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Multiple periodic solutions in shifts $\delta_\pm$ for an impulsive functional dynamic equation on time scales

Meng Hu* and Lili Wang

*Correspondence:
humeng2001@126.com
School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455000, People's Republic of China

Abstract

In this paper, based on the theory of calculus on time scales, by using a multiple fixed point theorem in cones, some criteria are established for the existence and multiplicity of positive periodic solutions in shifts $\delta_\pm$ for an impulsive functional dynamic equation on time scales of the following form: $x^\Delta(t) = -a(t)x(t) + b(t)f(t, x(g(t)))$, $t \neq t_j$, $t \in \mathbb{T}$, $x(t_j^+) = x(t_j^-) + I_j(x(t_j))$, where $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed. Finally, some numerical examples are presented to illustrate the feasibility and effectiveness of the results.

MSC: 34K13; 34K45; 34N05

Keywords: periodic solution; functional dynamic equation; impulse; shift operator; time scale

1 Introduction

The time scales approach unifies differential, difference, h-difference, and q-differences equations and more under dynamic equations on time scales. The theory of dynamic equations on time scales was introduced by Hilger in his PhD thesis in 1988 [1]. The existence problem of periodic solutions is an important topic in qualitative analysis of functional dynamic equations. Up to now, there are only a few results concerning periodic solutions of dynamic equations on time scales; see, for example, [2, 3]. In these papers, authors considered the existence of periodic solutions for dynamic equations on time scales satisfying the condition 'there exists a $\omega > 0$ such that $t \pm \omega \in \mathbb{T}$, $\forall t \in \mathbb{T}$'. Under this condition all periodic time scales are unbounded above and below. However, there are many time scales such as $\overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$ which do not satisfy the condition. Adıvar and Raffoul introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. They defined a new periodicity concept with the aid of shift operators $\delta_\pm$ which are first defined in [4] and then generalized in [5].

Recently, based on a fixed point theorem in cones, Çetin *et al.* studied the existence of positive periodic solutions in shifts $\delta_\pm$ for some nonlinear first-order functional dynamic equation on time scales; see [6, 7].

However, to the best of our knowledge, there are few papers published on the existence of positive periodic solutions in shifts $\delta_\pm$ for a functional dynamic equation with impulses.

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As we know, impulsive functional dynamic equation on time scales plays an important role in applications; see, for example, [8, 9].

Motivated by the above, in the present paper, we consider the following system:

$$\begin{cases} x^{\Delta}(t) = -a(t)x(t) + b(t)f(t, x(g(t))), & t \neq t_j, t \in \mathbb{T}, \\ x(t_j^+) = x(t_j^-) + I_j(x(t_j)), \end{cases} \tag{1.1}$$

where $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_{\pm}$ with period $P \in [t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed; $a, b \in C(\mathbb{T}, (0, \infty))$ are Δ-periodic in shifts $\delta_{\pm}$ with period ω and $-a \in \mathcal{R}^+$; $f \in C(\mathbb{T} \times (0, \infty), (0, \infty))$ is periodic in shifts $\delta_{\pm}$ with period ω with respect to the first variable; $g \in C(\mathbb{T}, \mathbb{T})$ is periodic in shifts $\delta_{\pm}$ with period ω; $x(t_j^+)$ and $x(t_j^-)$ represent the right and the left limit of $x(t_j)$ in the sense of time scales, in addition, if t_j is right-scattered, then $x(t_j^+) = x(t_j)$, whereas, if t_j is left-scattered, then $x(t_j^-) = x(t_j)$; $I_j \in C((0, \infty), [0, \infty))$, $j \in \mathbb{Z}$. Assume that there exists a positive constant q such that $t_{j+q} = \delta_+^{\omega}(t_j)$, $I_{j+q} = I_j$, $j \in \mathbb{Z}$. For each interval $\mathbb{I}$ of $\mathbb{R}$, we denote $\mathbb{I}_{\mathbb{T}} = \mathbb{I} \cap \mathbb{T}$, without loss of generality, set $[t_0, \delta_+^{\omega}(t_0))_{\mathbb{T}} \cap \{t_j, j \in \mathbb{Z}\} = \{t_1, t_2, \dots, t_q\}$.

The main purpose of this paper is to establish some sufficient conditions for the existence of at least three positive periodic solutions in shifts $\delta_{\pm}$ of system (1.1) using a multiple fixed point theorem (Avery-Peterson fixed point theorem) in cones.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions and state some preliminary results needed in later sections; then we give the Green's function of system (1.1), which plays an important role in this paper. In Section 3, we establish our main results for positive periodic solutions in shifts $\delta_{\pm}$ by applying Avery-Peterson fixed point theorem. In Section 4, some numerical examples are presented to illustrate that our results are feasible and more general.

2 Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T}) = C(\mathbb{T}, \mathbb{R})$.

For the basic theories of calculus on time scales, see [10].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}\big(r(\tau)\big) \Delta\tau \right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu pq, \qquad \ominus p = -\frac{p}{1 + \mu p}, \qquad p \ominus q = p \oplus (\ominus q).$$

Lemma 2.1 [10] *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $(e_{\ominus p}(t, s))^{\Delta} = (\ominus p)(t)e_{\ominus p}(t, s)$.

The following definitions and lemmas about the shift operators and the new periodicity concept for time scales can be found in [7, 11].

Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ be a fixed number, define operators $\delta_{\pm} : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_{\mathbb{T}}$ in $\delta_{\pm}(s, t)$ is called the shift size. The values $\delta_+(s, t)$ and $\delta_-(s, t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets

$$\mathbb{D}_{\pm} := \left\{(s, t) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_{\mp}(s, t) \in \mathbb{T}^*\right\}$$

are the domains of the shift operator $\delta_{\pm}$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_{\pm} : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 2.1 (Periodicity in shifts $\delta_{\pm}$ [11]) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_{\pm}$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_{\pm}$ if there exists $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p, t) \in \mathbb{D}_{\pm}$ for all $t \in \mathbb{T}^*$. Furthermore, if

$$P := \inf\left\{p \in (t_0, \infty)_{\mathbb{T}^*} : (p, t) \in \delta_{\pm}, \forall t \in \mathbb{T}^*\right\} \neq t_0,$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 2.2 (Periodic function in shifts $\delta_{\pm}$ [11]) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_{\pm}$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_{\pm}$ and $f(\delta_{\pm}^{\omega}(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_{\pm}^{\omega} := \delta_{\pm}(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.

Definition 2.3 (Δ-periodic function in shifts $\delta_{\pm}$ [11]) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_{\pm}$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_{\pm}$ for all $t \in \mathbb{T}^*$, the shifts $\delta_{\pm}^{\omega}$ are Δ-differentiable with rd-continuous derivatives and $f(\delta_{\pm}^{\omega}(t))\delta_{\pm}^{\Delta\omega}(t) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_{\pm}^{\omega} := \delta_{\pm}(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.

Lemma 2.2 [11] *$\delta_+^\omega(\sigma(t)) = \sigma(\delta_+^\omega(t))$ and $\delta_-^\omega(\sigma(t)) = \sigma(\delta_-^\omega(t))$ for all $t \in \mathbb{T}^*$.*

Lemma 2.3 [7] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^\omega$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P, \infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period ω. Then*

(i) *$e_p(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) = e_p(t, t_0)$ for $t, t_0 \in \mathbb{T}^*$;*
(ii) *$e_p(\delta_\pm^\omega(t), \sigma(\delta_\pm^\omega(s))) = e_p(t, \sigma(s)) = \frac{e_p(t,s)}{1+\mu(t)p(t)}$ for $t, s \in \mathbb{T}^*$.*

Lemma 2.4 [11] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P, and let f be a Δ-periodic function in shifts $\delta_\pm$ with the period $\omega \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{\mathrm{rd}}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_\pm^\omega(t_0)}^{\delta_\pm^\omega(t)} f(s)\Delta s.$$

Lemma 2.5 [10] *Suppose that r is regressive and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Let $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$, then the unique solution of the initial value problem*

$$y^\Delta = r(t)y + f(t), \qquad y(t_0) = y_0$$

is given by

$$y(t) = e_r(t, t_0)y_0 + \int_{t_0}^{t} e_r\bigl(t, \sigma(\tau)\bigr)f(\tau)\Delta\tau.$$

Define

$$PC(\mathbb{T}) = \bigl\{x : \mathbb{T} \to \mathbb{R} \mid x|_{(t_j, t_{j+1})} \in C(t_j, t_{j+1}), \exists x\bigl(t_j^-\bigr) = x(t_j), x\bigl(t_j^+\bigr), j \in \mathbb{Z}\bigr\}.$$

Set

$$X = \bigl\{x : x \in PC(\mathbb{T}), x\bigl(\delta_+^\omega(t)\bigr) = x(t)\bigr\}$$

with the norm $\|x\| = \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} |x(t)|$, then X is a Banach space.

Lemma 2.6 *$x(t) \in X$ is an ω-periodic solution in shifts $\delta_\pm$ of system* (1.1) *if and only if $x(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of*

$$x(t) = \int_{t}^{\delta_+^\omega(t)} G(t,s)b(s)f\bigl(s, x\bigl(g(s)\bigr)\bigr)\Delta s + \sum_{j:t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} G(t, t_j)e_{-a}\bigl(\sigma(t_j), t_j\bigr)I_j\bigl(x(t_j)\bigr), \tag{2.1}$$

where

$$G(t,s) = \frac{e_{-a}(t, \sigma(s))}{e_{-a}(t_0, \delta_+^\omega(t_0)) - 1}.$$

Proof If $x(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of system (1.1). For any $t \in \mathbb{T}$, there exists $j \in \mathbb{Z}$ such that t_j is the first impulsive point after t. By using Lemma 2.5, for $s \in [t, t_j]_{\mathbb{T}}$, we have

$$x(s) = e_{-a}(s,t)x(t) + \int_t^s e_{-a}\big(s,\sigma(\theta)\big)b(\theta)f\big(\theta,x\big(g(\theta)\big)\big)\Delta\theta,$$

then

$$x(t_j) = e_{-a}(t_j,t)x(t) + \int_t^{t_j} e_{-a}\big(t_j,\sigma(\theta)\big)b(\theta)f\big(\theta,x\big(g(\theta)\big)\big)\Delta\theta. \tag{2.2}$$

Again using Lemma 2.5 and (2.2), for $s \in (t_j,t_{j+1}]_{\mathbb{T}}$, then

$$\begin{aligned} x(s) &= e_{-a}(s,t_j)x\big(t_j^+\big) + \int_{t_j}^s e_{-a}\big(s,\sigma(\theta)\big)b(\theta)f\big(\theta,x\big(g(\theta)\big)\big)\Delta\theta \\ &= e_{-a}(s,t_j)x(t_j) + \int_{t_j}^s e_{-a}\big(s,\sigma(\theta)\big)b(\theta)f\big(\theta,x\big(g(\theta)\big)\big)\Delta\theta + e_{-a}(s,t_j)I_j\big(x(t_j)\big) \\ &= e_{-a}(s,t)x(t) + \int_t^s e_{-a}\big(s,\sigma(\theta)\big)b(\theta)f\big(\theta,x\big(g(\theta)\big)\big)\Delta\theta + e_{-a}(s,t_j)I_j\big(x(t_j)\big). \end{aligned}$$

