Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Oscillation criteria for third-order neutral dynamic equations with continuously distributed delay

Mehmet Tamer Şenel[1*] and Nadide Utku[2]

[*]Correspondence: senel@erciyes.edu.tr
[1]Department of Mathematics, Faculty of Sciences, Erciyes University, Kayseri, 38039, Turkey
Full list of author information is available at the end of the article

Abstract

It is the purpose of this paper to give oscillation criteria for the third-order neutral dynamic equations with continuously distributed delay,

$$\left[r(t)\left(\left[x(t)+\int_a^b p(t,\eta)x\left[\tau(t,\eta)\right]\Delta\eta\right]^{\Delta\Delta}\right)^{\gamma}\right]^{\Delta}+\int_c^d q(t,\xi)f\left(x\left[\phi(t,\xi)\right]\right)\Delta\xi=0,$$

on a time scale $\mathbb{T}$, where γ is the quotient of odd positive integers. By using a generalized Riccati transformation and an integral averaging technique, we establish some new sufficient conditions which ensure that every solution of this equation oscillates or converges to zero.

Keywords: oscillation; time scales; third-order neutral dynamic equation; asymptotic behavior

1 Introduction

We are concerned with the oscillatory behavior of third-order neutral dynamic equations with continuously distributed delay,

$$\left[r(t)\left(\left[x(t)+\int_a^b p(t,\eta)x\left[\tau(t,\eta)\right]\Delta\eta\right]^{\Delta\Delta}\right)^{\gamma}\right]^{\Delta}+\int_c^d q(t,\xi)f\left(x\left[\phi(t,\xi)\right]\right)\Delta\xi=0, \tag{1}$$

on an arbitrary time scale $\mathbb{T}$, where γ is a quotient of odd positive integers. Throughout this paper, we will assume the following hypotheses:

(H1) r and q are positive rd-continuous functions on $\mathbb{T}$ and

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t=\infty; \tag{2}$$

(H2) $p(t,\eta)\in C_{rd}([t_0,\infty)\times[a,b],\mathbb{R})$, $0\le p(t)\equiv\int_a^b p(t,\eta)\Delta\eta\le P<1$;

(H3) $\tau(t,\eta)\in C_{rd}([t_0,\infty)\times[a,b],\mathbb{T})$ is not a decreasing function for η and such that

$$\tau(t,\eta)\le t \quad\text{and}\quad \lim_{t\to\infty}\min_{\eta\in[a,b]}\tau(t,\eta)=\infty;$$

(H4) $\phi(t,\xi) \in C_{rd}([t_0,\infty) \times [c,d],\mathbb{T})$ is not decreasing function for ξ and such that

$$\phi(t,\xi) \le t \quad \text{and} \quad \lim_{t\to\infty} \min_{\xi\in[c,d]} \phi(t,\xi) = \infty;$$

(H5) the function $f \in C_{rd}(\mathbb{T},\mathbb{R})$ is assumed to satisfy $uf(u) > 0$ and there exists a positive rd-continuous function $\delta(t)$ on $\mathbb{T}$ such that $\frac{f(u)}{u^\gamma} \ge \delta$, for $u \ne 0$.

Define the function by

$$z(t) = x(t) + \int_a^b p(t,\eta)x\big[\tau(t,\eta)\big]\Delta\eta. \tag{3}$$

Furthermore, (1) is like the following:

$$\big[r(t)\big(\big[z(t)\big]^{\Delta\Delta}\big)^\gamma\big]^\Delta + \int_c^d q(t,\xi)f\big(x\big[\phi(t,\xi)\big]\big)\Delta\xi = 0. \tag{4}$$

A solution $x(t)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is non-oscillatory.

Much recent attention has been given to dynamic equations on time scales, or measure chains, and we refer the reader to the landmark paper of Hilger [1] for a comprehensive treatment of the subject. Since then, several authors have expounded various aspects of this new theory; see the survey paper by Agarwal *et al.* [2]. A book on the subject of time scales by Bohner and Peterson [3] also summarizes and organizes much of the time scale calculus. In the recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation and non-oscillation of solutions of various equations on time scales; we refer the reader to the papers [4–19]. Candan [20] considered oscillation of second-order neutral dynamic equations with distributed deviating arguments of the form

$$\big(r(t)\big(\big(y(t) + p(t)y\big(\tau(t)\big)\big)^\Delta\big)^\gamma\big)^\Delta + \int_c^d f\big(t, y\big(\theta(t,\xi)\big)\big)\Delta\xi = 0,$$

where $\gamma > 0$ is a ratio of odd positive integers with $r(t)$ and $p(t)$ real-valued rd-continuous positive functions defined on $\mathbb{T}$. He established some new oscillation criteria and gave sufficient conditions to ensure that all solutions of nonlinear neutral dynamic equation are oscillatory on a time scale $\mathbb{T}$.

To the best of our knowledge, there is very little known about the oscillatory behavior of third-order dynamic equations. Erbe *et al.* [21] are concerned with the oscillatory behavior of solutions of the third-order linear dynamic equation

$$x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0,$$

on an arbitrary time scale $\mathbb{T}$, where $p(t)$ is a positive real-valued rd-continuous function defined on $\mathbb{T}$. Li *et al.* [22] considered third-order nonlinear delay dynamic equation

$$x^{\Delta^3} + p(t)x^\gamma\big(\tau(t)\big) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is quotient of odd positive integers.

