Chen and He *Advances in Difference Equations* 2014, 2014:161
http://www.advancesindifferenceequations.com/content/2014/1/161

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Infinitely many homoclinic solutions for a class of second-order Hamiltonian systems

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Abstract

In this paper, we deal with the existence of infinitely many homoclinic solutions for a class of second-order Hamiltonian systems. By using the dual fountain theorem, we give some new criteria to guarantee that the second-order Hamiltonian systems have infinitely many homoclinic solutions. Some recent results are generalised and significantly improved.

MSC: 34B08; 34B15; 34B37; 58E30

Keywords: Hamiltonian systems; homoclinic solutions; variational methods; critical points

1 Introduction

Consider the following second-order Hamiltonian systems:

$$\ddot{u}(t) - L(t)u(t) + W_u\big(t, u(t)\big) = 0, \quad \forall t \in \mathbb{R}, \tag{1.1}$$

where $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix valued function, $u \in \mathbb{R}^N$ and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. As usual, we say that a solution u of system (1.1) is homoclinic to zero if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u \not\equiv 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

Inspired by the excellent monographs [1, 2], the existence of periodic solutions and homoclinic solutions for second-order Hamiltonian systems have been intensively studied in many recent papers via variational methods; see [3–30] and references therein. Recently, some researchers have begun to study the existence of solutions for second-order Hamiltonian systems with impulses by using some critical points theorems of [31, 32]; see [33, 34].

Homoclinic solutions of dynamical systems are very important in applications for a lot of reasons. They may be 'organizing centers' for the dynamics in their neighborhood. From their existence one may, under proper conditions, deduce the bifurcation behavior of periodic orbits or the existence of chaos nearby. In the past 20 years, with the aid of the variational methods, the existence and multiplicity of homoclinic solutions for system (1.1) have been extensively investigated by many authors; see [3, 4, 6–9, 11–30] and references therein. Most of them treated the superquadratic case [8, 15, 18, 19, 21, 24, 25, 27–29] treated subquadratic case and [9, 21, 30] treated asymptotically quadratic case. Particularly, Yang and Zhang [25] considered a superquadratic case and obtained system (1.1) has infinitely many homoclinic solutions by using the following conditions.

(A_1) $W(t,u)/|u|^2 \to +\infty$, as $|u| \to \infty$ uniformly for all $t \in \mathbb{R}$, and
(A_2) $W_u(t,u(t)) = o(|u|)$, as $|u| \to 0$ uniformly for all $t \in \mathbb{R}$.

Recently, Wei and Wang [22] dealt with a case that $W(t,u) = F(t,u) + G(t,u)$, where $F(t,u)$ is subquadratic and $G(t,u)$ is superquadratic, by using the following condition, they obtained system (1.1) has infinitely many homoclinic solutions.

(A_3) There exists $\nu < 1$ such that

$$l(t)|t|^{\nu-2} \to \infty \quad \text{as } |t| \to \infty,$$

where $l(t) = \inf_{|\xi|=1}(L(t)\xi, \xi)$.

In [23], Yang *et al.* obtained the following theorems by using the variant fountain theorem.

Theorem 1.1 ([23, Theorem 1.2]) *Assume that the following conditions are satisfied*:

(C_1) $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ *is a symmetric and positive definite matrix for all* $t \in \mathbb{R}$ *and there is a continuous function* $\beta : \mathbb{R} \to \mathbb{R}$ *such that* $\beta(t) > 0$ *for all* $t \in \mathbb{R}$ *and* $(L(t)u,u) \geq \beta(t)|u|^2$ *and* $\beta(t) \to \infty$ *as* $|t| \to \infty$.
(C_2)$'$ $c_1'(t)|u|^\gamma \leq (W_u(t,u),u)$, $|W_u(t,u)| \leq c_2'(t)|u|^{\gamma-1} + c_3'(t)|u|^{\sigma-1}$ *where* $c_1', c_2', c_3' : \mathbb{R} \to \mathbb{R}^+$ *are positive continuous functions such that* $c_1', c_2' \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R}^+)$, $c_3' \in L^{\frac{2}{2-\sigma}}(\mathbb{R}, \mathbb{R}^+)$ *and* $1 < \gamma < 2$, $1 < \sigma < 2$ *are constants*, $W(t,0) = 0$, $W(t,u) = W(t,-u)$ *for all* $(t,u) \in \mathbb{R} \times \mathbb{R}^N$.

Then system (1.1) *possesses infinitely many homoclinic solutions*.

Theorem 1.2 ([23, Theorem 1.3]) *Assume that* (C_1) *hold. Moreover, we assume that the following condition is satisfied*:

(C_3)$'$ $W(t,u) = m(t)|u|^\gamma + d|u|^q$ *where* $m : \mathbb{R} \to \mathbb{R}^+$ *is a positive continuous function such that* $m \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R}^+)$ *and* $1 < \gamma < 2$, $d \geq 0$, $q > 2$ *are constants*.

Then system (1.1) *possesses infinitely many homoclinic solutions*.

Motivated by the above facts, in this paper, we will improve and generalize some results in the references that we have mentioned above.

Now, we state our main results.

Theorem 1.3 *Assume that* (C_1) *hold. Moreover, we assume that the following conditions are satisfied*:

(C_2) $W(t,u) = F(t,u) + G(t,u)$ *where* $F(t,0) = 0$, $G(t,0) = 0$ *and* $F, G \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ *are even in* u.
(C_3) $c_1(t)|u|^\gamma \leq (F_u(t,u),u)$, $|F_u(t,u)| \leq c_2(t)|u|^{\gamma-1} + c_3(t)|u|^{\sigma-1}$ *where* $c_1, c_2, c_3 : \mathbb{R} \to \mathbb{R}^+$ *are positive continuous functions such that* $c_1, c_2 \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R}^+)$, $c_3 \in L^{\frac{2}{2-\sigma}}(\mathbb{R}, \mathbb{R}^+)$ *and* $1 < \gamma < 2$, $1 < \sigma < 2$ *are constants*.

(C_4) *$G(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$ and there exists $\mu > 2$ such that*

$$\left|G_u(t,u)\right| \leq a(t) + c(t)|u|^{\mu-1},$$

where $a, c : \mathbb{R} \to \mathbb{R}^+$ are positive continuous functions such that $a \in L^2(\mathbb{R}, \mathbb{R}^+)$ and $c \in L^\infty(\mathbb{R}, \mathbb{R}^+)$.

(C_5) *There exist $\rho > 2$ and $1 < \delta < 2$ such that*

$$\rho G(t,u) - \left(G_u(t,u), u\right) \leq h(t)|u|^\delta, \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N,$$

where $h : \mathbb{R} \to \mathbb{R}^+$ is a positive continuous function such that $h \in L^{\frac{2}{2-\delta}}(\mathbb{R}, \mathbb{R}^+)$.

Then system (1.1) *possesses infinitely many homoclinic solutions.*

Remark 1.1 It is clear that there are many functions satisfying (C_1) but do not satisfying (A_3); see [28].

