Advances in Difference Equations
a SpringerOpen Journal

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Multi-strip fractional q-integral boundary value problems for nonlinear fractional q-difference equations

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Abstract

In this article, we study the existence and uniqueness of solutions for multi-strip fractional q-integral boundary value problems of nonlinear fractional q-difference equations. By using the Banach contraction principle, Krasnoselskii's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder degree theory some interesting results are obtained. Some examples are presented to illustrate the results.

MSC: 34A08; 34B18; 39A13

Keywords: fractional q-calculus; boundary value problems; existence; uniqueness

1 Introduction

In this article, we investigate the following nonlinear fractional q-difference equation for multi-strip fractional q-integral boundary condition:

$$\begin{cases} D_q^\alpha u(t) = f(t,u(t)), & t \in (0,T), \\ u(0) = 0, & u(T) = \sum_{i=1}^m \gamma_i (I_{q_i}^{\beta_i} u)|_{\eta_i}^{\xi_i} = \sum_{i=1}^m \gamma_i (I_{q_i}^{\beta_i} u(\xi_i) - I_{q_i}^{\beta_i} u(\eta_i)), \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$, $0 < q, q_i < 1$, $\beta_i > 0$, $0 \leq \eta_i < \xi_i \leq T$, $\gamma_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, m$ are given constants, D_q^α is the fractional q-derivative of Riemann-Liouville type of order α, $I_{q_i}^{\beta_i}$ is the fractional q_i-integral of order β_i and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

q-Difference calculus or *quantum calculus* was initiated by Jackson [1]. Basic definitions and properties of quantum calculus can be found in the book [2]. The fractional q-difference calculus had its origin in the works by Al-Salam [3] and Agarwal [4]. For some recent work on the subject, we refer to [5–12] and the references cited therein.

Strip conditions appear in the mathematical modeling of certain real world problems. For motivation, discussion on multi-strip boundary conditions, examples and a consistent bibliography on these problems, we refer to the papers [13–20] and the references therein. As it is pointed out in [20], the boundary condition in (1.1) can be interpreted in the sense that a controller at the right-end of the considered interval is influenced by a discrete distribution of finite many nonintersecting strips of arbitrary length expressed in terms of fractional integral boundary conditions.

The significance of investigating problem (1.1) is that the multi-strip fractional q-integral boundary condition is very general and includes many conditions as special cases. In particular, if $\beta_i = 1$ for $i = 1, 2, \dots, m$, then the condition of (1.1) is reduced to the multi-strip q-integral condition as follows:

$$u(0) = 0, \qquad u(T) = \gamma_1 \int_{\eta_1}^{\xi_1} u(s)\, d_{q_1}s + \gamma_2 \int_{\eta_2}^{\xi_2} u(s)\, d_{q_2}s + \cdots + \gamma_m \int_{\eta_m}^{\xi_m} u(s)\, d_{q_m}s.$$

Moreover, we emphasize that we have different quantum numbers and as far as we know this is new in the literature.

The rest of the paper is organized as follows. In Section 2 we briefly give some basic notations, definitions and lemmas. In Section 3 we collect some auxiliary results needed in the proofs of our main results. Section 4 contains the main results concerning existence and uniqueness results for problem (1.1), which are shown by applying the Banach contraction principle, Krasnoselskii's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder degree theory. Some examples are presented in Section 5 to illustrate the results.

2 Preliminaries

To make this paper self-contained, below we recall some known facts on fractional q-calculus. The presentation here can be found in, for example, [21, 22].

For $q \in (0, 1)$, define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}. \tag{2.1}$$

The q-analogue of the power function $(a - b)^k$ with $k \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ is

$$(a - b)^{(0)} = 1, \qquad (a - b)^{(k)} = \prod_{i=0}^{k-1} \left(a - bq^i\right), \quad k \in \mathbb{N}, a, b \in \mathbb{R}. \tag{2.2}$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(a - b)^{(\gamma)} = a^\gamma \prod_{i=0}^{\infty} \frac{1 - (b/a)q^i}{1 - (b/a)q^{\gamma+i}}, \quad a \neq 0. \tag{2.3}$$

Note if $b = 0$, then $a^{(\gamma)} = a^\gamma$. We also use the notation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}. \tag{2.4}$$

Obviously, $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function h is defined by

$$(D_q h)(x) = \frac{h(x) - h(qx)}{(1 - q)x} \quad \text{for } x \neq 0 \quad \text{and} \quad (D_q h)(0) = \lim_{x \to 0} (D_q h)(x), \tag{2.5}$$

and q-derivatives of higher order are given by

$$\bigl(D_q^0 h\bigr)(x) = h(x) \quad \text{and} \quad \bigl(D_q^k h\bigr)(x) = D_q\bigl(D_q^{k-1} h\bigr)(x), \quad k \in \mathbb{N}. \tag{2.6}$$

The q-integral of a function h defined on the interval $[0, b]$ is given by

$$(I_q h)(x) = \int_0^x h(s)\, d_q s = x(1-q) \sum_{i=0}^{\infty} h\bigl(xq^i\bigr) q^i, \quad x \in [0, b]. \tag{2.7}$$

If $a \in [0, b]$ and h is defined in the interval $[0, b]$, then its integral from a to b is defined by

$$\int_a^b h(s)\, d_q s = \int_0^b h(s)\, d_q s - \int_0^a h(s)\, d_q s. \tag{2.8}$$

Similar to derivatives, an operator I_q^k is given by

$$\bigl(I_q^0 h\bigr)(x) = h(x) \quad \text{and} \quad \bigl(I_q^k h\bigr)(x) = I_q\bigl(I_q^{k-1} h\bigr)(x), \quad k \in \mathbb{N}. \tag{2.9}$$

The fundamental theorem of calculus applies to these operators D_q and I_q, *i.e.*,

$$(D_q I_q h)(x) = h(x), \tag{2.10}$$

and if h is continuous at $x = 0$, then

$$(I_q D_q h)(x) = h(x) - h(0). \tag{2.11}$$

Definition 2.1 Let $\nu \geq 0$ and h be a function defined on $[0, T]$. The fractional q-integral of Riemann-Liouville type is given by $(I_q^0 h)(x) = h(x)$ and

$$\bigl(I_q^\nu h\bigr)(x) = \frac{1}{\Gamma_q(\nu)} \int_0^x (x - qs)^{(\nu-1)} h(s)\, d_q s, \quad \nu > 0, x \in [0, T]. \tag{2.12}$$

Definition 2.2 The fractional q-derivative of Riemann-Liouville type of order $\nu \geq 0$ is defined by $(D_q^0 h)(x) = h(x)$ and

$$\bigl(D_q^\nu h\bigr)(x) = \bigl(D_q^l I_q^{l-\nu} h\bigr)(x), \quad \nu > 0, \tag{2.13}$$

where l is the smallest integer greater than or equal to ν.