Erbe *et al.* [23, 24] established some sufficient conditions which guarantee that every solution of the third-order nonlinear dynamic equation

$$\big(c(t)\big(a(t)x^{\Delta}(t)\big)^{\Delta}\big)^{\Delta} + q(t)f\big(x(t)\big) = 0,$$

and the third-order dynamic equation

$$\big(c(t)\big(\big(a(t)x^{\Delta}(t)\big)^{\Delta}\big)^{\gamma}\big)^{\Delta} + f\big(t, x(t)\big) = 0$$

oscillate or converge to zero. Li *et al.* [25] considered the third-order delay dynamic equations

$$\big(a(t)\big(\big[r(t)x^{\Delta}(t)\big]^{\Delta}\big)^{\gamma}\big)^{\Delta} + f\big(t, x\big(\tau(t)\big)\big) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is quotient of odd positive integers, a and r are positive rd-continuous functions on $\mathbb{T}$, and the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \le t$, and $\tau(t) \to \infty$ as $t \to \infty$, $f(x) \in C_{rd}(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ is assumed to satisfy $uf(t,u) > 0$, for $u \neq 0$, and there exists a function p on $\mathbb{T}$ such that $\frac{f(t,u)}{u^{\gamma}} \ge p(t) > 0$, for $u \neq 0$.

Saker [26] considered the third-order nonlinear functional dynamic equations

$$\big(p(t)\big(\big[r(t)x^{\Delta}(t)\big]^{\Delta}\big)^{\gamma}\big)^{\Delta} + q(t)f\big(x\big(\tau(t)\big)\big) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is quotient of odd positive integers. Recently Han *et al.* [27] and Grace *et al.* [28] considered the third-order neutral delay dynamic equation

$$\big(r(t)\big(x(t) - a(t)x\big(\tau(t)\big)\big)^{\Delta\Delta}\big)^{\Delta} + p(t)x^{\gamma}\big(\delta(t)\big) = 0,$$

on a time scale $\mathbb{T}$.

In this paper, we consider third-order neutral dynamic equation with continuously distributed delay on time scales which is not in literature. We obtain some conclusions which contribute to oscillation theory of third-order neutral dynamic equations.

2 Several lemmas

Before stating our main results, we begin with the following lemmas which play an important role in the proof of the main results. Throughout this paper, we let

$$d_+(t) := \max\{0, d(t)\}, \qquad d_-(t) := \max\{0, -d(t)\},$$

and

$$\beta(t) := b(t), \quad 0 < \gamma \le 1, \qquad \beta(t) := b^{\gamma}(t), \quad \gamma > 1,$$

$$b(t) = \frac{t}{\sigma(t)}, \qquad R(t, t_*) := \int_{t_*}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s,$$

where we have sufficiently large $t_* \in [t_0, \infty)_{\mathbb{T}}$.

In order to prove our main results, we will use the formula

$$\big(z^\gamma(t)\big)^\Delta = \gamma \int_0^1 \big[hz^\sigma + (1-h)z\big]^{\gamma-1} z^\Delta(t)\,dh,$$

where $z(t)$ is delta differentiable and eventually positive or eventually negative, which is a simple consequence of Keller's chain rule (see Bohner and Peterson [3]).

Lemma 2.1 *Let $x(t)$ be a positive solution of* (1), *$z(t)$ is defined as in* (3). *Then $z(t)$ has only one of the following two properties*:

(I) $z(t) > 0$, $z^\Delta(t) > 0$, $z^{\Delta\Delta}(t) > 0$,

(II) $z(t) > 0$, $z^\Delta(t) < 0$, $z^{\Delta\Delta}(t) > 0$,

with $t \geq t_1$, t_1 sufficiently large.

Proof Let $x(t)$ be a positive solution of (1) on $[t_0,\infty)$, so that $z(t) > x(t) > 0$, and

$$\big[r(t)\big(z^{\Delta\Delta}(t)\big)^\gamma\big]^\Delta = -\int_c^d q(t,\xi)f\big(x\big[\phi(t,\xi)\big]\big)\Delta\xi < 0.$$

Then $r(t)([z(t)]^{\Delta\Delta})^\gamma$ is a decreasing function and therefore eventually of one sign, so $z^{\Delta\Delta}(t)$ is either eventually positive or eventually negative on $t \geq t_1 \geq t_0$. We assert that $z^{\Delta\Delta}(t) > 0$ on $t \geq t_1 \geq t_0$. Otherwise, assume that $z^{\Delta\Delta}(t) < 0$, then there exists a constant $M > 0$, such that

$$r(t)\big(z^{\Delta\Delta}(t)\big)^\gamma \leq -M < 0.$$

By integrating the last inequality from t_1 to t, we obtain

$$z^\Delta(t) \leq z^\Delta(t_1) - M^{\frac{1}{\gamma}} \int_{t_1}^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s.$$

Let $t \to \infty$. Then from (H1), we have $(z(t))^\Delta \to -\infty$, and therefore eventually $z^\Delta(t) < 0$.

Since $z^{\Delta\Delta}(t) < 0$ and $z^\Delta(t) < 0$, we have $z(t) < 0$, which contradicts our assumption $z(t) > 0$. Therefore, $z(t)$ has only one of the two properties (I) and (II).

This completes the proof. □

Lemma 2.2 *Let $x(t)$ be an eventually positive solution of* (1), *correspondingly $z(t)$ has the property* (II). *Assume that* (2) *and*

$$\int_{t_0}^\infty \int_v^\infty \left[\frac{1}{r(u)} \int_u^\infty q_1(s)\Delta s\right]^{\frac{1}{\gamma}} \Delta u \Delta v = \infty \tag{5}$$

hold. Then $\lim_{t\to\infty} x(t) = 0$.