Repeating the above process for $s \in [t,\delta_+^\omega(t)]_{\mathbb{T}}$, we have

$$x(s) = e_{-a}(s,t)x(t) + \int_t^s e_{-a}\big(s,\sigma(\theta)\big)b(\theta)f\big(\theta,x\big(g(\theta)\big)\big)\Delta\theta + \sum_{j:t_j\in[t,s)_{\mathbb{T}}} e_{-a}(s,t_j)I_j\big(x(t_j)\big).$$

Let $s = \delta_+^\omega(t)$ in the above equality, we have

$$\begin{aligned} x\big(\delta_+^\omega(t)\big) &= e_{-a}\big(\delta_+^\omega(t),t\big)x(t) + \int_t^{\delta_+^\omega(t)} e_{-a}\big(\delta_+^\omega(t),\sigma(\theta)\big)b(\theta)f\big(\theta,x\big(g(\theta)\big)\big)\Delta\theta \\ &\quad + \sum_{j:t_j\in[t,\delta_+^\omega(t))_{\mathbb{T}}} e_{-a}\big(\delta_+^\omega(t),t_j\big)I_j\big(x(t_j)\big). \end{aligned}$$

Noticing that $x(\delta_+^\omega(t)) = x(t)$, $e_{-a}(t,\delta_+^\omega(t)) = e_{-a}(t_0,\delta_+^\omega(t_0))$, by Lemma 2.1, then x satisfies (2.1).

Let x be an ω-periodic solution in shifts $\delta_\pm$ of (2.1). If $t \neq t_i$, $i \in \mathbb{Z}$, then by (2.1) and Lemma 2.2, we have

$$\begin{aligned} x^\Delta(t) &= -a(t)x(t) + G\big(\sigma(t),\delta_+^\omega(t)\big)b\big(\delta_+^\omega(t)\big)\delta_+^{\Delta\omega}(t)f\big(\delta_+^\omega(t),x\big(g\big(\delta_+^\omega(t)\big)\big)\big) \\ &\quad - G\big(\sigma(t),t\big)b(t)f\big(t,x\big(g(t)\big)\big) \\ &= -a(t)x(t) + b(t)f\big(t,x\big(g(t)\big)\big). \end{aligned}$$

If $t = t_i$, $i \in \mathbb{Z}$, then by (2.1), we have

$$\begin{aligned} x\big(t_i^+\big) - x\big(t_i^-\big) &= \sum_{j:t_j\in[t_i^+,\delta_+^\omega(t_i^+))_{\mathbb{T}}} G(t_i,t_j)e_{-a}\big(\sigma(t_j),t_j\big)I_j\big(x(t_j)\big) \\ &\quad - \sum_{j:t_j\in[t_i^-,\delta_+^\omega(t_i^-))_{\mathbb{T}}} G(t_i,t_j)e_{-a}\big(\sigma(t_j),t_j\big)I_j\big(x(t_j)\big) \\ &= G\big(t_i,\delta_+^\omega(t_i)\big)e_{-a}\big(\sigma\big(\delta_+^\omega(t_i)\big),\delta_+^\omega(t_i)\big)I_i\big(x\big(\delta_+^\omega(t_i)\big)\big) \end{aligned}$$

$$
\begin{aligned}
&- G(t_i, t_i)e_{-a}\big(\sigma(t_i), t_i\big)I_i\big(x(t_i)\big) \\
&= I_i\big(x(t_i)\big).
\end{aligned}
$$

So, x is an ω-periodic solution in shifts $\delta_\pm$ of system (1.1). This completes the proof. □

It is easy to verify that the Green's function $G(t,s)$ satisfies the property

$$
0 < \frac{1}{\xi - 1} \le G(t,s) \le \frac{\xi}{\xi - 1}, \quad \forall s \in \big[t, \delta_+^\omega(t)\big]_{\mathbb{T}}, \tag{2.3}
$$

where $\xi = e_{-a}(t_0, \delta_+^\omega(t_0))$. By Lemma 2.3, we have

$$
G\big(\delta_+^\omega(t), \delta_+^\omega(s)\big) = G(t,s), \quad \forall t \in \mathbb{T}^*, s \in \big[t, \delta_+^\omega(t)\big]_{\mathbb{T}}. \tag{2.4}
$$

In order to obtain the existence of periodic solutions in shifts $\delta_\pm$ of system (1.1), we need the following concepts and Avery-Peterson fixed point theorem.

Let X be a Banach space and K be a cone in X, define $K_r = \{x \in K \mid \|x\| \le r\}$. A map α is said to be a nonnegative continuous concave functional on K if $\alpha : K \to [0, +\infty)$ is continuous and

$$
\alpha\big(\lambda x + (1-\lambda)y\big) \ge \lambda\alpha(x) + (1-\lambda)\alpha(y) \quad \text{for all } x, y \in K, 0 < \lambda < 1.
$$

Let γ and θ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K. Then for positive real numbers a, b, c, and d, we define the following convex sets:

$$
\begin{aligned}
&K(\gamma, d) = \big\{x \in K \mid \gamma(x) < d\big\}, \\
&K(\gamma, \alpha, b, d) = \big\{x \in K \mid b \le \alpha(x), \gamma(x) \le d\big\}, \\
&K(\gamma, \theta, \alpha, b, c, d) = \big\{x \in K \mid b \le \alpha(x), \theta(x) \le c, \gamma(x) \le d\big\},
\end{aligned}
$$

and a closed set $R(\gamma, \psi, a, d) = \{x \in K \mid a \le \psi(x), \gamma(x) \le d\}$.