Definition 2.3 For any $x, s > 0$,

$$B_q(x, s) = \int_0^1 u^{(x-1)} (1 - qu)^{(s-1)}\, d_q u \tag{2.14}$$

is called the q-beta function.

From [2], the expression of q-beta function in terms of the q-gamma function can be written as

$$B_q(x,s) = \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x+s)}.$$

Lemma 2.4 [4] *Let $\alpha, \beta \geq 0$ and f be a function defined in $[0,T]$. Then the following formulas hold*:

(1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 2.5 [22] *Let $\alpha > 0$ and ν be a positive integer. Then the following equality holds*:

$$\bigl(I_q^\alpha D_q^\nu f\bigr)(x) = \bigl(D_q^\nu I_q^\alpha f\bigr)(x) - \sum_{k=0}^{\nu-1} \frac{x^{\alpha-\nu+k}}{\Gamma_q(\alpha+k-\nu+1)} \bigl(D_q^k f\bigr)(0). \tag{2.15}$$

3 Some auxiliary lemmas

Lemma 3.1 *Let $\alpha, \beta > 0$ and $0 < q < 1$. Then we have*

$$\int_0^\eta (\eta - qs)^{(\alpha-1)} s^{(\beta)} \, d_q s = \eta^{\alpha+\beta} B_q(\alpha, \beta+1). \tag{3.1}$$

Proof Using the definitions of q-analogue of power function and q-beta function, we have

$$\begin{aligned}
\int_0^\eta (\eta - qs)^{(\alpha-1)} s^{(\beta)} \, d_q s &= (1-q)\eta \sum_{n=0}^\infty q^n \bigl(\eta - q\eta q^n\bigr)^{(\alpha-1)} \bigl(\eta q^n\bigr)^\beta \\
&= (1-q)\eta \sum_{n=0}^\infty q^n \eta^{\alpha-1} \bigl(1 - q q^n\bigr)^{(\alpha-1)} \eta^\beta q^{n\beta} \\
&= (1-q)\eta^{\alpha+\beta} \sum_{n=0}^\infty q^n \bigl(1 - q q^n\bigr)^{(\alpha-1)} q^{n\beta} \\
&= \eta^{\alpha+\beta} \int_0^1 (1-qs)^{(\alpha-1)} s^{(\beta)} \, d_q s \\
&= \eta^{\alpha+\beta} B_q(\alpha, \beta+1).
\end{aligned}$$

The proof is complete. □

Lemma 3.2 *Let $\alpha, \beta > 0$ and $0 < p, q < 1$. Then we have*

$$\int_0^\eta \int_0^x (\eta - px)^{(\alpha-1)} (x - qy)^{(\beta-1)} \, d_q y \, d_p x = \frac{\eta^{\alpha+\beta}}{[\beta]_q} \frac{\Gamma_p(\alpha)\Gamma_p(\beta+1)}{\Gamma_p(\alpha+\beta+1)}. \tag{3.2}$$

Proof Taking into account Lemma 3.1, we have

$$\begin{aligned}
&\int_0^\eta \int_0^x (\eta - px)^{(\alpha-1)} (x - qy)^{(\beta-1)} \, d_q y \, d_p x \\
&\quad = \int_0^\eta (\eta - px)^{(\alpha-1)} \int_0^x (x - qy)^{(\beta-1)} \, d_q y \, d_p x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{[\beta]_q} \int_0^\eta (\eta - px)^{(\alpha-1)} x^{(\beta)}\, d_p x \\
&= \frac{1}{[\beta]_q} \eta^{\alpha+\beta} B_p(\alpha, \beta+1) \\
&= \frac{\eta^{\alpha+\beta}}{[\beta]_q} \frac{\Gamma_p(\alpha)\Gamma_p(\beta+1)}{\Gamma_p(\alpha+\beta+1)}.
\end{aligned}
$$

This completes the proof. □

For convenience, we set a nonzero constant

$$
\Lambda = T^{\alpha-1} - \sum_{i=1}^m \frac{\gamma_i \Gamma_{q_i}(\alpha)}{\Gamma_{q_i}(\alpha+\beta_i)} \big(\xi_i^{\alpha+\beta_i-1} - \eta_i^{\alpha+\beta_i-1}\big). \tag{3.3}
$$

Lemma 3.3 *Let $\beta_i > 0$, $0 < q, q_i < 1$, $\gamma_i \in \mathbb{R}$, $\eta_i, \xi_i \in (0, T)$ and $\eta_i < \xi_i$ for all $i = 1, 2, \ldots, m$. Then, for a given $y \in C([0,1], \mathbb{R})$, the unique solution of the linear q-difference equation*

$$
D_q^\alpha u(t) = y(t), \quad t \in (0, T), 1 < \alpha \le 2, \tag{3.4}
$$

subject to the multi-strip fractional q-integral condition

$$
u(0) = 0, \qquad u(T) = \sum_{i=1}^m \gamma_i \big(I_{q_i}^{\beta_i} u\big)\big|_{\eta_i}^{\xi_i} = \sum_{i=1}^m \gamma_i \big(I_{q_i}^{\beta_i} u(\xi_i) - I_{q_i}^{\beta_i} u(\eta_i)\big), \tag{3.5}
$$

is given by

$$
\begin{aligned}
u(t) = {}& -\frac{t^{\alpha-1}}{\Lambda} \Bigg\{ \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s)\, d_q s \\
& - \sum_{i=1}^m \frac{\gamma_i}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)} \bigg(\int_0^{\xi_i} \int_0^s (\xi_i - q_i s)^{(\beta_i-1)} (s-qx)^{(\alpha-1)} y(x)\, d_q x\, d_{q_i} s \\
& + \int_0^{\eta_i} \int_0^s (\eta_i - q_i s)^{(\beta_i-1)} (s-qx)^{(\alpha-1)} y(x)\, d_q x\, d_{q_i} s \bigg) \Bigg\} \\
& + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s)\, d_q s,
\end{aligned} \tag{3.6}
$$

where Λ is defined by (3.3).

