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Barnes-type Daehee of the first kind and poly-Cauchy of the first kind mixed-type polynomials

Dae San Kim[1], Taekyun Kim[2*], Takao Komatsu[3] and Sang-Hun Lee[4]

*Correspondence: tkkim@kw.ac.kr
[2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract

In this paper, by considering Barnes-type Daehee polynomials of the first kind as well as poly-Cauchy polynomials of the first kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

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1 Introduction

In this paper, we consider the polynomials $D_n^{(k)}(x|a_1,\dots,a_r)$ called the Barnes-type Daehee of the first kind and poly-Cauchy of the first kind mixed-type polynomials, whose generating function is given by

$$\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^x=\sum_{n=0}^{\infty}D_n^{(k)}(x|a_1,\dots,a_r)\frac{t^n}{n!},\tag{1}$$

where $a_1,\dots,a_r\neq 0$. Here, $\mathrm{Lif}_k(x)$ ($k\in\mathbb{Z}$) is the polyfactorial function [1] defined by

$$\mathrm{Lif}_k(x)=\sum_{m=0}^{\infty}\frac{x^m}{m!(m+1)^k}.$$

When $x=0$, $D_n^{(k)}(a_1,\dots,a_r)=D_n^{(k)}(0|a_1,\dots,a_r)$ is called Barnes-type Daehee of the first kind and poly-Cauchy of the first kind mixed-type number.

Recall that the Barnes-type Daehee polynomials of the first kind, denoted by $D_n(x|a_1,\dots,a_r)$, are given by the generating function

$$\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^x=\sum_{n=0}^{\infty}D_n(x|a_1,\dots,a_r)\frac{t^n}{n!}.$$

If $a_1=\cdots=a_r=1$, then $D_n^{(r)}(x)=D_n(x|\underbrace{1,\dots,1}_{r})$ are the Daehee polynomials of the first kind of order r. Dahee polynomials were defined by the second author [2] and have been investigated in [3, 4].

The poly-Cauchy polynomials of the first kind, denoted by $c_n^{(k)}(x)$ [5, 6], are given by the generating function as

$$\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^x = \sum_{n=0}^{\infty} c_n^{(k)}(-x)\frac{t^n}{n!}.$$

In this paper, by considering Barnes-type Daehee polynomials of the first kind as well as poly-Cauchy polynomials of the first kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{2}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n \quad (n \geq 0). \tag{3}$$

In particular,

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of the umbral algebra. The order $O(f(t))$ of a power series $f(t)$ ($\neq 0$) is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a *delta series*; if $O(f(t)) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$, for $n,k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{5}$$

and

$$f(t)=\sum_{k=0}^{\infty}\langle f(t)|x^k\rangle\frac{t^k}{k!},\qquad p(x)=\sum_{k=0}^{\infty}\langle t^k|p(x)\rangle\frac{x^k}{k!}\tag{6}$$

[7, Theorem 2.2.5]. Thus, by (6), we get

$$t^k p(x)=p^{(k)}(x)=\frac{d^k p(x)}{dx^k}\quad\text{and}\quad e^{yt}p(x)=p(x+y).\tag{7}$$

Sheffer sequences are characterized by the generating function [7, Theorem 2.3.4].

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $(g(t),f(t))$ if and only if*

$$\frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)}=\sum_{k=0}^{\infty}\frac{s_k(y)}{k!}t^k\quad(y\in\mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x)\sim(g(t),f(t))$, we have the following equations [7, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]:

$$f(t)s_n(x)=ns_{n-1}(x)\quad(n\geq 0),\tag{8}$$

$$s_n(x)=\sum_{j=0}^{n}\frac{1}{j!}\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j|x^n\rangle x^j,\tag{9}$$

$$s_n(x+y)=\sum_{j=0}^{n}\binom{n}{j}s_j(x)p_{n-j}(y),\tag{10}$$

where $p_n(x)=g(t)s_n(x)$.