Lemma 2.7 (Avery-Peterson fixed point theorem [12]) *Let γ and θ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K satisfying $\psi(\rho x) \le \rho\psi(x)$ for $0 \le \rho \le 1$, such that for some positive numbers E and d,*

$$
\alpha(x) \le \psi(x) \quad \textit{and} \quad \|x\| \le E\gamma(x) \tag{2.5}
$$

for all $x \in \overline{K(\gamma, d)}$. Suppose $H : \overline{K(\gamma, d)} \to \overline{K(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that:

1. *$\{x \in K(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq \emptyset$ and $\alpha(Hx) > b$ for $x \in K(\gamma, \theta, \alpha, b, c, d)$,*
2. *$\alpha(Hx) > b$, for $x \in K(\gamma, \alpha, b, d)$ with $\theta(Hx) > c$,*
3. *$0 \,\overline{\in}\, R(\gamma, \psi, a, d)$ and $\psi(Hx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.*

Then H has at least three fixed points $x_1, x_2, x_3 \in \overline{K(\gamma, d)}$ such that

$$\gamma(x_i) \le d \quad \text{for } i = 1, 2, 3, \qquad b < \alpha(x_1),$$
$$a < \psi(x_2), \quad \text{with } \alpha(x_2) < b \quad \text{and} \quad \psi(x_3) < a.$$

Define K, a cone in X, by

$$K = \left\{ x \in X : x(t) \ge \frac{1}{\xi}\|x\|, \forall t \in \left[t_0, \delta_+^\omega(t_0)\right]_{\mathbb{T}} \right\} \tag{2.6}$$

and an operator $H : K \to X$ by

$$\begin{aligned}(Hx)(t) = {} & \int_t^{\delta_+^\omega(t)} G(t,s)b(s)f\bigl(s, x\bigl(g(s)\bigr)\bigr)\Delta s \\ & + \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} G(t, t_j)e_{-a}\bigl(\sigma(t_j), t_j\bigr)I_j\bigl(x(t_j)\bigr).\end{aligned} \tag{2.7}$$

For convenience, we denote

$$A^u := \sup_{t \in [t_0, \delta_+^\omega(t_0))_{\mathbb{T}}} a(t), \qquad B^u := \sup_{t \in [t_0, \delta_+^\omega(t_0))_{\mathbb{T}}} b(t), \qquad B := \int_{t_0}^{\delta_+^\omega(t_0)} b(s)\Delta s.$$

In the following, we shall give some lemmas concerning K and H defined by (2.6) and (2.7), respectively.

Lemma 2.8 *$H : K \to K$ is well defined.*

Proof For any $x \in K$, it is clear that $Hx \in PC(\mathbb{T})$. In view of (2.7), by Lemma 2.4 and (2.4), for $t \in \mathbb{T}$, we have

$$\begin{aligned}(Hx)\bigl(\delta_+^\omega(t)\bigr) &= \int_{\delta_+^\omega(t)}^{\delta_+^\omega(\delta_+^\omega(t))} G\bigl(\delta_+^\omega(t), s\bigr)b(s)f\bigl(s, x\bigl(g(s)\bigr)\bigr)\Delta s \\ &\quad + \sum_{j: t_j \in [\delta_+^\omega(t), \delta_+^\omega(\delta_+^\omega(t)))_{\mathbb{T}}} G\bigl(\delta_+^\omega(t), t_j\bigr)e_{-a}\bigl(\sigma(t_j), t_j\bigr)I_j\bigl(x(t_j)\bigr) \\ &= \int_t^{\delta_+^\omega(t)} G\bigl(\delta_+^\omega(t), \delta_+^\omega(s)\bigr)b\bigl(\delta_+^\omega(s)\bigr)\delta_+^{\Delta\omega}(s)f\bigl(\delta_+^\omega(s), x\bigl(g\bigl(\delta_+^\omega(s)\bigr)\bigr)\bigr)\Delta s \\ &\quad + \sum_{k: t_k \in [t, \delta_+^\omega(t))_{\mathbb{T}}} G\bigl(\delta_+^\omega(t), \delta_+^\omega(t_k)\bigr)e_{-a}\bigl(\sigma\bigl(\delta_+^\omega(t_k)\bigr), \delta_+^\omega(t_k)\bigr)I_k\bigl(x\bigl(\delta_+^\omega(t_k)\bigr)\bigr) \\ &= \int_t^{\delta_+^\omega(t)} G(t,s)b(s)f\bigl(s, x\bigl(g(s)\bigr)\bigr)\Delta s \\ &\quad + \sum_{k: t_k \in [t, \delta_+^\omega(t))_{\mathbb{T}}} G(t, t_k)e_{-a}\bigl(\sigma(t_k), t_k\bigr)I_k\bigl(x(t_k)\bigr) \\ &= (Hx)(t),\end{aligned}$$

that is, $Hx \in X$.

Furthermore, for any $x \in K$, $\forall t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, we have

$$\begin{aligned}
(Hx)(t) &\geq \frac{1}{\xi - 1} \int_t^{\delta_+^\omega(t)} b(s) f\big(s, x\big(g(s)\big)\big) \Delta s + \frac{1}{\xi - 1} \sum_{j=1}^{q} I_j\big(x(t_j)\big) \\
&= \frac{1}{\xi} \left[\frac{\xi}{\xi - 1} \int_{t_0}^{\delta_+^\omega(t_0)} b(s) f\big(s, x\big(g(s)\big)\big) \Delta s + \frac{\xi}{\xi - 1} \sum_{j=1}^{q} I_j\big(x(t_j)\big) \right] \\
&\geq \frac{1}{\xi} \|Hx\|,
\end{aligned}$$

that is, $Hx \in K$. This completes the proof. □

Lemma 2.9 *$H : K \to K$ is completely continuous.*

Proof Firstly, we show that H is continuous. Because of the continuity of f and I_j, $j \in \mathbb{Z}$, for any $\nu > 0$ and $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$\big\{\phi, \psi \in C\big(\mathbb{T}, (0, \infty)\big), \|\phi\| \leq \nu, \|\psi\| \leq \nu, \|\phi - \psi\| < \delta_0\big\}$$

imply

$$\big|f\big(s, \phi\big(g(s)\big)\big) - f\big(s, \psi\big(g(s)\big)\big)\big| < \frac{(\xi - 1)\varepsilon}{2\xi B}$$

and

$$\big|I_j(\phi) - I_j(\psi)\big| < \frac{(\xi - 1)\varepsilon}{2\xi q}, \quad j \in \mathbb{Z}.$$