Proof Since $1 < \alpha \le 2$, we take $n = 2$. In view of Definition 2.2 and Lemma 2.4, the linear q-difference equation (3.4) can be written as

$$
\big(I_q^\alpha D_q^2 I_q^{2-\alpha} u\big)(t) = \big(I_q^\alpha y\big)(t).
$$

Using Lemma 2.5, we obtain

$$
u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s)\, d_q s \tag{3.7}
$$

for some constants $c_1, c_2 \in \mathbb{R}$. Since $u(0) = 0$, we get $c_2 = 0$.

Applying the Riemann-Liouville fractional q_i-integral of order $\beta_i > 0$ with $c_2 = 0$ for (3.7) and taking into account Lemma 3.1, we have

$$\begin{aligned}
I_{q_i}^{\beta_i} u(\xi_i) &= \int_0^{\xi_i} \frac{(\xi_i - q_i s)^{(\beta_i - 1)}}{\Gamma_{q_i}(\beta_i)} \left(c_1 s^{\alpha-1} + \int_0^s \frac{(s-qx)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(x)\, d_q x \right) d_{q_i} s \\
&= \frac{1}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)} \int_0^{\xi_i} \int_0^s (\xi_i - q_i s)^{(\beta_i-1)} (s-qx)^{(\alpha-1)} y(x)\, d_q x\, d_{q_i} s \\
&\quad + \frac{c_1}{\Gamma_{q_i}(\beta_i)} \int_0^{\xi_i} (\xi_i - q_i s)^{(\beta_i-1)} s^{\alpha-1}\, d_{q_i} s \\
&= \frac{1}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)} \int_0^{\xi_i} \int_0^s (\xi_i - q_i s)^{(\beta_i-1)} (s-qx)^{(\alpha-1)} y(x)\, d_q x\, d_{q_i} s \\
&\quad + c_1 \frac{\Gamma_{q_i}(\alpha) \xi_i^{\alpha+\beta_i-1}}{\Gamma_{q_i}(\alpha+\beta_i)}.
\end{aligned} \tag{3.8}$$

Repeating the above process with $t = \eta_i$ and using the second condition of (3.5), we get a constant c_1 as follows:

$$\begin{aligned}
c_1 &= \frac{1}{\Lambda} \left\{ \sum_{i=1}^m \frac{\gamma_i}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)} \left(\int_0^{\xi_i} \int_0^s (\xi_i - q_i s)^{(\beta_i-1)} (s-qx)^{(\alpha-1)} y(x)\, d_q x\, d_{q_i} s \right.\right. \\
&\quad \left. - \int_0^{\eta_i} \int_0^s (\eta_i - q_i s)^{(\beta_i-1)} (s-qx)^{(\alpha-1)} y(x)\, d_q x\, d_{q_i} s \right) \\
&\quad \left. - \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s)\, d_q s \right\}.
\end{aligned} \tag{3.9}$$

Substituting the values of constants c_1 and c_2 in the linear solution (3.7), the desired result in (3.6) is obtained. □

4 Main results

Let $\mathcal{C} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the supremum norm defined by $\|u\| = \sup_{t\in[0,T]} |u(t)|$. In view of Lemma 3.3, we define an operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}
(\mathcal{A}u)(t) &= -\frac{t^{\alpha-1}}{\Lambda} \left\{ \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\bigl(s,u(s)\bigr)\, d_q s \right. \\
&\quad - \sum_{i=1}^m \frac{\gamma_i}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)} \left(\int_0^{\xi_i} \int_0^s (\xi_i - q_i s)^{(\beta_i-1)} (s-qx)^{(\alpha-1)} f\bigl(x,u(x)\bigr)\, d_q x\, d_{q_i} s \right. \\
&\quad \left.\left. + \int_0^{\eta_i} \int_0^s (\eta_i - q_i s)^{(\beta_i-1)} (s-qx)^{(\alpha-1)} f\bigl(x,u(x)\bigr)\, d_q x\, d_{q_i} s \right) \right\} \\
&\quad + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\bigl(s,u(s)\bigr)\, d_q s,
\end{aligned} \tag{4.1}$$

with $\Lambda \neq 0$. It should be noticed that problem (1.1) has solutions if and only if the operator $\mathcal{A}$ has fixed points.

For the sake of convenience, we put

$$\begin{aligned}\Phi &= \frac{T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)}\left(T^\alpha + \sum_{i=1}^m \frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} + \sum_{i=1}^m \frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)}\right) \\ &\quad + \frac{T^\alpha}{\Gamma_q(\alpha+1)}. \end{aligned} \tag{4.2}$$

The first existence and uniqueness result is based on the Banach contraction mapping principle.

Theorem 4.1 *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying the assumption*

(H_1) *there exists a constant* $L > 0$ *such that* $|f(t,u) - f(t,v)| \leq L|u - v|$ *for each* $t \in [0,T]$ *and* $u, v \in \mathbb{R}$.

If

$$L\Phi < 1, \tag{4.3}$$

where a constant Φ *is given by* (4.2), *then the multi-strip boundary value problem* (1.1) *has a unique solution on* $[0,T]$.

Proof We transform problem (1.1) into a fixed point problem, $u = \mathcal{A}u$, where the operator $\mathcal{A}$ is defined by (4.1). Applying the Banach contraction mapping principle, we will show that the operator $\mathcal{A}$ has a fixed point which is a unique solution of problem (1.1).

