq k \leq N, N\tau = T, 1 \leq n \leq M, Mh = l, \\ \frac{v_n^k - v_n^{k-1}}{\tau} - a(x_{n+1})\frac{v_{n+1}^k - v_n^k}{h} + \delta v_n^k = f_{2,n}^k, & f_{2,n}^k = f_2(t_k, x_n), \\ t_k = k\tau, x_n = nh, 1 \leq k \leq N, N\tau = T, 0 \leq n \leq M - 1, Mh = l, \\ u_0^k = \gamma u_M^k, \quad 0 \leq \gamma \leq 1, \quad \beta v_0^k = v_M^k, \quad 0 \leq \beta \leq 1, 0 \leq k \leq N, \\ u_n^0 = u_0(x_n), \quad v_n^0 = v_0(x_n), \quad x_n = nh, 0 \leq n \leq M, Mh = l. \end{cases} \tag{24}$$

We introduce the Banach space $\mathbb{C}([0, T]_\tau, E)$ of all continuous abstract mesh vector functions

$$u^\tau = \{u^k\}_{k=1}^N = \left\{ \begin{pmatrix} u_{1,n}^k \\ u_{2,n}^k \end{pmatrix}^h \right\}_{k=1}^N$$

defined on $[0, T]_\tau = \{t_k = k\tau, 1 \le k \le N, N\tau = T\}$ with values in E, equipped with the norm

$$\|u^\tau\|_{\mathbb{C}([0,T]_\tau, E)} = \max_{1 \le k \le N} \left\| \{u_{1,n}^k\}_{n=1}^M \right\|_E + \max_{1 \le k \le N} \left\| \{u_{2,n}^k\}_{n=1}^M \right\|_E.$$

Note that the problem (24) can be written in the form of the abstract Cauchy problem

$$\left\{ \left(\frac{\frac{u^k - u^{k-1}}{\tau}}{\frac{v^k - v^{k-1}}{\tau}} \right)^h \right\}_{k=1}^N + A_h^x \left\{ \begin{pmatrix} u^k \\ v^k \end{pmatrix}^h \right\}_{k=1}^N = \left\{ \begin{pmatrix} f_1^k \\ f_2^k \end{pmatrix}^h \right\}_{k=1}^N, \quad 1 \le k \le N, \tag{25}$$

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} u_n^0 \\ v_{n-1}^0 \end{pmatrix}_{n=1}^M$$

in a Banach space $E = \mathbb{C}_h$ with a positive operator A_h^x defined by (5). Here $\left\{ \begin{pmatrix} f_1^k \\ f_2^k \end{pmatrix}^h \right\}_{k=1}^N = \left\{ \begin{pmatrix} f_{1,n}^k \\ f_{2,n-1}^k \end{pmatrix}_{n=1}^M \right\}_{k=1}^N$ is the given abstract vector function defined on $[0, T]_\tau$ with values in E, $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} u_n^0 \\ v_{n-1}^0 \end{pmatrix}_{n=1}^M$ is the element of $D(A_h^x)$. It is well known that (see, for example [3]) the formula

$$\begin{pmatrix} u^k \\ v^k \end{pmatrix} = (I + \tau A_h^x)^{-k} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \sum_{j=1}^k (I + \tau A_h^x)^{-k+j-1} \begin{pmatrix} f_1^j \\ f_2^j \end{pmatrix} \tau \tag{26}$$

gives a solution of the problem (25) in $\mathbb{C}([0, T]_\tau, E)$.

Theorem 4.2 *For the solution of the problem* (25) *the stability inequality holds*:

$$\left\| \left\{ \begin{pmatrix} u^k \\ v^k \end{pmatrix} \right\}_{k=1}^N \right\|_{\mathbb{C}([0,T]_\tau, E)} \le M(a, \delta) \left[\left\| \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\|_E + \left\| \left\{ \begin{pmatrix} f_1^k \\ f_2^k \end{pmatrix} \right\}_{k=1}^N \right\|_{\mathbb{C}([0,T]_\tau, E)} \right].$$

The proof of Theorem 4.2 is based on the positivity of the operator A_h^x, equation (26) and the estimate of equation (23).

Applying the results of Theorems 2.1 and 4.2, we get the following theorem.