Proof Let $x(t)$ be an eventually positive solution of (1). Since $z(t)$ has the property (II), then there exists finite $\lim_{t\to\infty} z(t) = I$. We assert that $I = 0$. Assume that $I > 0$, then we have

$I + \epsilon > z(t) > I$ for all $\epsilon > 0$. Choosing $\epsilon < \frac{I(1-P)}{P}$ and using (3) and (H2), we obtain

$$\begin{aligned} x(t) &= z(t) - \int_a^b p(t,\eta)\big[x\big(\tau(t,\eta)\big)\big]\Delta\eta \\ &> I - \int_a^b p(t,\eta)\big[x\big(\tau(t,\eta)\big)\big]\Delta\eta \\ &\geq I - p(t)\big[z\big(\tau(t,a)\big)\big] \\ &\geq I - P(I+\epsilon) > Kz(t), \end{aligned} \tag{6}$$

where $K = \frac{I-P(1+\epsilon)}{I+\epsilon} > 0$. Using (H5) and (6), we find from (1) that

$$\begin{aligned} \big[r(t)\big(z^{\Delta\Delta}(t)\big)^{\gamma}\big]^{\Delta} &= -\int_c^d q(t,\xi)f\big(x\big[\phi(t,\xi)\big]\big)\Delta\xi \\ &\leq -\int_c^d q(t,\xi)\big(x\big[\phi(t,\xi)\big]\big)^{\gamma}\delta\Delta\xi \\ &\leq -K^{\gamma}\delta\int_c^d q(t,\xi)\big(z\big[\phi(t,\xi)\big]\big)^{\gamma}\Delta\xi. \end{aligned}$$

Note that $z(t)$ has property (II) and (H4), and we have

$$\big[r(t)\big(z^{\Delta\Delta}(t)\big)^{\gamma}\big]^{\Delta} \leq -K^{\gamma}\cdot\delta\cdot\big(z\big[\phi(t,d)\big]\big)^{\gamma}\int_c^d q(t,\xi)\Delta\xi = -q_1(t)\big(z\big(\phi_1(t)\big)\big)^{\gamma}, \tag{7}$$

where $q_1(t) = K^{\gamma}\delta\int_c^d q(t,\xi)\Delta\xi$, $\phi_1(t) = \phi(t,d)$. Integrating inequality (7) from t to ∞, we obtain

$$r(t)\big(z^{\Delta\Delta}(t)\big)^{\gamma} \geq \int_t^{\infty} q_1(s)\big(z\big(\phi_1(s)\big)\big)^{\gamma}\Delta s.$$

Using $(z(\phi_1(s)))^{\gamma} \geq I^{\gamma}$, we obtain

$$z^{\Delta\Delta}(t) \geq \frac{I}{r^{\frac{1}{\gamma}}}\left[\int_t^{\infty} q_1(s)\right]^{\frac{1}{\gamma}}\Delta(s). \tag{8}$$

Integrating inequality (8) from t to ∞, we have

$$-z^{\Delta}(t) \geq I\int_t^{\infty}\left[\frac{1}{r(u)}\int_u^{\infty} q_1(s)\Delta(s)\right]^{\frac{1}{\gamma}}\Delta u.$$

Integrating the last inequality from t_1 to ∞, we obtain

$$z(t_1) \geq I\int_{t_1}^{\infty}\int_v^{\infty}\left[\frac{1}{r(u)}\int_u^{\infty} q_1(s)\Delta(s)\right]^{\frac{1}{\gamma}}\Delta u\Delta v.$$

Because (7) and the last inequality contradict (5), we have $I = 0$. Since $0 \leq x(t) \leq z(t)$, $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Lemma 2.3 *Assume that $x(t)$ is a positive solution of* (1), *$z(t)$ is defined as in* (3) *such that $z^{\Delta\Delta}(t) > 0$, $z^{\Delta}(t) > 0$, on $[t_*, \infty)_{\mathbb{T}}$, $t_* \geq 0$. Then*

$$z^{\Delta}(t) \geq R(t,t_*)r^{\frac{1}{\gamma}}(t)z^{\Delta\Delta}(t). \tag{9}$$

Proof Since $r(t)(z^{\Delta\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_*, \infty)_{\mathbb{T}}$, we get for $t \in [t_*, \infty)_{\mathbb{T}}$

$$\begin{aligned} z^{\Delta}(t) &> z^{\Delta}(t) - z^{\Delta}(t_*) \\ &= \int_{t_*}^{t} \frac{(r(s)(z^{\Delta\Delta}(t))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \\ &\geq \bigl(r(t)\bigl(z^{\Delta\Delta}(t)\bigr)^{\gamma}\bigr)^{\frac{1}{\gamma}} \int_{t_*}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s. \end{aligned}$$

Using the definition of $R(t,t_*)$, we obtain

$$z^{\Delta}(t) > R(t,t_*)r^{\frac{1}{\gamma}}(t)z^{\Delta\Delta}(t) \quad \text{on } [t_*, \infty)_{\mathbb{T}}. \qquad \square$$

Lemma 2.4 *Assume that $x(t)$ is a positive solution of* (1), *correspondingly $z(t)$ has the property* (I). *Such that $z^{\Delta}(t) > 0$, $z^{\Delta\Delta}(t) > 0$, on $[t_*, \infty)_{\mathbb{T}}$, $t_* \geq t_0$. Furthermore,*

$$\int_{t_2}^{t} q_2(s)\phi_2^{\gamma}(s)\Delta s = \infty. \tag{10}$$

Then there exists a $T \in [t_*, \infty)_{\mathbb{T}}$, sufficiently large, so that

$$z(t) > tz^{\Delta}(t),$$

$z(t)/t$ is strictly decreasing, $t \in [T, \infty)_{\mathbb{T}}$.