Setting $\sup_{t\in[0,T]}|f(t,0)| = M_0 < \infty$ and choosing

$$r \geq \frac{M_0\Phi}{1 - L\Phi},$$

with $L\Phi$ satisfying (4.3), we will show that $\mathcal{A}B_r \subset B_r$, where the set $B_r = \{u \in \mathcal{C} : \|u\| \leq r\}$. For any $u \in B_r$, and taking into account Lemma 3.2, we have

$$\begin{aligned}\|\mathcal{A}u\| &\leq \sup_{t\in[0,T]}\left\{\frac{t^{\alpha-1}}{|\Lambda|}\left\{\int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big|f(s,u(s))\big|\,d_qs\right.\right. \\ &\quad + \sum_{i=1}^m \frac{|\gamma_i|}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)}\left(\int_0^{\xi_i}\int_0^s (\xi_i - q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\big|f(x,u(x))\big|\,d_qx\,d_{q_i}s\right. \\ &\quad \left.\left. + \int_0^{\eta_i}\int_0^s (\eta_i - q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\big|f(x,u(x))\big|\,d_qx\,d_{q_i}s\right)\right\} \\ &\quad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big|f(s,u(s))\big|\,d_qs\right\} \\ &\leq \frac{T^{\alpha-1}}{|\Lambda|}\left\{\int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big(\big|f(s,u(s)) - f(s,0)\big| + \big|f(s,0)\big|\big)\,d_qs\right. \\ &\quad + \sum_{i=1}^m \frac{|\gamma_i|}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)}\left(\int_0^{\xi_i}\int_0^s (\xi_i - q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\right.\end{aligned}$$

$$
\begin{aligned}
&\times \left(\left|f\left(x,u(x)\right)-f(x,0)\right|+\left|f(x,0)\right|\right)d_qx\,d_{q_i}s \\
&+\int_0^{\eta_i}\int_0^s (\eta_i-q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)} \\
&\times \left(\left|f\left(x,u(x)\right)-f(x,0)\right|+\left|f(x,0)\right|\right)d_qx\,d_{q_i}s\Bigg)\Bigg\} \\
&+\int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left(\left|f\left(s,u(s)\right)-f(s,0)\right|+\left|f(s,0)\right|\right)d_qs \\
\le{}& (Lr+M_0)\Bigg\{\frac{T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)}\Bigg(T^\alpha+\sum_{i=1}^m \frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} \\
&+\sum_{i=1}^m \frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)}\Bigg)+\frac{T^\alpha}{\Gamma_q(\alpha+1)}\Bigg\} \\
={}& (Lr+M_0)\Phi\le r.
\end{aligned}
$$

It follows that $\mathcal{A}B_r\subset B_r$.

For $u,v\in\mathcal{C}$ and for each $t\in[0,T]$, we have

$$
\begin{aligned}
&\left|\mathcal{A}u(t)-\mathcal{A}v(t)\right| \\
&\le \frac{T^{\alpha-1}}{|\Lambda|}\Bigg\{\int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left(\left|f\left(s,u(s)\right)-f\left(s,v(s)\right)\right|\right)d_qs \\
&\quad+\sum_{i=1}^m \frac{|\gamma_i|}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)}\Bigg(\int_0^{\xi_i}\int_0^s (\xi_i-q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)} \\
&\quad\times\left(\left|f\left(s,u(s)\right)-f\left(s,v(s)\right)\right|\right)d_qx\,d_{q_i}s \\
&\quad+\int_0^{\eta_i}\int_0^s (\eta_i-q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\left(\left|f\left(s,u(s)\right)-f\left(s,v(s)\right)\right|\right)d_qx\,d_{q_i}s\Bigg)\Bigg\} \\
&\quad+\int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left(\left|f\left(s,u(s)\right)-f\left(s,v(s)\right)\right|\right)d_qs \\
&\le L\|u-v\|\Bigg\{\frac{T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)}\Bigg(T^\alpha+\sum_{i=1}^m \frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} \\
&\quad+\sum_{i=1}^m \frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)}\Bigg)+\frac{T^\alpha}{\Gamma_q(\alpha+1)}\Bigg\} \\
&= L\Phi\|u-v\|.
\end{aligned}
$$

The above result leads to $\|\mathcal{A}u-\mathcal{A}v\|\le L\Phi\|u-v\|$. As $L\Phi<1$, by (4.3), therefore $\mathcal{A}$ is a contraction. Hence, by the Banach contraction mapping principle, we deduce that $\mathcal{A}$ has a fixed point which is the unique solution of problem (1.1). □

Next, we prove the existence of at least one solution by using Krasnoselskii's fixed point theorem.

Lemma 4.2 (Krasnoselskii's fixed point theorem [23]) *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) $Ax + By \in M$ *whenever* $x, y \in M$; (b) A *is compact and continuous*; (c) B *is a contraction mapping. Then there exists* $z \in M$ *such that* $z = Az + Bz$.

Theorem 4.3 *Assume that* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying assumption* (H_1). *In addition, we suppose that*

(H_2) $|f(t,u)| \leq \psi(t)$, $\forall (t,u) \in [0,T] \times \mathbb{R}$ *and* $\psi \in C([0,T], \mathbb{R}^+)$.

If the following condition holds

$$\frac{L}{\Gamma_q(\alpha+1)}\left(\frac{T^{2\alpha-1}}{|\Lambda|} + T^\alpha\right) < 1, \tag{4.4}$$

then the multi-strip boundary value problem (1.1) *has at least one solution on* $[0,T]$.