Remark 1.2 Obviously, Theorem 1.3 generalizes Theorem 1.2 in [23], Theorem 1.1 in [29] and Theorem 1.2 in [18]. In fact, let $G(t,u) \equiv 0$, then Theorem 1.3 coincides with Theorem 1.2 in [23], and contains Theorem 1.1 in [29] and Theorem 1.2 in [18]. Furthermore, there are many functions W satisfying our Theorem 1.3 and not satisfying Theorem 1.2 in [23], Theorem 1.1 in [29] and Theorem 1.2 in [18]. For example, the function

$$W(t,u) = F(t,u) + G(t,u), \tag{1.2}$$

where $F(t,u) = (\frac{1}{1+t^2})^{\frac{1}{3}}|u|^{\frac{4}{3}} + (\frac{1}{1+|t|})^{\frac{1}{3}}|u|^{\frac{5}{3}}$ and $G(t,u) = \frac{1}{1+t^6}|u|^4$.

Remark 1.3 It is easy to see that there are many functions W satisfying the conditions of Theorem 1.3 but not satisfying (A_1), (A_2), the condition $(C_3)'$ in Theorem 1.2 or conditions (R_2) and (R_4) in Theorem 1.1 in [22], for example, the function (1.2) since $(\frac{1}{1+t^2})^{\frac{1}{3}} \to 0$, $(\frac{1}{1+|t|})^{\frac{1}{3}} \to 0$ and $\frac{1}{1+t^6}|u|^4 \to 0$ as $|t| \to \infty$.

Theorem 1.4 *Assume that* (C_1)-(C_3) *hold. Moreover, we assume that the following conditions are satisfied*:

(C_6) *There exist $\lambda > 2$ and $0 \leq h_1 < \frac{\beta_0(\lambda-2)}{2}$ such that*

$$\lambda G(t,u) - \left(G_u(t,u), u\right) \leq h_1|u|^2, \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N,$$

where $\beta_0 = \min_{t\in\mathbb{R}} \beta(t)$.

(C_7) *There exists $0 < \theta < \beta_0$ such that $\lim_{u\to 0} \frac{G_u(t,u)}{|u|} < \theta$ uniformly for $t \in \mathbb{R}$.*

(C_8) *$G(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$ and there exist $h_2 > 0$ and $p_1 > 2$ such that*

$$\left|G_u(t,u)\right| \leq h_2\left(1 + |u|^{p_1-1}\right).$$

Then system (1.1) *possesses infinitely many homoclinic solutions.*

Remark 1.4 Obviously, all the conditions in Theorem 1.4 are more general than those in Theorem 1.2. Therefore, Theorem 1.4 is a complement of Theorem 1.2. On the other hand, there are many functions W satisfying our Theorem 1.4 and not satisfying Theorem 1.2. For example, the function

$$W(t,u) = F(t,u) + G(t,u), \tag{1.3}$$

where $F(t,u) = (\frac{1}{1+t^2})^{\frac{1}{3}}|u|^{\frac{4}{3}} + (\frac{1}{1+|t|})^{\frac{1}{3}}|u|^{\frac{5}{3}}$ and $G(t,u) = |u|^4 + \frac{\beta_0}{4}|u|^2$. Thus, $F(t,u)$ is subquadratic and $G(t,u)$ is superquadratic. To the best of our knowledge, with the exception of [22, 23], the study of this case has received considerably less attention. Furthermore, it is easy to see that the function (1.3) does not satisfy Theorem 1.1 in [22].

Remark 1.5 In Theorem 1.4, there are many functions W satisfying (A_1) and not satisfying (A_2), for example, the function (1.3).

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of Theorems 1.3 and 1.4.

2 Preliminaries

We will present some definitions and lemmas that will be used in the proofs of our results. Let

$$E = \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} \left[|\dot{u}(t)|^2 + \left(L(t)u(t), u(t)\right)\right] dt < +\infty \right\}$$

equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}} \left[|\dot{u}(t)|^2 + \left(L(t)u(t), u(t)\right)\right] dt \right)^{\frac{1}{2}}, \quad \forall u \in X, \tag{2.1}$$

and the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}} \left[\left(\dot{u}(t), \dot{v}(t)\right) + \left(L(t)u(t), v(t)\right)\right] dt, \quad \forall u, v \in X. \tag{2.2}$$

Then E is a Hilbert space with this inner product. Denote by E^* its dual space with the associated operator norm $\|\cdot\|_{E^*}$. Note that E is continuously embedded in $L^p(\mathbb{R}, \mathbb{R}^N)$ for all $p \in [2, +\infty]$. Therefore, there exists a constant $\delta_p > 0$ such that

$$\|u\|_p \le \delta_p \|u\|, \quad \forall u \in E, \tag{2.3}$$

where $\|\cdot\|_p$ denotes the usual norm on $L^p(\mathbb{R}, \mathbb{R}^N)$.

Lemma 2.1 (see [13]) *Suppose that L satisfies (C_1). Then the embedding of E in $L^2(\mathbb{R}, \mathbb{R}^N)$ is compact.*

Lemma 2.2 *Suppose that (C_1), (C_2), (C_3) and (C_4) are satisfied. If $u_k \rightharpoonup u$ in E, then $W_u(t, u_k) \to W_u(t,u)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$.*

Proof Assume that $u_k \rightharpoonup u$. In view of (C_2), (C_3) and (C_4), we get

$$\begin{aligned}
&\bigl|W_u(t,u_k) - W_u(t,u)\bigr| \\
&\quad\le c_2(t)\bigl(|u_k|^{\gamma-1} + |u|^{\gamma-1}\bigr) + c_3(t)\bigl(|u_k|^{\sigma-1} + |u|^{\sigma-1}\bigr) \\
&\qquad + \bigl|G_u(t,u_k) - G_u(t,u)\bigr| \\
&\quad\le c_2(t)\bigl(|u_k|^{\gamma-1} + |u|^{\gamma-1}\bigr) + c_3(t)\bigl(|u_k|^{\sigma-1} + |u|^{\sigma-1}\bigr) \\
&\qquad + c(t)\bigl(|u_k|^{\mu-1} + |u|^{\mu-1}\bigr) + 2a(t) \\
&\quad\le c_2(t)\bigl(|u_k - u|^{\gamma-1} + 2|u|^{\gamma-1}\bigr) + c_3(t)\bigl(|u_k - u|^{\sigma-1} + 2|u|^{\sigma-1}\bigr) \\
&\qquad + c(t)\bigl[2^{\mu-2}|u_k - u|^{\mu-1} + \bigl(2^{\mu-2}+1\bigr)|u|^{\mu-1}\bigr] + 2a(t) \\
&\quad\le c_2(t)\bigl(|u_k - u|^{\gamma-1} + 2|u|^{\gamma-1}\bigr) + c_3(t)\bigl(|u_k - u|^{\sigma-1} + 2|u|^{\sigma-1}\bigr) \\
&\qquad + 2^{\mu-1}c(t)\bigl(|u_k - u|^{\mu-1} + |u|^{\mu-1}\bigr) + 2a(t),
\end{aligned} \tag{2.4}$$

which yields

$$\begin{aligned}
\bigl|W_u(t,u_k) - W_u(t,u)\bigr|^2 \le{}& 7c_2^2(t)\bigl(|u_k - u|^{2\gamma-2} + 4|u|^{2\gamma-2}\bigr) \\
&+ 7c_3^2(t)\bigl(|u_k - u|^{2\sigma-2} + 4|u|^{2\sigma-2}\bigr) \\
&+ 28a^2(t) + 7 \times 2^{2\mu-2}c^2(t)\bigl(|u_k - u|^{2\mu-2} + |u|^{2\mu-2}\bigr).
\end{aligned} \tag{2.5}$$