Therefore, if $x, y, \in K$ with $\|x\| \leq \nu$, $\|y\| \leq \nu$, $\|x - y\| < \delta_0$, then

$$\begin{aligned}
&\big|(Hx)(t) - (Hy)(t)\big| \\
&\quad\leq \bigg| \int_t^{\delta_+^\omega(t)} G(t,s) b(s) f\big(s, x\big(g(s)\big)\big) \Delta s + \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} G(t, t_j) e_{-a}\big(\sigma(t_j), t_j\big) I_j\big(x(t_j)\big) \\
&\qquad - \int_t^{\delta_+^\omega(t)} G(t,s) b(s) f\big(s, y\big(g(s)\big)\big) \Delta s - \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} G(t, t_j) e_{-a}\big(\sigma(t_j), t_j\big) I_j\big(y(t_j)\big) \bigg| \\
&\quad\leq \frac{\xi}{\xi - 1} \int_t^{\delta_+^\omega(t)} b(s) \big|f\big(s, x\big(g(s)\big)\big) - f\big(s, y\big(g(s)\big)\big)\big| \Delta s \\
&\qquad + \frac{\xi}{\xi - 1} \sum_{j=1}^{q} \big|I_j\big(x(t_j)\big) - I_j\big(y(t_j)\big)\big| \\
&\quad< \frac{\xi}{\xi - 1} \left(B \frac{(\xi - 1)\varepsilon}{2\xi B} + q \frac{(\xi - 1)\varepsilon}{2\xi q} \right) \\
&\quad= \varepsilon,
\end{aligned}$$

which yields $\|Hx - Hy\| = \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} |(Hx)(t) - (Hy)(t)| \leq \varepsilon$, that is, H is continuous.

Next, we show that H maps any bounded sets in K into relatively compact sets. We first prove that f maps bounded sets into bounded sets. Indeed, let $\varepsilon = 1$, for any $\nu > 0$, there exists $\delta_0 > 0$ such that $\{x, y \in K, \|x\| \leq \nu, \|y\| \leq \nu, \|x - y\| < \delta_0, s \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}\}$ imply

$$\left|f\left(s, x\left(g(s)\right)\right) - f\left(s, y\left(g(s)\right)\right)\right| < 1$$

and

$$\left|I_j\left(x(t_j)\right) - I_j\left(y(t_j)\right)\right| < 1, \quad j \in \mathbb{Z}.$$

Choose a positive integer N such that $\frac{\nu}{N} < \delta_0$. Let $x \in K$ and define $x^k(\cdot) = \frac{x(\cdot)k}{N}$, $k = 0, 1, 2, \ldots, N$. If $\|x\| < \nu$, then

$$\left\|x^k - x^{k-1}\right\| = \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} \left|\frac{x(\cdot)k}{N} - \frac{x(\cdot)(k-1)}{N}\right| \leq \|x\| \frac{1}{N} \leq \frac{\nu}{N} < \delta_0.$$

Thus

$$\left|f\left(s, x^k\left(g(s)\right)\right) - f\left(s, x^{k-1}\left(g(s)\right)\right)\right| < 1$$

for all $s \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, and

$$\left|I_j\left(x^k(t_j)\right) - I_j\left(x^k(t_j)\right)\right| < 1, \quad j \in \mathbb{Z},$$

and these yield

$$\begin{aligned} f\left(s, x\left(g(s)\right)\right) &= f\left(s, x^N\left(g(s)\right)\right) \\ &\leq \sum_{k=1}^{N} \left|f\left(s, x^k\left(g(s)\right)\right) - f\left(s, x^{k-1}\left(g(s)\right)\right)\right| + f(s, 0) \\ &< N + \sup_{s \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} f(s, 0) =: W \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} I_j\left(x(t_j)\right) = I_j\left(x^N(t_j)\right) &\leq \sum_{k=1}^{N} \left|I_j\left(x^N(t_j)\right) - I_j\left(x^{N-1}(t_j)\right)\right| + I_j(0) \\ &< N + \max_{1 \leq j \leq q} I_j(0) =: U, \quad j \in \mathbb{Z}. \end{aligned} \tag{2.9}$$

It follows from (2.7), (2.8), and (2.9) that for $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$,

$$\begin{aligned} \|Hx\| &= \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} (Hx)(t) \\ &\leq \frac{\xi}{\xi - 1} \left(\int_{t_0}^{\delta_+^\omega(t_0)} b(s) f\left(s, x\left(g(s)\right)\right) \Delta s + \sum_{j=1}^{q} I_j\left(x(t_j)\right) \right) \\ &< \frac{\xi}{\xi - 1} (BW + qU) := D. \end{aligned}$$

Furthermore, for $t \in \mathbb{T}$, we have

$$(Hx)^{\Delta}(t) = -a(t)(Hx)(t) + b(t)f\bigl(t, x\bigl(g(t)\bigr)\bigr)$$

and

$$\bigl\|(Hx)^{\Delta}(t)\bigr\| \le \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} \bigl|-a(t)(Hx)(t) + b(t)f\bigl(t, x\bigl(g(t)\bigr)\bigr)\bigr| \le A^u D + B^u W.$$

To sum up, $\{Hx : x \in K, \|x\| \le \nu\}$ is a family of uniformly bounded and equicontinuous functionals on $[t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}$. By a theorem of Arzela-Ascoli, the functional H is completely continuous. This completes the proof. □

3 Main result

Now, we fix $\eta, l \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}$, $\eta \le l$, and let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals θ, γ, and the nonnegative continuous functional ψ be defined on the cone K by

$$\begin{aligned} &\alpha(x) = \inf_{t\in[\eta,l]_{\mathbb{T}}} x(t), \qquad \psi(x) = \theta(x) = \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} x(t), \\ &\gamma(x) = \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} (\Phi x)(t), \end{aligned} \tag{3.1}$$

respectively, where $(\Phi x)(t) = \int_{t_0}^{\delta_+^{\omega}(t_0)} h(t,s)x(s)\Delta s$, $h(t,s) \in C(\mathbb{T}^2, (0,\infty))$.