Proof We define $\sup_{t\in[0,T]} |\psi(t)| = \|\psi\|$ and choose a suitable constant R such that

$$R \geq \|\psi\|\Phi,$$

where Φ is defined by (4.2). Furthermore, we define the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on $B_R = \{u \in \mathcal{C} : \|u\| \leq R\}$ by

$$\begin{aligned}
&(\mathcal{A}_1 u)(t) \\
&\quad = \frac{t^{\alpha-1}}{\Lambda}\sum_{i=1}^m \frac{\gamma_i}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)}\int_0^{\xi_i}\int_0^s (\xi_i - q_i s)^{(\beta_i-1)}(s-qx)^{(\alpha-1)} f\bigl(x,u(x)\bigr)\,d_q x\,d_{q_i} s \\
&\qquad - \frac{t^{\alpha-1}}{\Lambda}\sum_{i=1}^m \frac{\gamma_i}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)}\int_0^{\eta_i}\int_0^s (\eta_i - q_i s)^{(\beta_i-1)}(s-qx)^{(\alpha-1)} f\bigl(x,u(x)\bigr)\,d_q x\,d_{q_i} s,
\end{aligned}$$

and

$$(\mathcal{A}_2 u)(t) = -\frac{t^{\alpha-1}}{\Lambda}\int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\bigl(s,u(s)\bigr)\,d_q s + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\bigl(s,u(s)\bigr)\,d_q s.$$

It should be noticed that $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$.

For any $u, v \in B_R$, we have

$$\begin{aligned}
\|\mathcal{A}_1 u + \mathcal{A}_2 v\| &\leq \|\psi\|\Biggl\{\frac{T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)}\Biggl(T^\alpha + \sum_{i=1}^m \frac{|\gamma_i|\xi_i^{\beta_i+\alpha} B_{q_i}(\beta_i,\alpha+1)}{\Gamma_{q_i}(\beta_i)} \\
&\qquad + \sum_{i=1}^m \frac{|\gamma_i|\eta_i^{\beta_i+\alpha} B_{q_i}(\beta_i,\alpha+1)}{\Gamma_{q_i}(\beta_i)}\Biggr) + \frac{T^\alpha}{\Gamma_q(\alpha+1)}\Biggr\} \\
&= \|\psi\|\Phi \\
&\leq R.
\end{aligned}$$

Therefore $(\mathcal{A}_1 u) + (\mathcal{A}_2 v) \in B_R$. Obviously, condition (4.4) implies that $\mathcal{A}_2$ is a contraction mapping.

Finally, we will show that $\mathcal{A}_1$ is compact and continuous. The continuity of f coupled with assumption (H_2) implies that the operator $\mathcal{A}_1$ is continuous and uniformly bounded on B_R. We define $\sup_{(t,u)\in[0,T]\times B_R}|f(t,u)| = M^* < \infty$. For $t_1, t_2 \in [0,T]$, $t_2 < t_1$ and $u \in B_R$, we have

$$\begin{aligned}
&\bigl|(\mathcal{A}_1u)(t_1) - (\mathcal{A}_1u)(t_2)\bigr| \\
&\quad\le \frac{|t_1^{\alpha-1} - t_2^{\alpha-1}|}{|\Lambda|}\sum_{i=1}^{m}\frac{|\gamma_i|}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)} \\
&\qquad\times \int_0^{\xi_i}\int_0^{s}(\xi_i - q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,d_qx\,d_{q_i}s \\
&\qquad+ \frac{|t_1^{\alpha-1} - t_2^{\alpha-1}|}{|\Lambda|}\sum_{i=1}^{m}\frac{|\gamma_i|}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)} \\
&\qquad\times \int_0^{\eta_i}\int_0^{s}(\eta_i - q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,d_qx\,d_{q_i}s \\
&\quad\le M^*\frac{|t_1^{\alpha-1} - t_2^{\alpha-1}|}{|\Lambda|\Gamma_q(\alpha+1)}\left\{\sum_{i=1}^{m}\frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} + \sum_{i=1}^{m}\frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)}\right\}.
\end{aligned}$$

Actually, as $|t_1 - t_2| \to 0$ the right-hand side of the above inequality tends to zero independently of u. So $\mathcal{A}_1$ is relatively compact on B_R. Therefore, by the Arzelá-Ascoli theorem, $\mathcal{A}_1$ is compact on B_R. Thus all the assumptions of Lemma 4.2 are satisfied. Thus, the boundary value problem (1.1) has at least one solution on $[0,T]$. The proof is complete. □

Remark 4.4 In the above theorem we can interchange the roles of the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ to obtain the second result replacing (4.4) by the following condition:

$$\frac{LT^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)}\left(\sum_{i=1}^{m}\frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} + \sum_{i=1}^{m}\frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)}\right) < 1.$$

Now, our third existence result is based on Leray-Schauder's nonlinear alternative.

Lemma 4.5 (Nonlinear alternative for single-valued maps [24]) *Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 4.6 *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In addition we suppose that:*

(H_3) *there exist a continuous nondecreasing function $\phi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T], \mathbb{R}^+)$ such that*

$$\bigl|f(t,u)\bigr| \le p(t)\phi\bigl(|u|\bigr) \quad \textit{for each } (t,u) \in [0,T] \times \mathbb{R};$$

(H$_4$) *there exists a constant* $N > 0$ *such that*

$$\frac{N}{\|p\|\phi(N)\Phi} > 1,$$

where Φ *is defined by* (4.2).

Then the multi-strip boundary value problem (1.1) *has at least one solution on* $[0, T]$.

Proof Firstly, we will show that the operator $\mathcal{A}$ defined by (4.1) *maps bounded sets (balls) into bounded sets in* $\mathcal{C}$. For a positive number ρ, let $B_\rho = \{u \in \mathcal{C} : \|u\| \leq \rho\}$ be a bounded ball in $\mathcal{C}$. Then, for $t \in [0, T]$, we have

$$\begin{aligned}
|\mathcal{A}u(t)| &\leq \frac{T^{\alpha-1}}{|\Lambda|}\Biggl\{\int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs \\
&\quad + \sum_{i=1}^m \frac{|\gamma_i|}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)}\biggl(\int_0^{\xi_i}\int_0^s (\xi_i - q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,d_qx\,d_{q_i}s \\
&\quad + \int_0^{\eta_i}\int_0^s (\eta_i - q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,d_qx\,d_{q_i}s\biggr)\Biggr\} \\
&\quad + \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs \\
&\leq \frac{T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)}\Biggl(\|p\|\phi\bigl(\|u\|\bigr)T^\alpha + \|p\|\phi\bigl(\|u\|\bigr)\sum_{i=1}^m \frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} \\
&\quad + \|p\|\phi\bigl(\|u\|\bigr)\sum_{i=1}^m \frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)}\Biggr) + \|p\|\phi\bigl(\|u\|\bigr)\frac{T^\alpha}{\Gamma_q(\alpha+1)} \\
&\leq \frac{\|p\|\phi(\rho)T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)}\Biggl(T^\alpha + \sum_{i=1}^m \frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} + \sum_{i=1}^m \frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)}\Biggr) \\
&\quad + \|p\|\phi(\rho)\frac{T^\alpha}{\Gamma_q(\alpha+1)} \\
&:= K.
\end{aligned}$$

Therefore, we deduce that $\|\mathcal{A}u\| \leq K$.