By virtue of (2.3), $u_k \rightharpoonup u$ and the Banach-Steinhaus Theorem, one has

$$\sup_{k\in\mathbb{N}} \|u_k\|_\infty \le M_1, \qquad \|u\|_\infty \le M_1, \tag{2.6}$$

where $M_1 > 0$ is a constant. In view of $2\mu - 4 > 0$, (2.5) and (2.6), we have

$$\begin{aligned}
&\bigl|W_u(t,u_k) - W_u(t,u)\bigr|^2 \\
&\quad\le 7c_2^2(t)\bigl(|u_k - u|^{2\gamma-2} + 4|u|^{2\gamma-2}\bigr) \\
&\qquad + 7c_3^2(t)\bigl(|u_k - u|^{2\sigma-2} + 4|u|^{2\sigma-2}\bigr) + 28a^2(t) \\
&\qquad + 7 \times 2^{2\mu-2}c^2(t)\bigl(|u_k - u|^{2\mu-2} + |u|^{2\mu-2}\bigr) \\
&\quad\le 7c_2^2(t)\bigl(|u_k - u|^{2\gamma-2} + 4|u|^{2\gamma-2}\bigr) \\
&\qquad + 7c_3^2(t)\bigl(|u_k - u|^{2\sigma-2} + 4|u|^{2\sigma-2}\bigr) + 28a^2(t) \\
&\qquad + 7 \times 2^{2\mu-2}c^2(t)\bigl[(2M_1)^{2\mu-4}|u_k - u|^2 + |u|^{2\mu-2}\bigr].
\end{aligned} \tag{2.7}$$

Since $u_k \rightharpoonup u$, then $u_k \to u$ in $L^2(\mathbb{R},\mathbb{R}^N)$, passing to a subsequence if necessary, we have

$$\|u_k - u\|_2 \to 0 \quad \text{as } k \to \infty.$$

So it can be assumed that

$$\sum_{k=1}^{\infty} \|u_k - u\|_2 < +\infty,$$

which implies that $u_k(t) \to u(t)$ for almost every $t \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} \bigl|u_k(t) - u(t)\bigr| = e(t) \in L^2(\mathbb{R}, \mathbb{R}).$$

Then we have

$$\begin{aligned}\bigl|W_u(t,u_k) - W_u(t,u)\bigr|^2 \le{} & 7c_2^2(t)\bigl(\bigl|e(t)\bigr|^{2\gamma-2} + 4|u|^{2\gamma-2}\bigr) \\ & + 7c_3^2(t)\bigl(\bigl|e(t)\bigr|^{2\sigma-2} + 4|u|^{2\sigma-2}\bigr) + 28a^2(t) \\ & + 7 \times 2^{2\mu-2}c^2(t)\bigl[(2M_1)^{2\mu-4}\bigl|e(t)\bigr|^2 + |u|^{2\mu-2}\bigr]. \end{aligned} \tag{2.8}$$

By (2.3), (2.8), $2\mu - 2 > 2$ and the Hölder inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}} \bigl|W_u(t,u_k) - W_u(t,u)\bigr|^2\,dt \\ &\quad\le 7\int_{\mathbb{R}} c_2^2(t)\bigl(\bigl|e(t)\bigr|^{2\gamma-2} + 4|u|^{2\gamma-2}\bigr)\,dt \\ &\qquad + 7\int_{\mathbb{R}} c_3^2(t)\bigl(\bigl|e(t)\bigr|^{2\sigma-2} + 4|u|^{2\sigma-2}\bigr)\,dt + 28\int_{\mathbb{R}} a^2(t)\,dt \\ &\qquad + 7 \times 2^{2\mu-2}\|c\|_\infty^2 \int_{\mathbb{R}} \bigl[(2M_1)^{2\mu-4}\bigl|e(t)\bigr|^2 + |u|^{2\mu-2}\bigr]\,dt \\ &\quad\le 7\|c_2\|_{\frac{2}{2-\gamma}}^2 \bigl(\|e\|_2^{2\gamma-2} + 4\|u\|_2^{2\gamma-2}\bigr) + 28\|a\|_2^2 \\ &\qquad + 7\|c_3\|_{\frac{2}{2-\sigma}}^2 \bigl(\|e\|_2^{2\sigma-2} + 4\|u\|_2^{2\sigma-2}\bigr) \\ &\qquad + 7 \times 2^{2\mu-2}\|c\|_\infty^2\bigl[(2M_1)^{2\mu-4}\|e\|_2^2 + \|u\|_{2\mu-2}^{2\mu-2}\bigr] \\ &\quad\le 7\|c_2\|_{\frac{2}{2-\gamma}}^2 \bigl(\|e\|_2^{2\gamma-2} + 4\delta_2^{2\gamma-2}\|u\|^{2\gamma-2}\bigr) + 28\|a\|_2^2 \\ &\qquad + 7\|c_3\|_{\frac{2}{2-\sigma}}^2 \bigl(\|e\|_2^{2\sigma-2} + 4\delta_2^{2\sigma-2}\|u\|^{2\sigma-2}\bigr) \\ &\qquad + 7 \times 2^{2\mu-2}\|c\|_\infty^2\bigl[(2M_1)^{2\mu-4}\|e\|_2^2 + \delta_{2\mu-2}^{2\mu-2}\|u\|^{2\mu-2}\bigr]. \end{aligned}$$

By using the Lebesgue dominated convergence theorem, the lemma is proved. □

Remark 2.1 Suppose that the condition (C_4) is replaced by conditions (C_7) and (C_8), then we can obtain the same conclusion.

Define the functional Ψ on E by

$$\begin{aligned}\Psi(u) &= \frac{1}{2}\int_{\mathbb{R}} \bigl[\bigl|\dot{u}(t)\bigr|^2 + \bigl(L(t)u(t), u(t)\bigr)\bigr]\,dt - \int_{\mathbb{R}} W\bigl(t, u(t)\bigr)\,dt \\ &= \frac{1}{2}\|u\|^2 - \varphi(u), \end{aligned} \tag{2.9}$$

where $\varphi(u) = \int_{\mathbb{R}} W(t, u(t))\,dt$.

Lemma 2.3 *Under conditions* (C_1)-(C_4), *we have*

$$\begin{aligned}\Psi'(u)v &= \int_{\mathbb{R}}\big[\big(\dot{u}(t),\dot{v}(t)\big)+\big(L(t)u(t),v(t)\big)\big]\,dt-\int_{\mathbb{R}}\big(W_u\big(t,u(t)\big),v(t)\big)\,dt\\ &= \int_{\mathbb{R}}\big[\big(\dot{u}(t),\dot{v}(t)\big)+\big(L(t)u(t),v(t)\big)\big]\,dt-\varphi'(u)v \end{aligned}\tag{2.10}$$

for any $u,v\in E$, *which yields*

$$\Psi'(u)u=\|u\|^2-\int_{\mathbb{R}}\big(W_u\big(t,u(t)\big),u(t)\big)\,dt.\tag{2.11}$$

Moreover, $\Psi\in C^1(E,\mathbb{R})$, $\varphi':E\to E^*$ *is compact and any critical point of* Ψ *on* E *is a classical solution for system* (1.1) *satisfying* $u\in C^2(\mathbb{R},\mathbb{R}^N)$, $u(t)\to 0$ *and* $\dot{u}(t)\to 0$ *as* $|t|\to\infty$.