The functionals defined above satisfy the following relations:

$$\alpha(x) \le \psi(x) = \theta(x), \quad \forall x \in K. \tag{3.2}$$

Lemma 3.1 *For $x \in K$, there exists a constant $E > 0$ such that*

$$\sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} x(t) \le E \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} (\Phi x)(t).$$

Proof For $x \in K$, we have

$$\begin{aligned} \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} (\Phi x)(t) &= \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} \int_{t_0}^{\delta_+^{\omega}(t_0)} h(t,s)x(s)\Delta s \\ &\ge \frac{1}{\xi}\|x\| \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} \int_{t_0}^{\delta_+^{\omega}(t_0)} h(t,s)\Delta s \\ &= \frac{L}{\xi} \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} x(t), \end{aligned}$$

where $L := \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} \int_{t_0}^{\delta_+^{\omega}(t_0)} h(t,s)\Delta s$. Setting $E := \frac{\xi}{L}$. This completes the proof. □

Moreover, for each $x \in K$,

$$\|x\| = \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} x(t) \le \frac{\xi}{L} \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} (\Phi x)(t) = E\gamma(x), \tag{3.3}$$

and $\psi(\rho x) = \rho\psi(x)$, $\forall \rho \in [0,1]$, for all $x \in K$. It follows from (3.2) and (3.3) that condition (2.5) in Lemma 2.7 is satisfied.

For convenience in the following discussion, we introduce the following notations:

$$I_1^M = \max_{0\le u\le Ed}\sum_{j=1}^{q} I_j(u), \qquad I_2^M = \max_{0\le u\le a}\sum_{j=1}^{q} I_j(u), \qquad I^m = \min_{b\le u\le b\xi}\sum_{j=1}^{q} I_j(u).$$

To present our main result, we assume that there exist constants $a, b, d > 0$ with $a < b < b\xi < \frac{d}{L}$ such that:

(S_1) $f(t,u) < \frac{(\xi-1)d}{\xi BL} - \frac{I_1^M}{B}$, for $0 \le u \le Ed$, $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$;
(S_2) $f(t,u) > \frac{(\xi-1)b}{B} - \frac{I^m}{B}$, for $b \le u \le b\xi$, $t \in [\eta, l]_{\mathbb{T}}$;
(S_3) $f(t,u) < \frac{(\xi-1)a}{\xi B} - \frac{I_2^M}{B}$, for $0 \le u \le a$, $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$.

Theorem 3.1 *Under assumptions* (S_1)-(S_3), *system* (1.1) *has at least three positive ω-periodic solutions x_1, x_2, and x_3 in shifts $\delta_\pm$ satisfying*

$$\sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} (\Phi x_i)(t) \le d, \quad i = 1,2,3, \qquad b < \inf_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} x_1(t),$$

$$a < \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} x_2(t), \quad \textit{with} \quad \inf_{t\in[\eta,l]_{\mathbb{T}}} x_2(t) < b \quad \textit{and} \quad \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} x_3(t) < a.$$

Proof For $x \in \overline{K(\gamma,d)}$, then $x \in K$ and $\gamma(x) = \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}}(\Phi x)(t) \le d$. From Lemma 3.1, we have $\sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} x(t) \le Ed$, that is, $0 \le x(t) \le Ed$, for $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$. Then, by Lemma 2.8 and assumption (S_1), for $x \in \overline{K(\gamma,d)}$, we have $Hx \in K$, and

$$\begin{aligned}
(Hx)(t) &= \int_t^{\delta_+^\omega(t)} G(t,s)b(s)f\bigl(s,x\bigl(g(s)\bigr)\bigr)\Delta s \\
&\quad + \sum_{j:t_j\in[t,\delta_+^\omega(t))_{\mathbb{T}}} G(t,t_j)e_{-a}\bigl(\sigma(t_j),t_j\bigr)I_j\bigl(x(t_j)\bigr) \\
&\le \frac{\xi}{\xi-1}\int_{t_0}^{\delta_+^\omega(t_0)} b(s)f\bigl(s,x\bigl(g(s)\bigr)\bigr)\Delta s + \frac{\xi}{\xi-1}\sum_{j=1}^{q} I_j\bigl(x(t_j)\bigr) \\
&\le \frac{B\xi}{\xi-1}\left(\frac{(\xi-1)d}{\xi BL} - \frac{I_1^M}{B}\right) + \frac{\xi}{\xi-1}I_1^M \\
&= \frac{d}{L},
\end{aligned}$$

then

$$\begin{aligned}
\gamma(Hx)(t) &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \Phi(Hx)(t) \\
&= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \int_{t_0}^{\delta_+^\omega(t_0)} h(t,s)(Hx)(s)\Delta s \\
&\le \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \left\{\int^{T} h(t,s)\Delta s\right\}\cdot\frac{d}{L} \\
&= d.
\end{aligned}$$

Therefore, $H : \overline{K(\gamma,d)} \to \overline{K(\gamma,d)}$.

To check condition (1) in Lemma 2.7, we take $\tilde{x} = b\xi$. It is easy to verify that $\tilde{x} \in K(\gamma,\theta,\alpha,b,b\xi,d)$, and $\alpha(\tilde{x}) = b\xi > b$, and so $\{x \in K(\gamma,\theta,\alpha,b,b\xi,d) \mid \alpha(x) > b\} \neq \emptyset$.