Secondly, we will show that $\mathcal{A}$ *maps bounded sets into equicontinuous sets of* $\mathcal{C}$. Let $\sup_{(t,u)\in[0,T]\times B_\rho}|f(t,u)| = K^* < \infty$, $\tau_1, \tau_2 \in [0, T]$ with $\tau_2 < \tau_1$ and $u \in B_\rho$. Then we have

$$\begin{aligned}
&\bigl|(\mathcal{A}u)(\tau_1) - (\mathcal{A}u)(\tau_2)\bigr| \\
&\quad \leq \frac{|\tau_1^{\alpha-1} - \tau_2^{\alpha-1}|}{|\Lambda|}\Biggl\{\int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs \\
&\qquad + \sum_{i=1}^m \frac{|\gamma_i|}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)}\biggl(\int_0^{\xi_i}\int_0^s (\xi_i - q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,d_qx\,d_{q_i}s \\
&\qquad + \int_0^{\eta_i}\int_0^s (\eta_i - q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,d_qx\,d_{q_i}s\biggr)\Biggr\} \\
&\qquad + \biggl|\int_0^{\tau_1} \frac{(\tau_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs - \int_0^{\tau_2} \frac{(\tau_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs\biggr|
\end{aligned}$$

$$\le \frac{|\tau_1^{\alpha-1} - \tau_2^{\alpha-1}|K^*}{|\Lambda|\Gamma_q(\alpha+1)}\left(T^\alpha + \sum_{i=1}^m \frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} + \sum_{i=1}^m \frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)}\right) + \frac{|\tau_1^{\alpha-1} - \tau_2^{\alpha-1}|K^*}{\Gamma_{q_i}(\alpha+1)}.$$

Obviously, the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $\tau_2 \to \tau_1$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Let u be a solution of problem (1.1). Then, for $t \in [0,T]$, and following similar computations as in the first step with (H_3), we have

$$\begin{aligned}\|u\| &\le \frac{T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)}\left(\|p\|\phi(\|u\|)T^\alpha + \|p\|\phi(\|u\|)\sum_{i=1}^m \frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)}\right. \\ &\quad \left. + \|p\|\phi(\|u\|)\sum_{i=1}^m \frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)}\right) + \|p\|\phi(\|u\|)\frac{T^\alpha}{\Gamma_q(\alpha+1)} \\ &= \|p\|\phi(\|u\|)\Phi.\end{aligned}$$

Consequently, we have

$$\frac{\|u\|}{\|p\|\phi(\|u\|)\Phi} \le 1.$$

In view of (H_4), there exists a constant $N > 0$ such that $\|u\| \ne N$. Let us set

$$U = \{x \in \mathcal{C} : \|u\| < N\}.$$

Note that the operator $\mathcal{A} : \overline{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda\mathcal{A}u$ for some $\lambda \in (0,1)$. Consequently, by nonlinear alternative of Leray-Schauder type (Lemma 4.5), we deduce that $\mathcal{A}$ has a fixed point in $\overline{U}$, which is a solution of the boundary value problem (1.1). This completes the proof. □

As the forth result, we prove the existence of solutions of (1.1) by using Leray-Schauder degree theory.

Theorem 4.7 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(H_5) *there exist constants $0 \le \omega < \Phi^{-1}$, where Φ are given by* (4.2), *and $\Psi > 0$ such that*

$$|f(t,u)| \le \omega|u| + \Psi \quad \textit{for each } (t,u) \in [0,T] \times \mathbb{R}.$$

Then the multi-strip boundary value problem (1.1) *has at least one solution on* $[0,T]$.

Proof Let $\mathcal{A}$ be the operator defined by (4.1). We will prove that there exists at least one solution $u \in \mathcal{C}$ of the operator equation $u = \mathcal{A}u$.

Setting a ball $B_{\rho^*} \subset \mathcal{C}$, where a constant radius $\rho^* > 0$, by

$$B_{\rho^*} = \left\{u \in \mathcal{C} : \sup_{t\in[0,T]} |u(t)| < \rho^*\right\},$$

it is sufficient to show that $\mathcal{A}:\overline{B}_{\rho^*}\to\mathcal{C}$ satisfies

$$u\neq\theta\mathcal{A}u,\quad \forall u\in\partial B_{\rho^*},\forall\theta\in[0,1]. \tag{4.5}$$

Now, we set

$$H(\theta,u)=\theta\mathcal{A}u,\quad u\in\mathcal{C},\theta\in[0,1].$$

As shown in Theorem 4.6, we have that the operator $\mathcal{A}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map $h_\theta(u)=u-H(\theta,u)=u-\theta\mathcal{A}u$ is completely continuous. If (4.5) holds, then the following Leray-Schauder degrees are well defined. From the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\theta,B_{\rho^*},0)&=\deg(I-\theta\mathcal{A},B_{\rho^*},0)=\deg(h_1,B_{\rho^*},0)\\&=\deg(h_0,B_{\rho^*},0)=\deg(I,B_{\rho^*},0)=1\neq0,\quad 0\in B_{\rho^*},\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, we have $h_1(u)=u-Au=0$ for at least one $u\in B_{\rho^*}$. Let us assume that $u=\theta\mathcal{A}u$ for some $\theta\in[0,1]$. Then, for all $t\in[0,T]$, we have