Proof We first show that $\Psi:E\to\mathbb{R}$. It follows from (C_2), (C_3), (C_4), (2.3), $\mu>2$ and the Hölder inequality that

$$\begin{aligned}0&\le\int_{\mathbb{R}}W\big(t,u(t)\big)\,dt\\ &\le\int_{\mathbb{R}}\bigg(\frac{1}{\gamma}c_2(t)\big|u(t)\big|^{\gamma}+\frac{1}{\sigma}c_3(t)\big|u(t)\big|^{\sigma}\bigg)\,dt+\int_{\mathbb{R}}\big(a(t)\big|u(t)\big|+c(t)\big|u(t)\big|^{\mu}\big)\,dt\\ &\le\frac{1}{\gamma}\|c_2\|_{\frac{2}{2-\gamma}}\|u\|_2^{\gamma}+\frac{1}{\sigma}\|c_3\|_{\frac{2}{2-\sigma}}\|u\|_2^{\sigma}+\|a\|_2\|u\|_2+\|c\|_\infty\|u\|_\mu^\mu\\ &\le\frac{1}{\gamma}\delta_2^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}}\|u\|^{\gamma}+\frac{1}{\sigma}\delta_2^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}}\|u\|^{\sigma}+\delta_2\|a\|_2\|u\|+\|c\|_\infty\delta_\mu^\mu\|u\|^\mu.\end{aligned}$$

Next, we prove that $\Psi\in C^1(E,\mathbb{R})$. It is sufficient to show that $\varphi\in C^1(E,\mathbb{R})$. At first, we will see that

$$\varphi'(u)v=\int_{\mathbb{R}}\big(W_u\big(t,u(t)\big),v(t)\big)\,dt\tag{2.12}$$

for any $u,v\in E$. For any given $u\in E$, let us define $\Phi(u):E\to\mathbb{R}$ as follows:

$$\Phi(u)v=\int_{\mathbb{R}}\big(W_u\big(t,u(t)\big),v(t)\big)\,dt,\quad\forall v\in E.$$

It is easy to see that $\Phi(u)$ is linear. In the following we show that $\Phi(u)$ is bounded. In fact, for any $u\in E$, by (C_2), (C_3), (C_4), (2.3) and the Hölder inequality, we have

$$\begin{aligned}\big|\Phi(u)v\big|&=\bigg|\int_{\mathbb{R}}\big(W_u\big(t,u(t)\big),v(t)\big)\,dt\bigg|\\ &\le\int_{\mathbb{R}}c_2(t)\big|u(t)\big|^{\gamma-1}\big|v(t)\big|\,dt+\int_{\mathbb{R}}c_3(t)\big|u(t)\big|^{\sigma-1}\big|v(t)\big|\,dt\\ &\quad+\int_{\mathbb{R}}a(t)\big|v(t)\big|\,dt+\int_{\mathbb{R}}c(t)\big|u(t)\big|^{\mu-1}\big|v(t)\big|\,dt\\ &\le\bigg(\int_{\mathbb{R}}c_2^2(t)\big|u(t)\big|^{2\gamma-2}\,dt\bigg)^{\frac{1}{2}}\|v\|_2+\bigg(\int_{\mathbb{R}}c_3^2(t)\big|u(t)\big|^{2\sigma-2}\,dt\bigg)^{\frac{1}{2}}\|v\|_2\end{aligned}$$

$$
\begin{aligned}
&\quad + \|a\|_2\|v\|_2 + \|c\|_\infty\left(\int_{\mathbb{R}}\bigl|u(t)\bigr|^{2\mu-2}\,dt\right)^{\frac{1}{2}}\|v\|_2\\
&\le \bigl(\|c_2\|_{\frac{2}{2-\gamma}}\|u\|_2^{\gamma-1} + \|c_3\|_{\frac{2}{2-\sigma}}\|u\|_2^{\sigma-1} + \|a\|_2 + \|c\|_\infty\|u\|_{2\mu-2}^{\mu-1}\bigr)\|v\|_2\\
&\le \delta_2\bigl(\delta_2^{\gamma-1}\|c_2\|_{\frac{2}{2-\gamma}}\|u\|^{\gamma-1} + \delta_2^{\sigma-1}\|c_3\|_{\frac{2}{2-\sigma}}\|u\|^{\sigma-1}\\
&\quad + \|a\|_2 + \|c\|_\infty\delta_{2\mu-2}^{\mu-1}\|u\|^{\mu-1}\bigr)\|v\|.
\end{aligned}
$$

Moreover, for any $u, v \in E$, by the mean value theorem, we have

$$
\int_{\mathbb{R}}\bigl[W\bigl(t,u(t)+v(t)\bigr) - W\bigl(t,u(t)\bigr)\bigr]\,dt = \int_{\mathbb{R}}\bigl(W_u\bigl(t,u(t)+\vartheta(t)v(t)\bigr), v(t)\bigr)\,dt,
$$

where $\vartheta(t) \in (0,1)$. Thus, by Lemma 2.2 and the Hölder inequality, one has

$$
\int_{\mathbb{R}}\bigl(W_u\bigl(t,u(t)+\vartheta(t)v(t)\bigr) - W_u\bigl(t,u(t)\bigr), v(t)\bigr)\,dt \to 0 \quad \text{as } v \to 0.
$$

Suppose that $u \to u_0$ in E and note that

$$
\varphi'(u)v - \varphi'(u_0)v = \int_{\mathbb{R}}\bigl(W_u\bigl(t,u(t)\bigr) - W_u\bigl(t,u_0(t)\bigr), v(t)\bigr)\,dt.
$$

Combining Lemma 2.2 and the Hölder inequality, we have

$$
\varphi'(u)v - \varphi'(u_0)v \to 0 \quad \text{as } u \to u_0.
$$

So φ' is continuous and $\Psi \in C^1(E,\mathbb{R})$. Let $u_k \rightharpoonup u$ in E, we get

$$
\begin{aligned}
\bigl\|\varphi'(u_k) - \varphi'(u)\bigr\|_{E^*} &= \sup_{\|v\|=1}\bigl\|\bigl(\varphi'(u_k) - \varphi'(u)\bigr)v\bigr\|\\
&= \sup_{\|v\|=1}\left|\int_{\mathbb{R}}\bigl(W_u\bigl(t,u_k(t)\bigr) - W_u\bigl(t,u(t)\bigr), v(t)\bigr)\,dt\right|\\
&\le \sup_{\|v\|=1}\left(\int_{\mathbb{R}}\bigl|W_u\bigl(t,u_k(t)\bigr) - W_u\bigl(t,u(t)\bigr)\bigr|^2\,dt\right)^{\frac{1}{2}}\|v\|_2\\
&\le \delta_2\left(\int_{\mathbb{R}}\bigl|W_u\bigl(t,u_k(t)\bigr) - W_u\bigl(t,u(t)\bigr)\bigr|^2\,dt\right)^{\frac{1}{2}} \to 0
\end{aligned}
$$

as $k \to \infty$. Consequently, φ' is weakly continuous. Therefore, φ' is compact by the weakly continuity of φ' since E is a Hilbert space.