For $x \in K(\gamma,\theta,\alpha,b,b\xi,d)$, we have

$$\inf_{t\in[\eta,l]_{\mathbb{T}}} x(t) \geq b, \qquad \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} x(t) \leq b\xi, \qquad \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} (\Phi x)(t) \leq d,$$

that is, $b \leq x(t) \leq b\xi$, $0 \leq (\Phi x)(t) \leq d$, for $t \in [\eta,l]_{\mathbb{T}}$.

Then, by assumption (S_2), we have

$$\begin{aligned}
\alpha(Hx)(t) &= \inf_{t\in[\eta,l]_{\mathbb{T}}} \left\{ \int_t^{\delta_+^{\omega}(t)} G(t,s)b(s)f\big(s,x\big(g(s)\big)\big)\Delta s \right.\\
&\quad \left. + \sum_{j:t_j\in[t,\delta_+^{\omega}(t))_{\mathbb{T}}} G(t,t_j)e_{-a}\big(\sigma(t_j),t_j\big)I_j\big(x(t_j)\big) \right\}\\
&\geq \inf_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} \left\{ \int_t^{\delta_+^{\omega}(t)} G(t,s)b(s)f(s,x_s)\Delta s \right.\\
&\quad \left. + \sum_{j:t_j\in[t,\delta_+^{\omega}(t))_{\mathbb{T}}} G(t,t_j)e_{-a}\big(\sigma(t_j),t_j\big)I_j\big(x(t_j)\big) \right\}\\
&\geq \frac{1}{\xi-1}\int_{t_0}^{\delta_+^{\omega}(t_0)} b(s)f\big(s,x\big(g(s)\big)\big)\Delta s + \frac{1}{\xi-1}\sum_{j=1}^{q} I_j\big(x(t_j)\big)\\
&> \frac{B}{\xi-1}\left(\frac{(\xi-1)b}{B} - \frac{I^m}{B}\right) + \frac{1}{\xi-1}I^m\\
&= b,
\end{aligned}$$

that is, $\alpha(Hx) > b$ for all $x \in K(\gamma,\theta,\alpha,b,b\xi,d)$. This shows that condition (1) in Lemma 2.7 is satisfied.

Secondly, by (3.1) and the cone K we defined in (2.6), we can get $\alpha(Hx) \geq \frac{1}{\xi}\theta(Hx) > \frac{1}{\xi}(b\xi) = b$ for all $x \in K(\gamma,\alpha,b,d)$ with $\theta(Hx) > b\xi$. Thus condition (2) in Lemma 2.7 is satisfied.

Finally, we show that condition (3) in Lemma 2.7 also holds. Clearly, as $\psi(0) = 0 < a$, we have $0 \overline{\in} R(\gamma,\psi,a,d)$. Suppose that $x \in R(\gamma,\psi,a,d)$ with $\psi(x) = a$, this implies that for $t \in [t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}$, there is $\sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} x(t) = a$, $\sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} (\Phi x)(t) \leq d$. Hence,

$$0 \leq x(t) \leq a, \qquad 0 \leq (\Phi x)(t) \leq d \quad \text{for } t \in \big[t_0,\delta_+^{\omega}(t_0)\big]_{\mathbb{T}}.$$

By assumption (S_3), we have

$$\begin{aligned}
\psi(Hx) &= \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}} (Hx)(t)\\
&\leq \frac{\xi}{\xi-1}\int^{\omega} b(s)f\big(s,x\big(g(s)\big)\big)\Delta s + \frac{\xi}{\xi-1}\sum_{j=1}^{q} I_j\big(x(t_j)\big)\\
&< \frac{B\xi}{\xi-1}\left(\frac{(\xi-1)a}{\xi B} - \frac{I_2^M}{B}\right) + \frac{\xi}{\xi-1}I_2^M\\
&= a.
\end{aligned}$$

So, condition (3) in Lemma 2.7 is satisfied.

To sum up, all conditions in Lemma 2.7 are satisfied. Hence, H has at least three fixed points, that is, system (1.1) has at least three positive ω-periodic solutions in shifts $\delta_\pm$. This completes the proof. □

4 Numerical examples

Consider the following system with impulses:

$$\begin{cases} x^\Delta(t) = -a(t)x(t) + b(t)f(t,x(g(t))), & t \neq t_j, t \in \mathbb{T}, \\ x(t_j^+) = x(t_j^-) + I_j(x(t_j)). \end{cases} \tag{4.1}$$

Example 1 Let

$$a(t) = 0.5, \qquad b(t) = 1 - 0.5\sin 2\pi t, \qquad f(t,x) = \begin{cases} \frac{|\sin \pi t|}{5} + \frac{x}{10}, & x \leq 5, \\ 10 + \frac{x}{15+5|\sin \pi t|}, & x > 5, \end{cases}$$

$$I_j\bigl(x(t_j)\bigr) = 0.01\bigl|\sin\bigl(x(t_j)\bigr)\bigr|, \quad j = 1, 2, \ldots, 10,$$

in system (4.1), then

$$\omega = 1; \qquad I_1^M, I_2^M, I^m \in [0, 0.1].$$

Case I: Let $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, then $\delta_+^\omega(t) = t + 1$. It is easy to verify $a(t)$, $b(t)$, $f(t,x)$ satisfy

$$a\bigl(\delta_+^\omega(t)\bigr)\delta_+^{\Delta\omega}(t) = a(t), \qquad b\bigl(\delta_+^\omega(t)\bigr)\delta_+^{\Delta\omega}(t) = b(t), \qquad f\bigl(\delta_+^\omega(t), x\bigr) = f(t,x), \quad \forall t \in \mathbb{T}^*,$$

and $-a \in \mathcal{R}^+$.