Assume that $p_n(x)\sim(1,f(t))$ and $q_n(x)\sim(1,g(t))$. Then the transfer formula [7, Corollary 3.8.2] is given by

$$q_n(x)=x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x)\quad(n\geq 1).$$

For $s_n(x)\sim(g(t),f(t))$ and $r_n(x)\sim(h(t),l(t))$, assume that

$$s_n(x)=\sum_{m=0}^{n}C_{n,m}r_m(x)\quad(n\geq 0).$$

Then we have [7, p.132]

$$C_{n,m}=\frac{1}{m!}\left\langle\frac{h(\bar{f}(t))}{g(\bar{f}(t))}l(\bar{f}(t))^m\middle|x^n\right\rangle.\tag{11}$$

3 Main results

From the definition (1), $D_n^{(k)}(x|a_1,\ldots,a_r)$ is the Sheffer sequence for the pair

$$g(t)=\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{t}\right)\frac{1}{\mathrm{Lif}_k(t)}\quad\text{and}\quad f(t)=e^t-1.$$

So,

$$D_n^{(k)}(x|a_1,\ldots,a_r)\sim\left(\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{t}\right)\frac{1}{\mathrm{Lif}_k(t)},e^t-1\right).\tag{12}$$

3.1 Explicit expressions

Recall that Barnes' multiple Bernoulli polynomials $B_n(x|a_1,\ldots,a_r)$ are defined by the generating function

$$\frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}e^{xt}=\sum_{n=0}^{\infty}B_n(x|a_1,\ldots,a_r)\frac{t^n}{n!},\tag{13}$$

where $a_1,\ldots,a_r\neq 0$ [8, 9]. Let $(n)_j=n(n-1)\cdots(n-j+1)$ $(j\geq 1)$ with $(n)_0=1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n=\sum_{m=0}^{n}S_1(n,m)x^m.$$

Theorem 1

$$\begin{aligned}
&D_n^{(k)}(x|a_1,\ldots,a_r)\\
&\quad=\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\frac{\binom{m}{l}}{(m-l+1)^k}B_l(x|a_1,\ldots,a_r) &(14)\\
&\quad=\sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}S_1(n-l,j)c_i^{(k)}D_{l-i}(a_1,\ldots,a_r)x^j &(15)\\
&\quad=\sum_{l=0}^{n}\binom{n}{l}D_{n-l}(a_1,\ldots,a_r)c_l^{(k)}(-x) &(16)\\
&\quad=\sum_{l=0}^{n}\binom{n}{l}c_{n-l}^{(k)}D_l(x|a_1,\ldots,a_r). &(17)
\end{aligned}$$

Proof Since

$$\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{t}\right)\frac{1}{\mathrm{Lif}_k(t)}D_n^{(k)}(x|a_1,\ldots,a_r)\sim\left(1,e^t-1\right)\tag{18}$$

and

$$(x)_n\sim\left(1,e^t-1\right),\tag{19}$$

we have

$$\begin{aligned}
D_n^{(k)}(x|a_1,\ldots,a_r) &= \prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right)\mathrm{Lif}_k(t)(x)_n \\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right)\mathrm{Lif}_k(t)x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right)\sum_{l=0}^{m}\frac{t^l}{l!(l+1)^k}x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right)\sum_{l=0}^{m}\frac{(m)_l}{l!(l+1)^k}x^{m-l} \\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(m)_l}{l!(l+1)^k}\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right)x^{m-l} \\
&= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{\binom{m}{l}}{(l+1)^k}B_{m-l}(x|a_1,\ldots,a_r) \\
&= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{\binom{m}{l}}{(m-l+1)^k}B_l(x|a_1,\ldots,a_r).
\end{aligned}$$

Thus, we get (14).