Finally, as in the discussion in Lemma 3.1 of [29], we find that the critical points of Ψ are classical solutions of system (1.1) satisfying $u \in C^2(\mathbb{R},\mathbb{R}^N)$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$. The proof is complete. □

Remark 2.2 If condition (C_4) is replaced by conditions (C_7) and (C_8), then we can obtain the same conclusion.

In the next section we shall prove our results applying the dual fountain theorem obtained in [35] (see also Proposition 2.1 of [24]). Assume that E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$, where X_j is a finite-dimensional subspace of E. For each $k \in \mathbb{N}$, let $Y_k = \bigoplus_{j=0}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. The functional Ψ is said to satisfy the $(PS)^*$ condition if for any sequence $\{u_j\}$ for which $\{\Psi(u_j)\}$ is bounded, $u_j \in Y_{k_j}$ for some k_j with $k_j \to \infty$ and $(\Psi|_{Y_{k_j}})'(u_j) \to 0$ as $j \to \infty$ has a subsequence converging to a critical point of Ψ.

Theorem 2.1 *Suppose that the functional $\Psi \in C^1(E,\mathbb{R})$ is even and satisfies the $(PS)^*$ condition. Assume that for each sufficiently large $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(H_1) $a_k := \inf_{u\in Z_k, \|u\|=\rho_k} \Psi(u) \geq 0$.
(H_2) $b_k := \max_{u\in Y_k, \|u\|=r_k} \Psi(u) < 0$.
(H_3) $d_k := \inf_{u\in Z_k, \|u\|\leq\rho_k} \Psi(u) \to 0$ *as* $k \to \infty$.

Then Ψ has a sequence of negative critical values converging to 0.

3 Proof of Theorem 1.3 and 1.4

Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3 We choose a completely orthonormal basis $\{e_j\}$ of X and define $X_j := \mathbb{R}e_j$, then Z_k and Y_k can be defined as that in Section 2. By (C_2) and Lemma 2.3, we see that $\Psi \in C^1(E,\mathbb{R})$ is even. In the following, we will check that all conditions in Theorem 2.1 are satisfied.

Step 1. We prove that Ψ satisfies the $(PS)^*$ condition. Let $\{u_j\}$ be a $(PS)^*$ sequence, that is, $\{\Psi(u_j)\}$ is bounded, $u_j \in Y_{k_j}$ for some k_j with $k_j \to \infty$ and $(\Psi|_{Y_{k_j}})'(u_j) \to 0$ as $j \to \infty$. Now we show that $\{u_j\}$ is bounded in E. By virtue of (C_2), (C_3) and (C_5), for j large enough, we have

$$\begin{aligned}
\rho M_2 + M_3\|u_j\| &\geq \rho\Psi(u_j) - \Psi'(u_j)u_j \\
&= \left(\frac{\rho}{2} - 1\right)\|u_j\|^2 + \int_{\mathbb{R}} \bigl[\bigl(W_u(t,u_j),u_j\bigr) - \rho W(t,u_j)\bigr]\,dt \\
&= \left(\frac{\rho}{2} - 1\right)\|u_j\|^2 + \int_{\mathbb{R}} \bigl[\bigl(F_u(t,u_j),u_j\bigr) - \rho F(t,u_j)\bigr]\,dt \\
&\quad + \int_{\mathbb{R}} \bigl[\bigl(G_u(t,u_j),u_j\bigr) - \rho G(t,u_j)\bigr]\,dt \\
&\geq \left(\frac{\rho}{2} - 1\right)\|u_j\|^2 - (\rho+1)\int_{\mathbb{R}} \bigl(c_2(t)\bigl|u_j(t)\bigr|^{\gamma} + c_3\bigl|u_j(t)\bigr|^{\sigma}\bigr)\,dt \\
&\quad - \int_{\mathbb{R}} h(t)\bigl|u_j(t)\bigr|^{\delta}\,dt \\
&\geq \left(\frac{\rho}{2} - 1\right)\|u_j\|^2 - (\rho+1)\bigl(\|c_2\|_{\frac{2}{2-\gamma}}\|u_j\|_2^{\gamma} + \|c_3\|_{\frac{2}{2-\sigma}}\|u_j\|_2^{\sigma}\bigr) \\
&\quad - \|h\|_{\frac{2}{2-\delta}}\|u_j\|_2^{\delta} \\
&\geq \left(\frac{\rho}{2} - 1\right)\|u_j\|^2 - (\rho+1)\bigl(\delta_2^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}}\|u_j\|^{\gamma} + \delta_2^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}}\|u_j\|^{\sigma}\bigr) \\
&\quad - \delta_2^{\delta}\|h\|_{\frac{2}{2-\delta}}\|u_j\|^{\delta}
\end{aligned} \tag{3.1}$$

for some $M_2 > 0$, $M_3 > 0$. Since $\rho > 2$ and $\gamma, \sigma, \delta < 2$, it follows that $\{u_j\}$ is bounded in E.

From the reflexivity of E, we may extract a weakly convergent subsequence, which, for simplicity, we call $\{u_j\}$, $u_j \rightharpoonup u_1$ in E. In view of the Riesz representation theorem, $(\Psi|_{Y_{k_j}})' : Y_{k_j} \to Y_{k_j}^*$ and $\varphi' : E \to E^*$ can be viewed as $(\Psi|_{Y_{k_j}})' : Y_{k_j} \to Y_{k_j}$ and $\varphi' : E \to E$, respectively, where $Y_{k_j}^*$ is the dual space of Y_{k_j}. Note that

$$(\Psi|_{Y_{k_j}})'(u_j) = u_j - P_{k_j}\varphi'(u_j), \quad \forall j \in \mathbb{N}, \tag{3.2}$$

where $P_{k_j} : E \to Y_{k_j}$ is the orthogonal projection for all $j \in \mathbb{N}$. That is,

$$u_j = (\Psi|_{Y_{k_j}})'(u_j) + P_{k_j}\varphi'(u_j), \quad \forall j \in \mathbb{N}. \tag{3.3}$$

Due to the compactness of φ' and $u_j \rightharpoonup u_1$, the right hand side of (3.3) converges strongly in E and hence $u_j \to u_1$ in E.