Proof Let $U(t) = z(t) - tz^{\Delta}(t)$. Hence $U^{\Delta}(t) = -\sigma(t)z^{\Delta\Delta}(t) < 0$. We claim there exists a $t_1 \in [t_*, \infty)_{\mathbb{T}}$ such that $U(t) > 0$, $z(\phi(t,\xi)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not. Then $U(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{z(t)}{t}\right)^{\Delta} = \frac{tz^{\Delta}(t) - z(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

which implies that $z(t)/t$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Pick $t_2 \in [t_1, \infty)_{\mathbb{T}}$ so that $\phi(t,\xi) \geq \phi(t_1,\xi)$, for $t \geq t_2$. Then

$$\frac{z(\phi(t,\xi))}{\phi(t,\xi)} \geq \frac{z(\phi(t_1,\xi))}{\phi(t_1,\xi)} = d > 0,$$

so that $z(\phi(t,\xi)) > d\phi(t,\xi)$, for $t \geq t_2$. By (1), (3), and (H2), we obtain

$$\begin{aligned} x(t) &= z(t) - \int_a^b p(t,\eta)x\bigl[\tau(t,\eta)\bigr]\Delta\eta \\ &\geq z(t) - \int_a^b p(t,\eta)z\bigl[\tau(t,\eta)\bigr]\Delta\eta \end{aligned}$$

$$\begin{aligned}
&\geq z(t) - z\big[\tau(t,b)\big] \int_a^b p(t,\eta)\Delta\eta \\
&\geq \left(1 - \int_a^b p(t,\eta)\Delta\eta\right) z(t) \\
&\geq (1-P)z(t).
\end{aligned} \tag{11}$$

Using (11), (H4), and (H5), we have

$$\begin{aligned}
\big[r(t)\big(\big[z(t)\big]^{\Delta\Delta}\big)^{\gamma}\big]^{\Delta} &= -\int_c^d q(t,\xi) f\big(x\big[\phi(t,\xi)\big]\big)\Delta\xi \\
&\leq -\delta(1-P)^{\gamma} \int_c^d q(t,\xi) z^{\gamma}\big(\phi(t,\xi)\big)\Delta\xi \\
&\leq -\delta(1-P)^{\gamma} z^{\gamma}\big(\phi(t,c)\big) \int_c^d q(t,\xi)\Delta\xi \\
&\leq -q_2(t) z^{\gamma}\big(\phi_2(t)\big),
\end{aligned} \tag{12}$$

where $q_2(t) = \delta(1-P)^{\gamma} \int_c^d q(t,\xi)\Delta\xi$, $\phi_2(t) = \phi(t,c)$.

Now by integrating both sides of last equation from t_2 to t, we have

$$r(t)\big(z^{\Delta\Delta}(t)\big)^{\gamma} - r(t_2)\big(z^{\Delta\Delta}(t_2)\big)^{\gamma} + \int_{t_2}^t q_2(s) z^{\gamma}\big(\phi_2(s)\big)\Delta s \leq 0.$$

This implies that

$$r(t_2)\big(z^{\Delta\Delta}(t_2)\big)^{\gamma} \geq \int_{t_2}^t q_2(s)\big(z\big(\phi_2(s)\big)\big)^{\gamma}\Delta s \geq d^{\gamma} \int_{t_2}^t q_2(s)\phi_2^{\gamma}(s)\Delta s,$$

which contradicts (10). So $U(t) > 0$ on $t \in [t_1,\infty)_{\mathbb{T}}$ and consequently,

$$\left(\frac{z(t)}{t}\right)^{\Delta} = \frac{t z^{\Delta}(t) - z(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \quad t \in [t_1,\infty)_{\mathbb{T}},$$

and we find that $z(t)/t$ is strictly decreasing on $t \in [t_1,\infty)_{\mathbb{T}}$. The proof is now complete. □

3 Main results

In this section we give some new oscillation criteria for (1).

Theorem 3.1 *Assume that* (2), (5), *and* (10) *hold. Furthermore, assume that there exists a positive function* $\rho \in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, *for all sufficiently large* $T_1 \in [t_0,\infty)_{\mathbb{T}}$, *there is a* $T > T_1$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left[\rho^{\sigma}(s) q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^{\gamma} - \frac{((\rho^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,t_*))^{\gamma}}\right]\Delta s = \infty. \tag{13}$$

Then every solution of (1) *is either oscillatory or tends to zero.*

Proof Assume (1) has a non-oscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t)>0$, $t\geq t_1$; $x(\tau(t,\eta))>0$, $(t,\eta)\in[t_1,\infty)\times[a,b]$ and $x(\phi(t,\xi))>0$, $(t,\xi)\in[t_1,\infty)\times[c,d]$ for all $t_1\in[t_0,\infty)_{\mathbb{T}}$. $z(t)$ is defined as in (3). We suppose that $z(t)>0$. We shall consider only this case, since the proof when $z(t)$ is eventually negative is similar. Therefore Lemma 2.1 and Lemma 2.2, we get

$$\big[r(t)\big(\big[z(t)\big]^{\Delta\Delta}\big)^{\gamma}\big]^{\Delta}<0,\quad z^{\Delta\Delta}(t)>0, t\in[t_1,\infty)_{\mathbb{T}},$$

and either $z^{\Delta}(t)>0$ for $t\geq t_2\geq t_1$ or $\lim_{t\to\infty}x(t)=0$. Let $z^{\Delta}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$.

By (11) and (12), we have

$$\big[r(t)\big(\big[z(t)\big]^{\Delta\Delta}\big)^{\gamma}\big]^{\Delta}\leq -q_2(t)z^{\gamma}\big(\phi_2(t)\big),$$

where $q_2(t)=\delta(1-P)^{\gamma}\int_c^d q(t,\xi)\Delta\xi$, $\phi_2(t)=\phi(t,c)$.