By a direct calculation, we can get

$$\xi = e^{0.5} = 1.6487, \qquad B = \int_0^1 (1 - 0.5\sin 2\pi t)\Delta t = \left(t + \frac{\cos 2\pi t}{4\pi}\right)\bigg|_0^1 = 1.$$

Choose $a = 5$, $b = 10$, $L = 1$, $d = 50$, then

$$\begin{aligned} &f(t,x) \leq 10 + 5.4957 = 15.4957 < 19.6735 - I_1^M \quad \text{for } x \in [0, 82.4361], \\ &f(t,x) > 10 + \frac{10}{20} = 10.5 > 6.4872 - I^m \quad \text{for } x \in [10, 16.4872], \\ &f(t,x) < \frac{1}{5} + \frac{5}{10} = 0.7 < 1.9673 - I_2^M \quad \text{for } x \in [0, 5]. \end{aligned}$$

According to Theorem 3.1, when $\mathbb{T} = \mathbb{R}$, for system (4.1) there exist at least three positive periodic solutions $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ in shifts $\delta_\pm$ with period $\omega = 1$, and

$$\sup_{t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}} \hat{x}_3(t) < 5 < \sup_{t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}} \hat{x}_2(t), \qquad \inf_{t \in [\eta, l]_\mathbb{T}} \hat{x}_2(t) < 10 < \inf_{t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}} \hat{x}_1(t).$$

Case II: Let $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, then $\delta_+^\omega(t) = t + 1$. It is easy to verify $a(t)$, $b(t)$, $f(t,x)$ satisfy

$$a\bigl(\delta_+^\omega(t)\bigr)\delta_+^{\Delta\omega}(t) = a(t), \qquad b\bigl(\delta_+^\omega(t)\bigr)\delta_+^{\Delta\omega}(t) = b(t), \qquad f\bigl(\delta_+^\omega(t), x\bigr) = f(t,x), \quad \forall t \in \mathbb{T}^*,$$

and $-a \in \mathcal{R}^+$.

By a direct calculation, we can get

$$\xi = (0.5)^{-1} = 2, \qquad B = \int_0^1 (1 - 0.5\sin 2\pi t)\Delta t = (t - 0.5t\sin 2\pi t)|_0^1 = 1.$$

Choose $a = 5$, $b = 10$, $L = 1$, $d = 30$, then

$$f(t,x) \le 10 + 4 = 14 < 15 - I_1^M \quad \text{for } x \in [0,60],$$
$$f(t,x) > 10 + \frac{10}{20} = 10.5 > 10 - I^m \quad \text{for } x \in [10,20],$$
$$f(t,x) < \frac{1}{5} + \frac{5}{10} = 0.7 < 2.5 - I_2^M \quad \text{for } x \in [0,5].$$

According to Theorem 3.1, when $\mathbb{T} = \mathbb{Z}$, for system (4.1) there exist at least three positive periodic solutions $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ in shifts $\delta_\pm$ with period $\omega = 1$, and

$$\sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \tilde{x}_3(t) < 5 < \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \tilde{x}_2(t), \qquad \inf_{t\in[\eta,l]_\mathbb{T}} \tilde{x}_2(t) < 10 < \inf_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \tilde{x}_1(t).$$

Example 2 Let

$$a(t) = \frac{1}{5t}, \qquad b(t) = \frac{1}{b_0 t}, \qquad f(t,x) = \begin{cases} \frac{|\sin\frac{\pi}{4}t|}{5} + \frac{x}{10}, & x \le 5, \\ 12 + \frac{x}{15+5|\sin\frac{\pi}{4}t|}, & x > 5, \end{cases}$$
$$I_j\big(x(t_j)\big) = 0.01\big|\sin\big(x(t_j)\big)\big|, \quad j = 1,2,\ldots,10,$$

in system (4.1), where $b_0 = \int_1^4 \frac{1}{t}\Delta t$, then

$$\omega = 4; \qquad I_1^M, I_2^M, I^m \in [0,0.1].$$

Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $t_0 = 1$, then $\delta_+^\omega(t) = 4t$. It is easy to verify $a(t)$, $b(t)$, $f(t,x)$ satisfy

$$a\big(\delta_+^\omega(t)\big)\delta_+^{\Delta\omega}(t) = a(t), \qquad b\big(\delta_+^\omega(t)\big)\delta_+^{\Delta\omega}(t) = b(t), \qquad f\big(\delta_+^\omega(t),x\big) = f(t,x), \quad \forall t \in \mathbb{T}^*,$$

and $-a \in \mathcal{R}^+$.

By a direct calculation, we can get

$$\xi = \prod_{t\in[1,4)} \left(1 + \frac{t}{5-t}\right) = 2.0833, \qquad B = \int_1^4 \frac{1}{b_0 t}\Delta t = 1.$$

Choose $a = 5$, $b = 10$, $L = 1$, $d = 50$, then

$$f(t,x) \le 12 + 6.9444 = 18.9444 < 26 - I_1^M \quad \text{for } x \in [0,104.1667],$$
$$f(t,x) > 12 + \frac{10}{20} = 12.5 > 10.8333 - I^m \quad \text{for } x \in [10,20.8333],$$
$$f(t,x) < \frac{1}{5} + \frac{5}{10} = 0.7 < 2.6 - I_2^M \quad \text{for } x \in [0,5].$$

According to Theorem 3.1, when $\mathbb{T} = 2^{\mathbb{N}_0}$, for system (4.1) there exist at least three positive periodic solutions $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ in shifts $\delta_\pm$ with period $\omega = 4$, and

$$\sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \hat{x}_3(t) < 5 < \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \hat{x}_2(t), \qquad \inf_{t\in[\eta,l]_\mathbb{T}} \hat{x}_2(t) < 10 < \inf_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} \hat{x}_1(t).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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