Step 2. We verify condition (H_1) in Theorem 2.1. Set $\beta_k = \sup_{u\in Z_k, \|u\|=1} \|u\|_2$, then $\beta_k \to 0$ as $k \to \infty$ since E is compactly embedded into $L^2(\mathbb{R}, \mathbb{R}^N)$. By (C_2), (C_3) and (C_4), we have

$$\begin{aligned}
\Psi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W\bigl(t, u(t)\bigr)\,dt \\
&\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \bigl(c_2(t)\bigl|u(t)\bigr|^{\gamma} + c_3(t)\bigl|u(t)\bigr|^{\sigma}\bigr)\,dt \\
&\quad - \int_{\mathbb{R}} \bigl(a(t)\bigl|u(t)\bigr| + c(t)\bigl|u(t)\bigr|^{\mu}\bigr)\,dt \\
&\geq \frac{1}{2}\|u\|^2 - \|c_2\|_{\frac{2}{2-\gamma}}\|u\|_2^{\gamma} - \|c_3\|_{\frac{2}{2-\sigma}}\|u\|_2^{\sigma} - \|a\|_2\|u\|_2 - \|c\|_{\infty}\|u\|_{\mu}^{\mu} \\
&\geq \frac{1}{2}\|u\|^2 - \beta_k^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}}\|u\|^{\gamma} - \beta_k^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}}\|u\|^{\sigma} - \beta_k\|a\|_2\|u\| \\
&\quad - \delta_{\mu}^{\mu}\|c\|_{\infty}\|u\|^{\mu}.
\end{aligned} \tag{3.4}$$

In view of (3.4), $\mu > 2$ and $\gamma, \sigma > 1$, one has

$$\Psi(u) \geq \frac{1}{4}\|u\|^2 - \bigl(\beta_k^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}} + \beta_k^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}} + \beta_k\|a\|_2\bigr)\|u\| \tag{3.5}$$

for $\|u\|$ small enough. Let $\rho_k = 8(\beta_k^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}} + \beta_k^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}} + \beta_k\|a\|_2)$, it is easy to see that $\rho_k \to 0$ as $k \to \infty$. Thus, for each sufficiently large $k \in \mathbb{N}$, by (3.5), we get

$$a_k \geq \frac{1}{8}\rho_k^2 > 0.$$

Step 3. We verify condition (H_3) in Theorem 2.1. By (3.5), for any $u \in Z_k$ with $\|u\| \leq \rho_k$, we have

$$\Psi(u) \geq -\bigl(\beta_k^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}} + \beta_k^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}} + \beta_k\|a\|_2\bigr)\|u\|. \tag{3.6}$$

Therefore, by (C_2), we obtain

$$0 \geq d_k \geq -\bigl(\beta_k^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}} + \beta_k^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}} + \beta_k\|a\|_2\bigr)\|u\|. \tag{3.7}$$

Since $\beta_k, \rho_k \to 0$ as $k \to \infty$, one has

$$d_k = \inf_{u \in Z_k, \|u\| \le \rho_k} \Psi(u) \to 0 \quad \text{as } k \to \infty.$$

Step 4. We verify condition (H_2) in Theorem 2.1. Firstly, we claim that there exists $\varepsilon > 0$ such that

$$\operatorname{meas}\{t \in \mathbb{R} : c_1(t)|u(t)|^\gamma \ge \varepsilon \|u\|^\gamma\} \ge \varepsilon, \quad \forall u \in Y_k \setminus \{0\}. \tag{3.8}$$

If not, there exists a sequence $\{u_n\} \subset Y_k$ with $\|u_n\| = 1$ such that

$$\operatorname{meas}\left\{t \in \mathbb{R} : c_1(t)|u_n(t)|^\gamma \ge \frac{1}{n}\right\} \le \frac{1}{n}. \tag{3.9}$$

Since $\dim Y_k < \infty$, it follows from the compactness of the unit sphere of Y_k that there exists a subsequence, say $\{u_n\}$, such that u_n converges to some u_0 in Y_k. Hence, we have $\|u_0\| = 1$. Since all norms are equivalent in the finite-dimensional space, we have $u_n \to u_0$ in $L^2(\mathbb{R}, \mathbb{R}^N)$. By the Hölder inequality, one has

$$\int_{\mathbb{R}} c_1(t)|u_n - u_0|^\gamma \, dt \le \|c_1\|_{\frac{2}{2-\gamma}} \left(\int_{\mathbb{R}} |u_n - u_0|^2 \, dt\right)^{\frac{\gamma}{2}} \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

Thus there exist $\epsilon_1, \epsilon_2 > 0$ such that

$$\operatorname{meas}\{t \in \mathbb{R} : c_1(t)|u_0(t)|^\gamma \ge \epsilon_1\} \ge \epsilon_2. \tag{3.11}$$

In fact, if not, we have

$$\operatorname{meas}\left\{t \in \mathbb{R} : c_1(t)|u_0(t)|^\gamma \ge \frac{1}{n}\right\} = 0, \tag{3.12}$$

for all positive integers n, which implies that

$$\int_{\mathbb{R}} c_1(t)|u_0(t)|^{\gamma+2} \, dt < \frac{1}{n}\|u_0\|_2^2 \le \frac{\delta_2^2}{n}\|u_0\|^2 = \frac{\delta_2^2}{n} \to 0$$

as $n \to \infty$. Hence $u_0 = 0$, which contradicts $\|u_0\| = 1$. Therefore, (3.11) holds. Thus, define

$$\Omega_0 = \{t \in \mathbb{R} : c_1(t)|u_0(t)|^\gamma \ge \epsilon_1\}, \qquad \Omega_n = \left\{t \in \mathbb{R} : c_1(t)|u_n(t)|^\gamma < \frac{1}{n}\right\}$$

and $\Omega_n^c = \mathbb{R} \setminus \Omega_n = \{t \in \mathbb{R} : c_1(t)|u_n(t)|^\gamma \ge \frac{1}{n}\}$. Combining (3.9) and (3.11), we have

$$\begin{aligned} \operatorname{meas}(\Omega_n \cap \Omega_0) &= \operatorname{meas}(\Omega_0 \setminus \Omega_n^c \cap \Omega_0) \\ &\ge \operatorname{meas}(\Omega_0) - \operatorname{meas}(\Omega_n^c \cap \Omega_0) \\ &\ge \epsilon_2 - \frac{1}{n} \end{aligned}$$

for all positive integers n. Let n be large enough such that $\epsilon_2 - \frac{1}{n} \geq \frac{1}{2}\epsilon_2$ and $\frac{1}{2^{\gamma-1}}\epsilon_1 - \frac{1}{n} \geq \frac{1}{2^\gamma}\epsilon_1$. Then we have

$$\begin{aligned}\int_{\mathbb{R}} c_1(t)|u_n - u_0|^\gamma\,dt &\geq \int_{\Omega_n\cap\Omega_0} c_1(t)|u_n - u_0|^\gamma\,dt \\ &\geq \frac{1}{2^{\gamma-1}}\int_{\Omega_n\cap\Omega_0} c_1(t)|u_0|^\gamma\,dt - \int_{\Omega_n\cap\Omega_0} c_1(t)|u_n|^\gamma\,dt \\ &\geq \left(\frac{1}{2^{\gamma-1}}\epsilon_1 - \frac{1}{n}\right)\operatorname{meas}(\Omega_n\cap\Omega_0) \\ &\geq \frac{\epsilon_1\epsilon_2}{2^{\gamma+1}}\end{aligned}$$

for all large n, which is a contradiction to (3.10). Therefore, (3.8) holds. For the ε given in (3.8), let

$$\Omega_u = \left\{t \in \mathbb{R} : c_1(t)\left|u(t)\right|^\gamma \geq \varepsilon\|u\|^\gamma\right\}, \quad \forall u \in Y_k \setminus \{0\}. \tag{3.13}$$

By (3.8), we obtain

$$\operatorname{meas}(\Omega_u) \geq \varepsilon, \quad \forall u \in Y_k \setminus \{0\}. \tag{3.14}$$