Define the function $w(t)$ by the Riccati substitution

$$w(t)=\rho(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)}. \tag{14}$$

Then

$$\begin{aligned}
w^{\Delta}(t)&=\rho^{\Delta}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)}+\rho^{\sigma}(t)\left[\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)}\right]^{\Delta}\\
&=\rho^{\Delta}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)}+\rho^{\sigma}(t)\frac{[r(t)([z(t)]^{\Delta\Delta})^{\gamma}]^{\Delta}}{z^{\gamma\sigma}(t)}\\
&\quad-\rho^{\sigma}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma\sigma}(t)}.
\end{aligned}$$

From (1), the definition of $w(t)$ and using the fact $z(t)/t$ is strictly decreasing for $t\in[t_3,\infty)_{\mathbb{T}}$, $t_3\geq t_2$, it follows that

$$\begin{aligned}
w^{\Delta}(t)&\leq\frac{\rho^{\Delta}(t)}{\rho(t)}w(t)-\rho^{\sigma}(t)q_2(t)\frac{z^{\gamma}(\phi_2(t))}{z^{\gamma\sigma}(t)}-\rho^{\sigma}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma\sigma}(t)},\\
w^{\Delta}(t)&\leq\frac{\rho^{\Delta}(t)}{\rho(t)}w(t)-\rho^{\sigma}(t)q_2(t)\left(\frac{\phi_2(t)}{\sigma(t)}\right)^{\gamma}-\rho^{\sigma}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma\sigma}(t)}.
\end{aligned} \tag{15}$$

Now we consider the following two cases: $0<\gamma\leq 1$ and $\gamma>1$. In the first case $0<\gamma\leq 1$. Using the Keller chain rule (see [3]), we have

$$\big(z^{\gamma}(t)\big)^{\Delta}=\gamma\int_0^1\big[hz^{\sigma}+(1-h)z\big]^{\gamma-1}z^{\Delta}(t)\,dh\geq\gamma\big(z^{\sigma}(t)\big)^{\gamma-1}z^{\Delta}(t), \tag{16}$$

in view of (16), Lemma 2.2, Lemma 2.3, and (9), we have

$$\begin{aligned}
w^{\Delta}(t)&\leq-\rho^{\sigma}(t)q_2(t)\left(\frac{\phi_2(t)}{\sigma(t)}\right)^{\gamma}+\frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t)-\gamma\rho^{\sigma}(t)\frac{r(t)(z^{\Delta\Delta}(t))^{\gamma}z^{\Delta}(t)z(t)}{z^{\gamma+1}(t)z^{\sigma}(t)}\\
&\leq-\rho^{\sigma}(t)q_2(t)\left(\frac{\phi_2(t)}{\sigma(t)}\right)^{\gamma}+\frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t)
\end{aligned}$$

$$\begin{aligned}&-\gamma\rho^{\sigma}(t)R(t,t_*)\frac{r^{\frac{\gamma+1}{\gamma}}(t)(z^{\Delta\Delta}(t))^{\gamma+1}z(t)}{z^{\gamma+1}(t)z(\sigma(t))}\\&\le-\rho^{\sigma}(t)q_2(t)\left(\frac{\phi_2(t)}{\sigma(t)}\right)^{\gamma}+\frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t)-\gamma\rho^{\sigma}(t)R(t,t_*)\frac{t}{\sigma(t)}\frac{w^{\frac{\gamma+1}{\gamma}}(t)}{\rho^{\frac{\gamma+1}{\gamma}}(t)}.\end{aligned}\tag{17}$$

In the second case $\gamma > 1$. Applying the Keller chain rule, we have

$$\left(z^{\gamma}(t)\right)^{\Delta}=\gamma\int_0^1\left[hz^{\sigma}+(1-h)z\right]^{\gamma-1}z^{\Delta}(t)\,dh\ge\gamma\left(z(t)\right)^{\gamma-1}z^{\Delta}(t),\tag{18}$$

in the view of (18), Lemma 2.2, Lemma 2.3, and (9), we have

$$\begin{aligned}w^{\Delta}(t)&\le-\rho^{\sigma}(t)q_2(t)\left(\frac{\phi_2(t)}{\sigma(t)}\right)^{\gamma}+\frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t)\\&\quad-\gamma\rho^{\sigma}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}z^{\Delta}(t)z^{\gamma}(t)}{z^{\gamma+1}(t)z^{\gamma\sigma}(t)},\\w^{\Delta}(t)&\le-\rho^{\sigma}(t)q_2(t)\left(\frac{\phi_2(t)}{\sigma(t)}\right)^{\gamma}+\frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t)\\&\quad-\gamma\rho^{\sigma}(t)\left(\frac{t}{\sigma(t)}\right)^{\gamma}R(t,t_*)\frac{w^{\frac{\gamma+1}{\gamma}}(t)}{\rho^{\frac{\gamma+1}{\gamma}}(t)}.\end{aligned}\tag{19}$$

By (17), (19), and the definition of $b(t)$ and $\beta(t)$, we have, for $\gamma > 0$,

$$w^{\Delta}(t)\le-\rho^{\sigma}(t)q_2(t)\left(\frac{\phi_2(t)}{\sigma(t)}\right)^{\gamma}+\frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t)-\gamma\rho^{\sigma}(t)\beta(t)R(t,t_*)\frac{w^{\lambda}(t)}{\rho^{\lambda}(t)},\tag{20}$$

where $\lambda := \frac{\gamma+1}{\gamma}$. Define $A \ge 0$ and $B \ge 0$ by

$$\begin{aligned}A^{\lambda}&:=\gamma\rho^{\sigma}(t)\beta(t)R(t,t_*)\frac{w^{\lambda}(t)}{\rho^{\lambda}(t)},\\B^{\lambda-1}&:=\frac{\rho^{\Delta}(t)}{\lambda(\gamma\rho^{\sigma}(t)\beta(t)R(t,t_*))^{\frac{1}{\lambda}}}.\end{aligned}$$

Then using the inequality [15]

$$\lambda AB^{\lambda-1}-A^{\lambda}\le(\lambda-1)B^{\lambda},\tag{21}$$

which yields

$$\frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t)-\gamma\rho^{\sigma}(t)\beta(t)R(t,t_*)\frac{w^{\lambda}(t)}{\rho^{\lambda}(t)}\le\frac{((\rho^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(t)\rho^{\sigma}(t)R(t,t_*))^{\gamma}}.$$

From this last inequality and (20), we find

$$w^{\Delta}(t)\le-\rho^{\sigma}(t)q_2(t)\left(\frac{\phi_2(t)}{\sigma(t)}\right)^{\gamma}+\frac{((\rho^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(t)\rho^{\sigma}(t)R(t,t_*))^{\gamma}}.$$

Integrating both sides from T to t, we get

$$\int_T^t \left[\rho^\sigma(s) q_2(s) \left(\frac{\phi_2(s)}{\sigma(s)} \right)^\gamma - \frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,t_*))^\gamma} \right] \Delta s \le w(T) - w(t) \le w(T),$$

which contradicts assumption (13). This completes the proof of Theorem 3.1. □

Remark 3.1 From Theorem 3.1, we can obtain different conditions for oscillation of (1) with different choices of $\rho(t)$.