$$\begin{aligned}\bigl|u(t)\bigr|&=\bigl|\theta(\mathcal{A}u)(t)\bigr|\\&\le\frac{T^{\alpha-1}}{|\Lambda|}\Biggl\{\int_0^T\frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs\\&\quad+\sum_{i=1}^m\frac{|\gamma_i|}{\Gamma_{q_i}(\beta_i)\Gamma_q(\alpha)}\biggl(\int_0^{\xi_i}\int_0^s(\xi_i-q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,d_qx\,d_{q_i}s\\&\quad+\int_0^{\eta_i}\int_0^s(\eta_i-q_is)^{(\beta_i-1)}(s-qx)^{(\alpha-1)}\bigl|f\bigl(x,u(x)\bigr)\bigr|\,d_qx\,d_{q_i}s\biggr)\Biggr\}\\&\quad+\int_0^T\frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs\\&\le\bigl(\omega|u|+\Psi\bigr)\Biggl\{\frac{T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)}\Biggl(T^\alpha+\sum_{i=1}^m\frac{|\gamma_i|\xi_i^{\beta_i+\alpha}B_{q_i}(\beta_i,\alpha+1)}{\Gamma_{q_i}(\beta_i)}\\&\quad+\sum_{i=1}^m\frac{|\gamma_i|\eta_i^{\beta_i+\alpha}B_{q_i}(\beta_i,\alpha+1)}{\Gamma_{q_i}(\beta_i)}\Biggr)+\frac{T^\alpha}{\Gamma_q(\alpha+1)}\Biggr\}\\&=\bigl(\omega|u|+\Psi\bigr)\Phi.\end{aligned}$$

Taking norm $\sup_{t\in[0,T]}|u(t)|=\|u\|$ and solving for $\|u\|$, we get

$$\|u\|\le\frac{\Psi\Phi}{1-\omega\Phi}.$$

Choosing $\rho^*=\frac{\Psi\Phi}{1-\omega\Phi}+1$, then we deduce that (4.5) holds. This completes the proof. □

5 Examples

In this section, we present some examples to illustrate our results.

Example 5.1 Consider the following multi-strip fractional q-integral boundary value problem:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{7}{4}}u(t) = \frac{4|u(t)|}{(5+t)^2(1+|u(t)|)}, \quad t \in (0,1), \\ u(0) = 0, \\ u(1) = 2(I_{\frac{1}{4}}^{\frac{2}{3}}u)|_{\frac{1}{6}}^{\frac{1}{5}} + \frac{3}{4}(I_{\frac{1}{2}}^{\frac{4}{5}}u)|_{\frac{2}{5}}^{\frac{2}{3}} + 10(I_{\frac{1}{5}}^{\frac{7}{6}}u)|_{\frac{1}{2}}^{1}. \end{cases} \tag{5.1}$$

Here $\alpha = 7/4$, $q = 1/2$, $T = 1$, $m = 3$, $\gamma_1 = 2$, $\gamma_2 = 3/4$, $\gamma_3 = 10$, $\beta_1 = 2/3$, $\beta_2 = 4/5$, $\beta_3 = 7/6$, $q_1 = 1/4$, $q_2 = 1/2$, $q_3 = 1/5$, $\xi_1 = 1/5$, $\xi_2 = 2/3$, $\xi_3 = 1$, $\eta_1 = 1/6$, $\eta_2 = 2/5$, $\eta_3 = 1/2$ and $f(t,u) = (4|u(t)|)/((5+t)^2(1+|u(t)|))$. Since

$$\bigl|f(t,u) - f(t,v)\bigr| \leq \frac{4}{25}|u-v|,$$

then (H_1) is satisfied with $L = 4/25$. Using the Maple program, we find that

$$\begin{aligned} \Lambda &= T^{\alpha-1} - \sum_{i=1}^{m} \frac{\gamma_i \Gamma_{q_i}(\alpha)}{\Gamma_{q_i}(\alpha+\beta_i)} \bigl(\xi_i^{\alpha+\beta_i-1} - \eta_i^{\alpha+\beta_i-1}\bigr) \\ &\approx -5.259895840, \\ \Phi &= \frac{T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)} \left(T^{\alpha} + \sum_{i=1}^{m} \frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} + \sum_{i=1}^{m} \frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} \right) \\ &\quad + \frac{T^{\alpha}}{\Gamma_q(\alpha+1)} \\ &\approx 2.200354723. \end{aligned}$$

Therefore, we get

$$L\Phi = \frac{4}{25}(2.200354723) \approx 0.352056756 < 1.$$

Hence, by Theorem 4.1, the boundary value problem (5.1) has a unique solution on $[0,1]$.

Example 5.2 Consider the following multi-strip fractional q-integral boundary value problem:

$$\begin{cases} D_{\frac{5}{8}}^{\frac{4}{3}}u(t) = \frac{1}{u^2+3\pi^2}\sin(\frac{\pi u}{4}) + \frac{1}{20\pi}(5+\sin(\pi t)), \quad t \in (0,2), \\ u(0) = 0, \\ u(2) = (I_{\frac{1}{3}}^{\frac{1}{2}}u)|_{\frac{1}{5}}^{\frac{1}{4}} + \frac{2}{3}(I_{\frac{1}{2}}^{\frac{6}{5}}u)|_{1}^{2} - 3(I_{\frac{1}{8}}^{\frac{3}{2}}u)|_{\frac{1}{3}}^{1} + \frac{4}{5}(I_{\frac{1}{4}}^{\frac{3}{4}}u)|_{\frac{2}{5}}^{\frac{1}{2}}. \end{cases} \tag{5.2}$$

Here $\alpha = 4/3$, $q = 5/8$, $T = 2$, $m = 4$, $\gamma_1 = 1$, $\gamma_2 = 2/3$, $\gamma_3 = -3$, $\gamma_4 = 4/5$, $\beta_1 = 1/2$, $\beta_2 = 6/5$, $\beta_3 = 3/2$, $\beta_4 = 3/4$, $q_1 = 1/3$, $q_2 = 1/2$, $q_3 = 1/8$, $q_4 = 1/4$, $\xi_1 = 1/4$, $\xi_2 = 2$, $\xi_3 = 1$, $\xi_4 = 1/2$, $\eta_1 =$