For any $u_k \in Y_k$, by (C_2), (C_3), (C_4), (3.13) and (3.14), we have

$$\begin{aligned}\Psi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W\big(t, u(t)\big)\,dt \\ &\leq \frac{1}{2}\|u\|^2 - \frac{1}{\gamma}\int_{\mathbb{R}} c_1(t)\left|u(t)\right|^\gamma\,dt - \int_{\mathbb{R}} G\big(t, u(t)\big)\,dt \\ &\leq \frac{1}{2}\|u\|^2 - \frac{1}{\gamma}\int_{\mathbb{R}} c_1(t)\left|u(t)\right|^\gamma\,dt \\ &\leq \frac{1}{2}\|u\|^2 - \frac{1}{\gamma}\int_{\Omega_u} c_1(t)\left|u(t)\right|^\gamma\,dt \\ &\leq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{\gamma}\|u\|^\gamma\operatorname{meas}(\Omega_u) \\ &\leq \frac{1}{2}\|u\|^2 - \frac{\varepsilon^2}{\gamma}\|u\|^\gamma.\end{aligned}$$

Choose $0 < r_k < \min\{\rho_k, (\varepsilon^2/\gamma)^{\frac{1}{2-\gamma}}\}$. Direct computation shows that

$$b_k \leq -\frac{r_k^2}{2} < 0, \quad \forall k \in \mathbb{N}.$$

Thus, by Theorem 2.1, Ψ has infinitely many nontrivial critical points, that is, system (1.1) possesses infinitely many homoclinic solutions. □

Now we give the proof of Theorem 1.4.

Proof of Theorem 1.4 Step 1. We prove that Ψ satisfies the $(PS)^*$ condition. Let $\{u_j\}$ be a $(PS)^*$ sequence, that is, $\{\Psi(u_j)\}$ is bounded, $u_j \in Y_{k_j}$ for some k_j with $k_j \to \infty$ and

$(\Psi|_{Y_{k_j}})'(u_j) \to 0$ as $j \to \infty$. Now we show that $\{u_j\}$ is bounded in E. In view of (C_1), (C_2), (C_3) and (C_6), for j large enough, we obtain

$$\begin{aligned}
\lambda M_4 + M_5\|u_j\| &\geq \lambda\Psi(u_j) - \Psi'(u_j)u_j \\
&= \left(\frac{\lambda}{2}-1\right)\|u_j\|^2 + \int_{\mathbb{R}} \left[\left(W_u(t,u_j),u_j\right) - \lambda W(t,u_j)\right] dt \\
&= \left(\frac{\lambda}{2}-1\right)\|u_j\|^2 + \int_{\mathbb{R}} \left[\left(F_u(t,u_j),u_j\right) - \lambda F(t,u_j)\right] dt \\
&\quad + \int_{\mathbb{R}} \left[\left(G_u(t,u_j),u_j\right) - \lambda G(t,u_j)\right] dt \\
&\geq \left(\frac{\lambda}{2}-1\right)\|u_j\|^2 - (\lambda+1)\int_{\mathbb{R}} \left(c_2(t)\left|u_j(t)\right|^{\gamma} + c_3\left|u_j(t)\right|^{\sigma}\right) dt \\
&\quad - h_1 \int_{\mathbb{R}} \left|u_j(t)\right|^2 dt \\
&\geq \left(\frac{\lambda}{2}-1\right)\|u_j\|^2 - (\lambda+1)\left(\|c_2\|_{\frac{2}{2-\gamma}}\|u_j\|_2^{\gamma} + \|c_3\|_{\frac{2}{2-\sigma}}\|u_j\|_2^{\sigma}\right) \\
&\quad - \frac{h_1}{\beta_0}\int_{\mathbb{R}} \left(L(t)u_j(t),u_j(t)\right) dt \\
&\geq \left(\frac{\lambda-2}{2} - \frac{h_1}{\beta_0}\right)\|u_j\|^2 \\
&\quad - (\lambda+1)\left(\delta_2^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}}\|u_j\|^{\gamma} + \delta_2^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}}\|u_j\|^{\sigma}\right)
\end{aligned} \tag{3.15}$$

for some $M_4 > 0$, $M_5 > 0$. Since $0 \leq h_1 < \frac{\beta_0(\lambda-2)}{2}$ and $1 < \gamma, \sigma < 2$, it follows that $\{u_j\}$ is bounded in E. In the following, the proof of the $(PS)^*$ condition is the same as that in Theorem 1.1, and we omit it here.

Step 2. We verify condition (H_1) in Theorem 2.1. Set $\beta_k = \sup_{u\in Z_k,\|u\|=1}\|u\|_2$, then $\beta_k \to 0$ as $k \to \infty$ since E is compactly embedded into $L^2(\mathbb{R},\mathbb{R}^N)$. By (C_8), we have

$$\left|G(t,u)\right| \leq h_2\left(|u| + |u|^{p_1}\right). \tag{3.16}$$

It follows from (C_7) that there exists $\delta_0 > 0$ such that

$$\left|G(t,u)\right| \leq \frac{\theta}{2}|u|^2, \quad \forall |u| \leq \delta_0, \forall t \in \mathbb{R}. \tag{3.17}$$

Combining (3.16) and (3.17), we get

$$\left|G(t,u)\right| \leq \frac{\theta}{2}|u|^2 + h_3|u|^{p_1}, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N, \tag{3.18}$$

where $h_3 = h_2(1 + \delta_0^{1-p_1})$. By (C_1), (C_2), (C_3) and (3.18), we have

$$\begin{aligned}
\Psi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W\left(t,u(t)\right) dt \\
&\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \left(c_2(t)\left|u(t)\right|^{\gamma} + c_3(t)\left|u(t)\right|^{\sigma}\right) dt - \int_{\mathbb{R}} \left(\frac{\theta}{2}\left|u(t)\right|^2 + h_3\left|u(t)\right|^{p_1}\right) dt
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2}\|u\|^2 - \|c_2\|_{\frac{2}{2-\gamma}}\|u\|_2^{\gamma} - \|c_3\|_{\frac{2}{2-\sigma}}\|u\|_2^{\sigma} - h_3\|u\|_{p_1}^{p_1} \\
&\quad - \frac{\theta}{2\beta_0}\int_{\mathbb{R}}\bigl(L(t)u_j(t), u_j(t)\bigr)\,dt \\
&\geq \left(\frac{1}{2} - \frac{\theta}{2\beta_0}\right)\|u\|^2 - \beta_k^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}}\|u\|^{\gamma} - \beta_k^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}}\|u\|^{\sigma} - h_3\delta_{p_1}^{p_1}\|u\|^{p_1}. \qquad (3.19)
\end{aligned}$$

Take $\theta_0 = \frac{1}{2} - \frac{\theta}{2\beta_0}$, by ($C_7$), we obtain $\theta_0 > 0$. By virtue of (3.19), $p_1 > 2$ and $\gamma, \sigma > 1$, one has

$$\Psi(u) \geq \frac{\theta_0}{2}\|u\|^2 - \bigl(\beta_k^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}} + \beta_k^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}}\bigr)\|u\| \qquad (3.20)$$

for $\|u\|$ small enough. Let $\rho_k = \frac{\theta_0}{4}(\beta_k^{\gamma}\|c_2\|_{\frac{2}{2-\gamma}} + \beta_k^{\sigma}\|c_3\|_{\frac{2}{2-\sigma}})$, it is easy to see that $\rho_k \to 0$ as $k \to \infty$. Thus, for each sufficiently large $k \in \mathbb{N}$, by (3.20), we get

$$a_k \geq \frac{\theta_0}{4}\rho_k^2 > 0.$$

Step 3. We verify condition (H_3) in Theorem 2.1. The proof is similar to the Step 3 in the proof of Theorem 1.3, and we omit it.