Remark 3.2 The conclusion of Theorem 3.1 remains intact if assumption (13) is replaced by the two conditions

$$\limsup_{t\to\infty} \int_T^t \rho^\sigma(s) q_2(s) \left(\frac{\phi_2(s)}{\sigma(s)} \right)^\gamma \Delta s = \infty,$$
$$\limsup_{t\to\infty} \int_T^t \frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)\psi(s,t_*))^\gamma} \Delta s < \infty.$$

For example, let $\rho(t) = t$. Now Theorem 3.1 yields the following results.

Corollary 3.1 *Assume that* (H1)-(H5), (5), *and* (10) *hold. If*

$$\limsup_{t\to\infty} \int_T^t \left[\sigma(s) q_2(s) \left(\frac{\phi_2(s)}{\sigma(s)} \right)^\gamma - \frac{1}{(\gamma+1)^{\gamma+1}(\beta(s)\sigma(s)R(s,t_*))^\gamma} \right] \Delta s = \infty \tag{22}$$

holds, then every solution (1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

For example, let $\rho(t) = 1$. Now Theorem 3.1 yields the following results.

Corollary 3.2 *Assume that* (H1)-(H5), (5), *and* (10) *hold. If*

$$\limsup_{t\to\infty} \int_T^t q_2(s) \left(\frac{\phi_2(s)}{\sigma(s)} \right)^\gamma \Delta s = \infty, \tag{23}$$

then every solution (1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Theorem 3.2 *Assume that* (2), (5), *and* (10) *hold. Furthermore, suppose that there exist functions* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv (t,s) : t \ge s \ge t_0$ *such that*

$$H(t,t) = 0, \quad t \ge 0,$$
$$H(t,s) > 0, \quad t > s \ge t_0,$$

and H *has a nonpositive continuous* Δ*-partial derivative* $H^{\Delta s}(t,s)$ *with respect to the second variable and satisfies*

$$H^{\Delta s}(\sigma(t),s) + H(\sigma(t),\sigma(s)) \frac{\rho^\Delta(s)}{\rho(s)} = -\frac{h(t,s)}{\rho(s)} H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}, \tag{24}$$

and for all sufficiently large $T_1 \in [t_0,\infty)_{\mathbb{T}}$, there is a $T > T_1$ such that

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),T)} \int_T^{\sigma(t)} K(t,s) = \infty, \tag{25}$$

where ρ is a positive Δ-differentiable function and

$$K(t,s) = H\big(\sigma(t),\sigma(s)\big)\rho^\sigma(s)q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^\gamma - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\Delta s = \infty.$$

Then every solution of (1) *is either oscillatory or tends to zero.*

Proof Suppose that $x(t)$ is a non-oscillatory solution of (1) and $z(t)$ is defined as in (3). Without loss of generality, we may assume that there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ sufficiently large so that the conclusions of Lemma 2.1 hold and (24) holds for $t_2 > t_1$. If case (1) of Lemma 2.1 holds then proceeding as in the proof of Theorem 3.1, we see that (20) holds for $t > t_2$. Multiplying both sides of (20) by $H(\sigma(t),\sigma(s))$ and integrating from T to $\sigma(t)$, we get

$$\begin{aligned}
&\int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\rho^\sigma(s)q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^\gamma \Delta s \\
&\quad\le -\int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)w^\Delta(s)\Delta s + \int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\frac{\rho^\Delta(s)}{\rho(s)}w(s)\Delta s \\
&\quad\quad - \int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s \quad \left(\lambda = \frac{\gamma+1}{\gamma}\right).
\end{aligned} \tag{26}$$

Integrating by parts and using $H(t,t) = 0$, we obtain

$$\int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)w^\Delta(s)\Delta s = -H\big(\sigma(t),T\big)w(T) - \int_T^{\sigma(t)} H^{\Delta s}\big(\sigma(t),s\big)w(s)\Delta s.$$

It then follows from (26) that

$$\begin{aligned}
&\int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\rho^\sigma(s)q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^\gamma \Delta s \\
&\quad\le H\big(\sigma(t),T\big)w(T) + \int_T^{\sigma(t)} H^{\Delta s}\big(\sigma(t),s\big)w(s)\Delta s \\
&\quad\quad + \int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\frac{\rho^\Delta(s)}{\rho(s)}w(s)\Delta s \\
&\quad\quad - \int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s, \\
&\int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\rho^\sigma(s)q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^\gamma \Delta s \\
&\quad\le H\big(\sigma(t),T\big)w(T) \\
&\quad\quad + \left[\int_T^{\sigma(t)} H^{\Delta s}\big(\sigma(t),s\big) + H\big(\sigma(t),\sigma(s)\big)\frac{\rho^\Delta(s)}{\rho(s)}\right]w(s)\Delta s \\
&\quad\quad - \int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s.
\end{aligned}$$