By (9) with (12), we get

$$\begin{aligned}
&\left\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j \middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^j \middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\big(\ln(1+t)\big) \middle| \sum_{l=0}^{\infty}\frac{j!}{(l+j)!}S_1(l+j,j)t^{l+j}x^n\right\rangle \\
&= \sum_{l=0}^{n-j}\frac{j!}{(l+j)!}S_1(l+j,j)(n)_{l+j}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\big(\ln(1+t)\big) \middle| x^{n-l-j}\right\rangle \\
&= \sum_{l=0}^{n-j} j!\binom{n}{l+j}S_1(l+j,j)\left\langle \sum_{i=0}^{\infty}D_i^{(k)}(a_1,\ldots,a_r)\frac{t^i}{i!} \middle| x^{n-l-j}\right\rangle \\
&= \sum_{l=0}^{n-j} j!\binom{n}{l+j}S_1(l+j,j)D_{n-l-j}^{(k)}(a_1,\ldots,a_r) \\
&= \sum_{l=0}^{n-j} j!\binom{n}{l}S_1(n-l,j)D_l^{(k)}(a_1,\ldots,a_r).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j \middle| x^n\right\rangle \\
&= \sum_{l=0}^{n-j}\frac{j!}{(l+j)!}S_1(l+j,j)(n)_{l+j}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \mathrm{Lif}_k\big(\ln(1+t)\big)x^{n-l-j}\right\rangle
\end{aligned}$$

$$= \sum_{l=0}^{n-j} j! \binom{n}{l+j} S_1(l+j,j) \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| \sum_{i=0}^{n-l-j} c_i^{(k)} \frac{t^i}{i!} x^{n-l-j} \right\rangle$$

$$= \sum_{l=0}^{n-j} j! \binom{n}{l+j} S_1(l+j,j) \sum_{i=0}^{n-l-j} c_i^{(k)} \frac{(n-l-j)_i}{i!} \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| x^{n-l-j-i} \right\rangle$$

$$= \sum_{l=0}^{n-j} j! \binom{n}{l+j} S_1(l+j,j) \sum_{i=0}^{n-l-j} c_i^{(k)} \frac{(n-l-j)_i}{i!} \left\langle \sum_{m=0}^{\infty} D_m(a_1,\ldots,a_r) \frac{t^m}{m!} \middle| x^{n-l-j-i} \right\rangle$$

$$= \sum_{l=0}^{n-j} \sum_{i=0}^{n-l-j} j! \binom{n}{l+j} \binom{n-l-j}{i} S_1(l+j,j) c_i^{(k)} D_{n-l-j-i}(a_1,\ldots,a_r)$$

$$= \sum_{l=0}^{n-j} \sum_{i=0}^{l} j! \binom{n}{l} \binom{l}{i} S_1(n-l,j) c_i^{(k)} D_{l-i}(a_1,\ldots,a_r).$$

Thus, we obtain

$$\begin{aligned} &D_n^{(k)}(x|a_1,\ldots,a_r) \\ &\quad = \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{l} S_1(n-l,j) D_l^{(k)}(a_1,\ldots,a_r) x^j \\ &\quad = \sum_{j=0}^{n} \sum_{l=0}^{n-j} \sum_{i=0}^{l} \binom{n}{l} \binom{l}{i} S_1(n-l,j) c_i^{(k)} D_{l-i}(a_1,\ldots,a_r) x^j, \end{aligned}$$

which is the identity (15).

Next,

$$\begin{aligned} D_n^{(k)}(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} D_i^{(k)}(y|a_1,\ldots,a_r) \frac{t^i}{i!} \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| \mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| \sum_{l=0}^{n} c_l^{(k)}(-y) \frac{t^l}{l!} x^n \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} c_l^{(k)}(-y) \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| x^{n-l} \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} c_l^{(k)}(-y) \left\langle \sum_{i=0}^{\infty} D_i(a_1,\ldots,a_r) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} c_l^{(k)}(-y) D_{n-l}(a_1,\ldots,a_r). \end{aligned}$$

Thus, we obtain (16).