Step 4. We verify condition (H_2) in Theorem 2.1. The proof is the same as that the Step 4 in the proof of Theorem 1.3, and we omit it here.

Thus, by Theorem 2.1, Ψ has infinitely many nontrivial critical points, that is, system (1.1) possesses infinitely many homoclinic solutions. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the referees for their valuable suggestions and comments which lead to improvement of the manuscript.

Received: 5 February 2014 Accepted: 12 May 2014 Published: 02 Jun 2014

References
1. Mawhin, J, Willem, M: Critical Point Theory and Hamiltonian Systems. Springer, Berlin (1989)
2. Rabinowitz, PH: Variational methods for Hamiltonian systems. In: Handbook of Dynamical Systems, part 1, Chapter 14, vol. 1, pp. 1091-1127. North-Holland, Amsterdam (2002)
3. Bonanno, G, Livrea, R: Multiple periodic solutions for Hamiltonian systems with not coercive potential. J. Math. Anal. Appl. **363**, 627-638 (2010)
4. Caldiroli, P, Montecchiari, P: Homoclinic orbits for second order Hamiltonian systems with potential changing sign. Commun. Appl. Nonlinear Anal. **1**(2), 97-129 (1994)
5. Chen, P, Tang, XH, Agarwal, RP: Infinitely many homoclinic solutions for nonautonomous *p*(*t*)-Laplacian Hamiltonian systems. Comput. Math. Appl. **63**, 751-763 (2012)
6. Cordaro, G, Rao, G: Three periodic solutions for perturbed second order Hamiltonian systems. J. Math. Anal. Appl. **359**, 780-785 (2009)
7. Coti Zelati, V, Rabinowitz, PH: Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials. J. Am. Math. Soc. **4**, 693-727 (1991)
8. Ding, Y: Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems. Nonlinear Anal. **25**, 1095-1113 (1995)
9. Ding, Y, Lee, C: Homoclinics for asymptotically quadratic and superquadratic Hamiltonian systems. Nonlinear Anal. **71**, 1395-1413 (2009)
10. Guo, C, Agarwal, RP, Wang, C, O'Regan, D: The existence of homoclinic orbits for a class of first-order superquadratic Hamiltonian systems. Mem. Differ. Equ. Math. Phys. **61**, 83-102 (2014)
11. Korman, P, Lazer, AC: Homoclinic orbits for a class of symmetric Hamiltonian systems. Electron. J. Differ. Equ. **1994**, 1-10 (1994)

12. Lv, X, Lu, S, Jiang, J: Homoclinic solutions for a class of second-order Hamiltonian systems. Nonlinear Anal., Real World Appl. **13**, 176-185 (2012)
13. Omana, W, Willem, M: Homoclinic orbits for a class of Hamiltonian systems. Differ. Integral Equ. **5**, 1115-1120 (1992)
14. Paturel, E: Multiple homoclinic orbits for a class of Hamiltonian systems. Calc. Var. Partial Differ. Equ. **12**, 117-143 (2001)
15. Qu, Z, Tang, C: Existence of homoclinic solutions for the second order Hamiltonian systems. J. Math. Anal. Appl. **291**, 203-213 (2004)
16. Rabinowitz, PH: Homoclinic orbits for a class of Hamiltonian systems. Proc. R. Soc. Edinb., Sect. A **114**, 33-38 (1990)
17. Rabinowitz, PH, Tanaka, K: Some results on connecting orbits for a class of Hamiltonian systems. Math. Z. **206**, 473-499 (1991)
18. Sun, J, Chen, H, Nieto, JJ: Homoclinic solutions for a class of subquadratic second-order Hamiltonian systems. J. Math. Anal. Appl. **373**, 20-29 (2011)
19. Tang, X, Lin, X: Existence of infinitely many homoclinic orbits in Hamiltonian systems. Proc. R. Soc. Edinb., Sect. A **141**, 1103-1119 (2011)
20. Tang, X, Lin, X: Homoclinic solutions for a class of second-order Hamiltonian systems. J. Math. Anal. Appl. **354**, 539-549 (2009)
21. Wang, J, Xu, J, Zhang, F: Homoclinic orbits for a class of Hamiltonian systems with superquadratic or asymptotically quadratic potentials. Commun. Pure Appl. Anal. **10**, 269-286 (2011)
22. Wei, J, Wang, J: Infinitely many homoclinic orbits for second order Hamiltonian systems with general potentials. J. Math. Anal. Appl. **366**, 694-699 (2010)
23. Yang, L, Chen, H, Sun, J: Infinitely many homoclinic solutions for some second order Hamiltonian systems. Nonlinear Anal. **74**, 6459-6468 (2011)
24. Yang, M, Han, Z: Infinitely many homoclinic solutions for second-order Hamiltonian systems with odd nonlinearities. Nonlinear Anal. **74**, 2635-2646 (2011)
25. Yang, J, Zhang, F: Infinitely many homoclinic orbits for the second-order Hamiltonian systems with super-quadratic potentials. Nonlinear Anal., Real World Appl. **10**, 1417-1423 (2009)
26. Yu, JS: Subharmonic solutions with prescribed minimal period of a class of nonautonomous Hamiltonian systems. J. Dyn. Differ. Equ. **20**, 787-796 (2008)
27. Zhang, Q, Liu, C: Infinitely many homoclinic solutions for second order Hamiltonian systems. Nonlinear Anal. **72**, 894-903 (2010)
28. Zhang, Z, Yuan, R: Homoclinic solutions of some second order non-autonomous systems. Nonlinear Anal. **71**, 5790-5798 (2009)
29. Zhang, Z, Yuan, R: Homoclinic solutions for a class of non-autonomous subquadratic second-order Hamiltonian systems. Nonlinear Anal. **71**, 4125-4130 (2009)
30. Zou, W, Li, S: Infinitely many homoclinic orbits for the second-order Hamiltonian systems. Appl. Math. Lett. **16**, 1283-1287 (2003)
31. Ricceri, B: A three critical points theorem revisited. Nonlinear Anal. **70**, 3084-3089 (2009)
32. Ricceri, B: A further three critical points theorem. Nonlinear Anal. **71**, 4151-4157 (2009)
33. Chen, HW, He, ZM: New results for perturbed Hamiltonian systems with impulses. Appl. Math. Comput. **218**, 9489-9497 (2012)
34. Sun, J, Chen, H, Nieto, JJ, Otero-Novoa, M: The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects. Nonlinear Anal. **72**, 4575-4586 (2010)
35. Bartsch, T, Willem, M: On an elliptic equation with concave and convex nonlinearities. Proc. Am. Math. Soc. **123**, 3555-3561 (1995)

10.1186/1687-1847-2014-161
Cite this article as: Chen and He: **Infinitely many homoclinic solutions for a class of second-order Hamiltonian systems.** *Advances in Difference Equations* 2014, 2014:161