It then follows from (24) that

$$\begin{aligned}
&\int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\rho^\sigma(s)q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^\gamma \Delta s \\
&\quad\le H\big(\sigma(t),T\big)w(T) \\
&\qquad + \int_T^{\sigma(t)} \left[-\frac{h(t,s)}{\rho(s)} H\big(\sigma(t),\sigma(s)\big)^{\frac{\gamma}{\gamma+1}}\right] w(s)\Delta s \\
&\qquad - \int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s \\
&\quad\le H\big(\sigma(t),T\big)w(T) + \int_T^{\sigma(t)} \left[\frac{h(t,s)}{\rho(s)} H\big(\sigma(t),\sigma(s)\big)^{\frac{\gamma}{\gamma+1}}\right] w(s)\Delta s \\
&\qquad - \int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s.
\end{aligned}$$

Therefore, as in Theorem 3.1, by letting

$$\begin{aligned}
&A^\lambda := H\big(\sigma(t),\sigma(s)\big)\gamma\rho^\sigma(t)\beta(t)R(t,T_1)\frac{w^\lambda(t)}{\rho^\lambda(t)}, \\
&B^{\lambda-1} := \frac{h_-(t,s)}{\lambda(\gamma\rho^\sigma(t)\beta(t)R(t,T_1))^{\frac{1}{\lambda}}}.
\end{aligned}$$

Then using the inequality [15]

$$\lambda A B^{\lambda-1} - A^\lambda \le (\lambda-1)B^\lambda.$$

We have

$$\begin{aligned}
&\int_T^{\sigma(t)} \left[\frac{h_-(t,s)}{\rho(s)} H\big(\sigma(t),\sigma(s)\big)^{\frac{\gamma}{\gamma+1}}\right] w(s)\Delta s \\
&\qquad - \int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s \\
&\quad = \int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(t,T_1))^\gamma}\Delta s, \\
&\int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\rho^\sigma(s)q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^\gamma \Delta s \\
&\quad\le H\big(\sigma(t),T\big)w(T) + \int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(t,T_1))^\gamma}\Delta s.
\end{aligned}$$

Then for $T > T_1$ we have

$$\begin{aligned}
&\int_T^{\sigma(t)} \left[H\big(\sigma(t),\sigma(s)\big)\rho^\sigma(s)q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^\gamma - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\right]\Delta s \\
&\quad\le H\big(\sigma(t),T\big)w(T),
\end{aligned}$$

and this implies that

$$\frac{1}{H(\sigma(t),T)}\int_T^{\sigma(t)}\Bigg[H\big(\sigma(t),\sigma(s)\big)\rho^\sigma(s)q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^\gamma - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\Bigg]\Delta s < w(T),$$

for all large T, which contradicts (25). This completes the proof of Theorem 3.2. □

Remark 3.3 The conclusion of Theorem 3.2 remains intact if assumption (25) is replaced by the two conditions

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),T)}\int_T^{\sigma(t)} H\big(\sigma(t),\sigma(s)\big)\rho^\sigma(s)q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^\gamma\Delta s = \infty,$$
$$\liminf_{t\to\infty}\frac{1}{H(\sigma(t),T)}\int_T^{\sigma(t)}\frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\Delta s < \infty.$$

Remark 3.4 Define w as (14), we also get

$$w^\Delta(t) = r^\sigma(t)\big(z^{\Delta\Delta}(t)\big)^{\gamma\sigma}\left[\frac{\rho(t)}{z^\gamma(t)}\right]^\Delta + \frac{\rho(t)}{z^\gamma}\big[r(t)\big(z^{\Delta\Delta}(t)\big)^\gamma\big]^\Delta,$$

similar to the proofs of Theorem 3.1, we can obtain different results. We leave the details to the reader.

Example 3.1 Consider the following third-order neutral dynamic equation $t \in [t_0,\infty)_{\mathbb{T}}$:

$$\left(x(t) + \int_a^b e^{-t}x(t-\eta)\Delta\eta\right)^{\Delta\Delta\Delta} + \int_c^d \frac{\beta\cdot t}{(t^2-t\xi)(t^2-t\xi)^\sigma}x(t-\xi)\Delta\xi = 0, \tag{27}$$

where $\gamma = 1$, $r(t) = 1$, $\tau(t,\eta) = t-\eta$, $\phi(t,\xi) = t-\xi$, $\delta = 1$, $q_2(t) = \frac{\beta}{t\phi_2(t)}$, $p(t,\eta) = e^{-t}$, $q(t,\xi) = \beta\cdot t/(t^2-t\xi)(t^2-t\xi)^\sigma$.

It is clear that condition (2), (5), and (10) hold. Therefore, by Theorem 3.1, picking $\rho(t) = t$, we have

$$\limsup_{t\to\infty}\int_T^t\left[\rho^\sigma(s)q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^\gamma - \frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,t_*))^\gamma}\right]\Delta s$$
$$= \limsup_{t\to\infty}\int_T^t\left[\frac{\beta}{s} - \frac{1}{(\gamma+1)^{(\gamma+1)}s(s-t_*)}\right]\Delta s = \infty.$$

Hence, by Theorem 3.1 every solution of (27) is oscillatory or tends to zero if $\beta > 0$.

Example 3.2 Consider the following third-order neutral dynamic equation $t \in [t_0,\infty)_{\mathbb{T}}$:

$$\left[\frac{1}{t}\left(\left[x(t) + \int_a^b \frac{1}{2}x\left[\tau\left(\frac{t}{2}\right)\right]\Delta\eta\right]^{\Delta\Delta}\right)^3\right]^\Delta + \int_c^d q(t,\xi)f\left(x\left[\phi\left(\frac{t}{2}\right)\right]\right)\Delta\xi = 0, \tag{28}$$

where $\gamma = 3$, $r(t) = \frac{1}{t}$, $\tau(t,\eta) = \frac{t}{2}$, $\phi(t,\xi) = \frac{t}{2}$, $\delta = 1$, $q_2(t) = \frac{\beta}{t}\frac{\sigma^3(s)}{\phi_2^3(t)}$, $p(t,\eta) = \frac{1}{2}$.