Finally, we obtain

$$\begin{aligned}
D_n^{(k)}(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} D_i^{(k)}(y|a_1,\ldots,a_r)\frac{t^i}{i!} \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(\ln(1+t)\big) \middle| \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^y x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(\ln(1+t)\big) \middle| \sum_{l=0}^{n} D_l(y|a_1,\ldots,a_r)\frac{t^l}{l!}x^n \right\rangle \\
&= \sum_{l=0}^{n} D_l(y|a_1,\ldots,a_r)\binom{n}{l}\left\langle \mathrm{Lif}_k\big(\ln(1+t)\big)|x^{n-l}\right\rangle \\
&= \sum_{l=0}^{n} D_l(y|a_1,\ldots,a_r)\binom{n}{l}\left\langle \sum_{i=0}^{\infty} c_i^{(k)}\frac{t^i}{i!} \middle| x^{n-l}\right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l} D_l(y|a_1,\ldots,a_r)c_{n-l}^{(k)}.
\end{aligned}$$

Thus, we get the identity (17). □

3.2 Sheffer identity

Theorem 2

$$D_n^{(k)}(x+y|a_1,\ldots,a_r) = \sum_{j=0}^{n}\binom{n}{j}D_j^{(k)}(x|a_1,\ldots,a_r)(y)_{n-j}. \tag{20}$$

Proof By (12) with

$$\begin{aligned}
p_n(x) &= \prod_{j=1}^{r}\left(\frac{e^{a_j t}-1}{t}\right)\frac{1}{\mathrm{Lif}_k(t)}D_n(x|a_1,\ldots,a_r) \\
&= (x)_n \sim \big(1, e^t-1\big),
\end{aligned}$$

using (10), we have (20). □

3.3 Difference relations

Theorem 3

$$D_n^{(k)}(x+1|a_1,\ldots,a_r) - D_n^{(k)}(x|a_1,\ldots,a_r) = nD_{n-1}^{(k)}(x|a_1,\ldots,a_r). \tag{21}$$

Proof By (8) with (12), we get

$$\big(e^t-1\big)D_n^{(k)}(x|a_1,\ldots,a_r) = nD_{n-1}^{(k)}(x|a_1,\ldots,a_r).$$

By (7), we have (21). □

3.4 Recurrence

Theorem 4

$$\begin{aligned} D_{n+1}^{(k)}(x|a_1,\ldots,a_r) &= xD_n^{(k)}(x-1|a_1,\ldots,a_r) \\ &\quad - \sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=1}^{m+1}\sum_{i=0}^{l}\frac{\binom{m+1}{l}\binom{l}{i}}{(m+1)(l-i+1)^k}S_1(n,m) \\ &\quad \times (-a_j)^{m+1-l}B_{m+1-l}B_i(x-1|a_1,\ldots,a_r) \\ &\quad + \sum_{m=0}^{n}\sum_{l=0}^{m}\frac{\binom{m}{l}}{(m+2-l)^k}S_1(n,m)B_l(x-1|a_1,\ldots,a_r), \end{aligned} \tag{22}$$

where B_n is the nth ordinary Bernoulli number.

Proof By applying

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x) \tag{23}$$

[7, Corollary 3.7.2] with (12), we get

$$D_{n+1}^{(k)}(x|a_1,\ldots,a_r) = xD_n^{(k)}(x-1|a_1,\ldots,a_r) - e^{-t}\frac{g'(t)}{g(t)}D_n^{(k)}(x|a_1,\ldots,a_r).$$

Now,

$$\begin{aligned} \frac{g'(t)}{g(t)} &= \left(\ln g(t)\right)' \\ &= \left(\sum_{j=1}^{r}\ln\left(e^{a_jt}-1\right) - r\ln t - \ln \mathrm{Lif}_k(t)\right)' \\ &= \sum_{j=1}^{r}\frac{a_je^{a_jt}}{e^{a_jt}-1} - \frac{r}{t} - \frac{\mathrm{Lif}_k'(t)}{\mathrm{Lif}_k(t)} \\ &= \frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{t\prod_{j=1}^{r}(e^{a_jt}-1)} - \frac{\mathrm{Lif}_k'(t)}{\mathrm{Lif}_k(t)}. \end{aligned}$$