Theorem 4.3 *The solution of the problem* (24) *satisfies the following estimate*:

$$\max_{1 \le k \le N} \max_{1 \le n \le M} |u_n^k| + \max_{1 \le k \le N} \max_{0 \le n \le M-1} |v_n^k|$$

$$\le M(a, \delta) \left[\max_{1 \le n \le M} |u_n^0| + \max_{0 \le n \le M-1} |v_n^0| + \max_{1 \le k \le N} \max_{1 \le n \le M} |f_{1,n}^k| + \max_{1 \le k \le N} \max_{0 \le n \le M-1} |f_{2,n}^k| \right].$$

Applying results of Theorems 3.2, 4.1, and 4.2, we get the following theorem.

Theorem 4.4 *Assume that*

$$f_{1,0}^k = \gamma f_{1,M}^k, \quad 0 \le \gamma \le 1,$$

$$\beta f_{2,0}^k = f_{2,M}^k, \quad 0 \le \beta \le 1, 1 \le k \le N.$$

Then the solution of the problem (24) *satisfies the following estimate*:

$$
\max_{1 \le k \le N} \left(\max_{1 \le n \le M} |u_n^k| + \sup_{1 \le n < n+m \le N} \frac{|u_{n+m}^k - u_n^k|}{(m\tau)^\alpha} \right)
$$

$$
+ \max_{1 \le k \le N} \left(\max_{0 \le n \le M-1} |v_n^k| + \sup_{0 \le n < n+m \le M-1} \frac{|v_{n+m}^k - v_n^k|}{(m\tau)^\alpha} \right)
$$

$$
\le M(a, \delta, \alpha) \left[\max_{1 \le n \le M} |u_n^0| + \sup_{1 \le n < n+m \le N} \frac{|u_{n+m}^0 - u_n^0|}{(m\tau)^\alpha} \right.
$$

$$
+ \max_{0 \le n \le M-1} |v_n^0| + \sup_{0 \le n < n+m \le M-1} \frac{|v_{n+m}^0 - v_n^0|}{(m\tau)^\alpha}
$$

$$
+ \max_{1 \le k \le N} \left(\max_{1 \le n \le M} |f_{1,n}^k| + \sup_{1 \le n < n+m \le N} \frac{|f_{1,n+m}^k - f_{1,n}^k|}{(m\tau)^\alpha} \right)
$$

$$
\left. + \max_{1 \le k \le N} \left(\max_{0 \le n \le M-1} |f_{2,n}^k| + \sup_{0 \le n < n+m \le M-1} \frac{|f_{2,n+m}^k - f_{2,n}^k|}{(m\tau)^\alpha} \right) \right].
$$

Finally, one has not been able to obtain a sharp estimate for the constants figuring in the stability estimates. Therefore, our interest in the present paper is studying the difference scheme equation (24) by numerical experiments. Applying this difference scheme, the numerical method is proposed in the following section for the numerical solution of the hyperbolic system of equations with nonlocal boundary conditions. The method is illustrated by a numerical example.

5 Numerical results

For the numerical result, the initial value problem

$$
\begin{cases}
\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} - (u - v) + (1 + 2\pi)\cos(2\pi x + t), & 0 < t < 1, 0 < x < 1, \\
\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} - v + (1 - 2\pi)\cos(2\pi x + t) + \sin(2\pi x + t), & 0 < t < 1, 0 < x < 1, \\
u(t, 0) = u(t, 1), \qquad v(t, 0) = v(t, 1), & 0 \le t \le 1, \\
u(0, x) = v(0, x) = \sin 2\pi x, & 0 \le x \le 1
\end{cases}
\tag{27}
$$

for the hyperbolic system of equations with nonlocal boundary conditions is considered. Applying the difference scheme equation (24), we obtain

$$
\begin{cases}
\frac{u_n^k - u_n^{k-1}}{\tau} = -\frac{u_n^k - u_{n-1}^k}{h} - u_n^k + v_n^k + \varphi_n^k, \\
\quad 1 \le n \le M, 1 \le k \le N, Mh = 1, N\tau = 1, \\
\frac{v_n^k - v_n^{k-1}}{\tau} = \frac{v_{n+1}^k - v_n^k}{h} - v_n^k + \psi_n^k, \\
\quad 0 \le n \le M - 1, 1 \le k \le N, \\
u_0^k = u_M^k, \qquad v_0^k = v_M^k, \quad 0 \le k \le N, \\
u_n^0 = v_n^0 = \sin(2\pi nh), \quad 0 \le n \le M,
\end{cases}
\tag{28}
$$

where

$$
\begin{cases}
\varphi_n^k = (1 + 2\pi)\cos(2\pi nh + k\tau), \\
\psi_n^k = (1 - 2\pi)\cos(2\pi nh + k\tau) + \sin(2\pi nh + k\tau).
\end{cases}
$$