It is clear that condition (2), (5), and (10) hold. Therefore, by Theorem 3.1, picking $\rho(t) = 1$, we have

$$\limsup_{t\to\infty} \int_T^t q_2(s)\left(\frac{\phi_2(s)}{\sigma(s)}\right)^3 \Delta s = \limsup_{t\to\infty} \int_T^t \frac{\beta}{s} \Delta s = \infty.$$

Hence, by Theorem 3.1 every solution of (28) is oscillatory or tends to zero if $\beta > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Sciences, Erciyes University, Kayseri, 38039, Turkey. [2]Institute of Sciences, Erciyes University, Kayseri, 38039, Turkey.

Received: 17 January 2014 Accepted: 19 July 2014 Published: 05 Aug 2014

References

1. Hilger, S: Analysis on measure chains a unified approach to continuous and discrete calculus. Results Math. **18**, 18-56 (1990)
2. Agarwal, RP, Bohner, M, O'Regan, D, Peterson, A: Dynamic equations on time scales: a survey. J. Comput. Appl. Math. **141**, 1-26 (2002)
3. Bohner, M, Peterson, A: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
4. Agarwal, RP, O' Regan, D, Saker, SH: Oscillation criteria for second-order nonlinear neutral delay dynamic equations. J. Math. Anal. Appl. **300**(1), 203-217 (2004)
5. Şahiner, Y: Oscillation of second-order neutral delay and mixed-type dynamic equations on time scales. Adv. Differ. Equ. **2006**, Article ID 65626 (2006)
6. Wu, H-W, Zhuang, R-K, Mathsen, RM: Oscillation criteria for second-order nonlinear neutral variable delay dynamic equations. Appl. Math. Comput. **178**(2), 321-331 (2006)
7. Saker, SH: Oscillation of second-order nonlinear neutral delay dynamic equations on time scales. J. Comput. Appl. Math. **187**(2), 123-141 (2006)
8. Zhang, SY, Wang, QR: Oscillation of second-order nonlinear neutral dynamic equations on time scales. Appl. Math. Comput. **216**(10), 2837-2848 (2010)
9. Şenel, MT: Kamenev-type oscillation criteria for the second-order nonlinear dynamic equations with damping on time scales. Abstr. Appl. Anal. **2012**, Article ID 253107 (2012)
10. Şenel, MT: Oscillation theorems for dynamic equation on time scales. Bull. Math. Anal. Appl. **3**, 101-105 (2011)
11. Li, T, Agarwal, RP, Bohner, M: Some oscillation results for second-order neutral dynamic equations. Hacet. J. Math. Stat. **41**, 715-721 (2012)
12. Erbe, L, Hassan, TS, Peterson, A: Oscillation of third order nonlinear functional dynamic equations on time scales. Differ. Equ. Dyn. Syst. **18**, 199-227 (2010)
13. Şenel, MT: Behavior of solutions of a third-order dynamic equation on time scales. J. Inequal. Appl. **2013**, Article ID 47 (2013)
14. Zhang, Q, Gao, L, Yu, Y: Oscillation criteria for third order neutral differential equations with continuously distributed delay. Appl. Math. Lett. **10**, 10-16 (2012)
15. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge (1988)
16. Graef, J, Saker, SH: Oscillation of third-order nonlinear neutral functional dynamic equations. Dyn. Syst. Appl. **21**, 583-606 (2012)
17. Saker, SH: On oscillation of a certain class of third-order nonlinear functional dynamic equations on time scales. Bull. Math. Soc. Sci. Math. Roum. **54**, 365-389 (2011)
18. Saker, SH: Oscillation Theory of Dynamic Equations on Time Scales. Lambert Academic Publishing, Colne (2010)
19. Agarwal, RP, O'Regan, D, Saker, SH: Philos-type oscillation criteria of second-order half-linear dynamic equations on time scales. Rocky Mt. J. Math. **37**, 1085-1104 (2007)
20. Candan, T: Oscillation of second order nonlinear neutral dynamic equations on time scales with distributed deviating arguments. Comput. Math. Appl. **62**, 4118-4125 (2011)
21. Erbe, L, Peterson, A, Saker, SH: Hille and Nehari type criteria for third-order dynamic equations. J. Math. Anal. Appl. **329**, 112-131 (2007)
22. Li, T, Han, Z, Zhang, C, Sun, Y: Oscillation criteria for third-order nonlinear delay dynamic equations on time scales. Bull. Math. Anal. Appl. **3**, 52-60 (2011)
23. Erbe, L, Peterson, A, Saker, SH: Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales. J. Comput. Appl. Math. **181**, 92-102 (2005)
24. Erbe, L, Peterson, A, Saker, SH: Oscillation and asymptotic behavior of a third-order nonlinear dynamic equation. Can. Appl. Math. Q. **14**(2), 124-147 (2006)

25. Li, T, Han, Z, Sun, S, Zhao, Y: Oscillation results for third order nonlinear delay dynamic equations on time scales. Bull. Malays. Math. Soc. **34**, 639-648 (2011)
26. Saker, SH: Oscillation of third-order functional dynamic equations on time scales. Sci. China Math. **54**, 2597-2614 (2011)
27. Han, Z, Li, T, Sun, S, Zhang, C: Oscillation behavior of third order neutral Emden-Fowler delay dynamic equations on time scales. Adv. Differ. Equ. **2010**, Article ID 586312 (2010)
28. Grace, SR, Graef, JR, El-Beltagy, MA: On the oscillation of third order delay dynamic equations on time scales. Appl. Math. Comput. **63**, 775-782 (2012)

10.1186/1687-1847-2014-220
Cite this article as: Şenel and Utku: Oscillation criteria for third-order neutral dynamic equations with continuously distributed delay. *Advances in Difference Equations* 2014, 2014:220