$1/5$, $\eta_2 = 1$, $\eta_3 = 1/3$, $\eta_4 = 2/5$ and $f(t,u) = ((\sin(\pi u/4))/(u^2+3\pi^2)) + ((5+\sin(\pi t))/(20\pi))$. By using the Maple program, we find that

$$\begin{aligned}\Lambda &= T^{\alpha-1} - \sum_{i=1}^{m} \frac{\gamma_i \Gamma_{q_i}(\alpha)}{\Gamma_{q_i}(\alpha+\beta_i)} \bigl(\xi_i^{\alpha+\beta_i-1} - \eta_i^{\alpha+\beta_i-1}\bigr) \\ &\approx 2.448357686, \\ \Phi &= \frac{T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)} \left(T^{\alpha} + \sum_{i=1}^{m} \frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} + \sum_{i=1}^{m} \frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} \right) \\ &\quad + \frac{T^{\alpha}}{\Gamma_q(\alpha+1)} \\ &\approx 5.843174987.\end{aligned}$$

Clearly,

$$\bigl|f(t,u)\bigr| = \left| \frac{1}{u^2+3\pi^2} \sin\left(\frac{\pi u}{4}\right) + \frac{5+\sin(\pi t)}{20\pi} \right| \le \bigl(5+\sin(\pi t)\bigr)\left(\frac{5|u|+3}{60\pi}\right).$$

Choosing $p(t) = 5+\sin(\pi t)$ and $\psi(|u|) = (5|u|+3)/(60\pi)$, we can show that

$$\frac{N}{(6)(\frac{5N+3}{60\pi})(3.493621065)} > 1,$$

which implies that $N > 7.967778981$. Hence, by Theorem 4.6, the boundary value problem (5.2) has at least one solution on $[0,2]$.

Example 5.3 Consider the following multi-strip fractional q-integral boundary value problem:

$$\begin{cases} D_{\frac{3}{8}}^{\frac{7}{5}} u(t) = \frac{1}{12\pi}\tan^{-1}(3u) + \frac{2|u(t)|}{1+|u(t)|}, \quad t \in (0,3), \\ u(0) = 0, \\ u(3) = -2(I_{\frac{1}{2}}^{\frac{6}{5}} u)|_{\frac{3}{2}}^{\frac{5}{3}} + 11(I_{\frac{1}{4}}^{\frac{7}{3}} u)|_{\frac{1}{5}}^{\frac{7}{4}} + (\frac{9}{5})(I_{\frac{2}{3}}^{\frac{6}{7}} u)|_{\frac{5}{4}}^{\frac{5}{2}} \\ \qquad + 5(I_{\frac{2}{5}}^{\frac{3}{4}} u)|_{\frac{3}{7}}^{\frac{1}{2}} + (\frac{4}{7})(I_{\frac{3}{5}}^{\frac{2}{3}} u)|_{\frac{1}{3}}^{\frac{9}{5}}. \end{cases} \tag{5.3}$$

Here $\alpha = 7/5$, $q = 3/8$, $T = 3$, $m = 5$, $\gamma_1 = -2$, $\gamma_2 = 11$, $\gamma_3 = 9/5$, $\gamma_4 = 5$, $\gamma_5 = 4/7$, $\beta_1 = 6/5$, $\beta_2 = 7/3$, $\beta_3 = 6/7$, $\beta_4 = 3/4$, $\beta_5 = 2/3$, $q_1 = 1/2$, $q_2 = 1/4$, $q_3 = 2/3$, $q_4 = 2/5$, $q_5 = 3/5$, $\xi_1 = 5/3$, $\xi_2 = 7/4$, $\xi_3 = 5/2$, $\xi_4 = 1/2$, $\xi_5 = 9/5$, $\eta_1 = 3/2$, $\eta_2 = 1/5$, $\eta_3 = 5/4$, $\eta_4 = 3/7$, $\eta_5 = 1/3$ and $f(t,u) = ((\tan^{-1}(3u))/(12\pi)) + ((2|u(t)|)/(1+|u(t)|))$. By using the Maple program, we find that

$$\begin{aligned}\Lambda &= T^{\alpha-1} - \sum_{i=1}^{m} \frac{\gamma_i \Gamma_{q_i}(\alpha)}{\Gamma_{q_i}(\alpha+\beta_i)} \bigl(\xi_i^{\alpha+\beta_i-1} - \eta_i^{\alpha+\beta_i-1}\bigr) \\ &\approx -33.26381181, \\ \Phi &= \frac{T^{\alpha-1}}{|\Lambda|\Gamma_q(\alpha+1)} \left(T^{\alpha} + \sum_{i=1}^{m} \frac{|\gamma_i|\xi_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} + \sum_{i=1}^{m} \frac{|\gamma_i|\eta_i^{\beta_i+\alpha}\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\beta_i+1)} \right)\end{aligned}$$

$$+ \frac{T^{\alpha}}{\Gamma_q(\alpha+1)}$$
$$\approx 7.22159847.$$

We observe that

$$\bigl|f(t,u)\bigr| = \left|\frac{1}{12\pi}\tan^{-1}(3u) + \frac{2|u(t)|}{1+|u(t)|}\right| \le \frac{|u|}{4\pi} + 2.$$

Therefore, we have $\Psi = 2$ and

$$\omega = 1/4\pi < \Phi^{-1} = 0.13847350.$$

Hence, by Theorem 4.7, the boundary value problem (5.3) has at least one solution on $[0,3]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
The research of J. Tariboon and S. Asawasamrit is supported by King Mongkut's University of Technology North Bangkok, Thailand.

Received: 8 May 2014 Accepted: 2 July 2014 Published: 23 Jul 2014

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10.1186/1687-1847-2014-193
Cite this article as: Pongarm et al.: **Multi-strip fractional q-integral boundary value problems for nonlinear fractional q-difference equations.** *Advances in Difference Equations* 2014, 2014:193