We get the system of equations in the matrix form

$$J_1 v^k = B v^{k-1} + \varphi^k, \quad 1 \le k \le N, v^0 = \psi,$$

$$J_2 u^k = B u^{k-1} + v^k + \psi^k, \quad 1 \le k \le N, u^0 = \varphi,$$

where

$$v^k = \begin{pmatrix} v_0^k \\ v_1^k \\ \vdots \\ v_M^k \end{pmatrix}, \qquad \psi = \begin{pmatrix} 0 \\ \sin 2\pi h \\ \sin 4\pi h \\ \vdots \\ \sin 2M\pi h \end{pmatrix},$$

$$u_k = \begin{pmatrix} u_0^k \\ u_1^k \\ \vdots \\ u_M^k \end{pmatrix}, \qquad \varphi = \begin{pmatrix} 0 \\ \sin 2\pi h \\ \sin 4\pi h \\ \vdots \\ \sin 2M\pi h \end{pmatrix},$$

$$J_1 = \begin{pmatrix} a & b & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & b & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & b & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & a & b \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}_{(N+1)\times(M+1)},$$

$$B = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{(M+1)\times(M+1)},$$

$$J_2 = \begin{pmatrix} a & f & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & f & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & f & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & a & f \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}_{(M+1)\times(M+1)},$$

$$a = \frac{1}{\tau} - \frac{1}{h} + 1, f = -\frac{1}{h}, b = \frac{1}{h}, c = \frac{1}{\tau},$$

$$\varphi^k = \begin{pmatrix} \varphi_0^k \\ \varphi_1^k \\ \vdots \\ \varphi_{M-1}^k \\ 0 \end{pmatrix}, \qquad \psi^k = \begin{pmatrix} \psi_0^k \\ \psi_1^k \\ \vdots \\ \psi_{M-1}^k \\ 0 \end{pmatrix}.$$

Table 1 Difference scheme

	$M = N = 50$	$M = N = 100$
Comparison of errors for u	0.0444	0.0226
Comparison of errors for v	0.0875	0.0436

Thus, we have the first-order difference equation with respect to k matrix coefficients. To solve this difference equation we have the following procedure:

$$\begin{cases} v^k = J_1^{-1} B v^{k-1} + J_1^{-1} \varphi^k, & 1 \le k \le N, v^0 = \psi, \\ u^k = J_2^{-1} B u^{k-1} + J_2^{-1} v^k + J_2^{-1} \psi^k, & 1 \le k \le N, u^0 = \varphi. \end{cases} \tag{29}$$

For their comparison, the errors are computed by

$$E_u = \max_{1 \le k \le N-1, 1 \le n \le M-1} \left| u(t_k, x_n) - u_n^k \right|,$$

$$E_v = \max_{1 \le k \le N-1, 1 \le n \le M-1} \left| v(t_k, x_n) - v_n^k \right|$$

of the numerical solutions. The numerical solutions are recorded for different values of $N = M$; u_n^k, v_n^k represent the numerical solutions of these difference schemes at (t_k, x_n). The errors are given in Table 1 for $N = M = 50$, and $N = M = 100$, respectively.

6 Conclusion

In the present study, the finite-difference method for the initial-boundary value problem for the hyperbolic system of equations with nonlocal boundary conditions is studied. The positivity of the difference analogy of the space operator generated by this problem in the space with maximum norm is established. The structure interpolation spaces generated by this difference operator is investigated. The positivity of this difference operator in Hölder spaces is established. In practice, stability estimates for the solution of the difference scheme for the hyperbolic system of equations with nonlocal boundary conditions are obtained. A numerical example is applied. Moreover, applying this approach we can construct the stable difference schemes for numerical solutions of the initial-boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + a(x) \frac{\partial u(t,x)}{\partial x} + \delta(u(t,x) - v(t,x)) = f_1(t,x;u(t,x),v(t,x)), \\ \quad 0 < x < l, 0 < t < T, \\ \frac{\partial v(t,x)}{\partial t} - a(x) \frac{\partial v(t,x)}{\partial x} + \delta v(t,x) = f_2(t,x;v(t,x)), \\ \quad 0 < x < l, 0 < t < T, \\ u(t,0) = \gamma u(t,l), \quad 0 \le \gamma \le 1, \\ \beta v(t,0) = v(t,l), \quad 0 \le \beta \le 1, 0 \le t \le T, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad 0 \le x \le l \end{cases} \tag{30}$$

for the hyperbolic system of semilinear equations with nonlocal boundary conditions. Of course, convergence estimates for the solution of these difference schemes can be obtained.

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