Observe that

$$\begin{aligned} &\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{\prod_{j=1}^{r}(e^{a_jt}-1)} \\ &\quad = \frac{\frac{1}{2}(\sum_{j=1}^{r}a_1\cdots a_{j-1}a_j^2a_{j+1}\cdots a_r)t^{r+1}+\cdots}{(a_1\cdots a_r)t^r+\cdots} \\ &\quad = \frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots \end{aligned}$$

is a series with order ≥ 1. Since

$$D_n^{(k)}(x|a_1,\dots,a_r) = \prod_{j=1}^{r}\left(\frac{t}{e^{a_jt}-1}\right)\mathrm{Lif}_k(t)(x)_n = \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{t}{e^{a_jt}-1}\right)\mathrm{Lif}_k(t)x^m,$$

we have

$$\begin{aligned}\frac{g'(t)}{g(t)}D_n^{(k)}(x|a_1,\dots,a_r) &= \sum_{m=0}^{n} S_1(n,m)\frac{g'(t)}{g(t)}\left(\prod_{j=1}^{r}\frac{t}{e^{a_jt}-1}\right)\mathrm{Lif}_k(t)x^m \\ &= \sum_{m=0}^{n} S_1(n,m)\,\mathrm{Lif}_k(t)\left(\prod_{j=1}^{r}\frac{t}{e^{a_jt}-1}\right) \\ &\quad\times \frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{t\prod_{j=1}^{r}(e^{a_jt}-1)}x^m \\ &\quad- \sum_{m=0}^{n} S_1(n,m)\,\mathrm{Lif}'_k(t)\left(\prod_{j=1}^{r}\frac{t}{e^{a_jt}-1}\right)x^m. \qquad (24)\end{aligned}$$

Since

$$\begin{aligned}&\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{t\prod_{j=1}^{r}(e^{a_jt}-1)}x^m \\ &= \frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{x^{m+1}}{m+1} \\ &= \frac{1}{m+1}\sum_{j=1}^{r}\left(\frac{a_jte^{a_jt}}{e^{a_jt}-1}-1\right)x^{m+1} \\ &= \frac{1}{m+1}\sum_{j=1}^{r}\left(\sum_{l=0}^{\infty}\frac{(-1)^lB_la_j^l}{l!}t^l-1\right)x^{m+1} \\ &= \frac{1}{m+1}\sum_{j=1}^{r}\left(\sum_{l=0}^{m+1}\binom{m+1}{l}(-a_j)^lB_lx^{m+1-l}-x^{m+1}\right) \\ &= \frac{1}{m+1}\sum_{j=1}^{r}\sum_{l=1}^{m+1}\binom{m+1}{l}(-a_j)^lB_lx^{m+1-l} \\ &= \frac{1}{m+1}\sum_{j=1}^{r}\sum_{l=1}^{m+1}\binom{m+1}{l}(-a_j)^{m+1-l}B_{m+1-l}x^l,\end{aligned}$$

the first term in (24) is

$$\begin{aligned}&\sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}\sum_{j=1}^{r}\sum_{l=1}^{m+1}\binom{m+1}{l}(-a_j)^{m+1-l}B_{m+1-l}\,\mathrm{Lif}_k(t)\left(\prod_{j=1}^{r}\frac{t}{e^{a_jt}-1}\right)x^l \\ &= \sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}\sum_{j=1}^{r}\sum_{l=1}^{m+1}\binom{m+1}{l}(-a_j)^{m+1-l}B_{m+1-l}\sum_{i=0}^{l}\frac{t^i}{i!(i+1)^k}B_l(x|a_1,\dots,a_r)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \sum_{j=1}^{r} \sum_{l=1}^{m+1} \binom{m+1}{l} (-a_j)^{m+1-l} B_{m+1-l} \sum_{i=0}^{l} \frac{\binom{l}{i}}{(i+1)^k} B_{l-i}(x|a_1,\ldots,a_r) \\
&= \sum_{m=0}^{n} \sum_{j=1}^{r} \sum_{l=1}^{m+1} \sum_{i=0}^{l} \frac{\binom{m+1}{l}\binom{l}{i}}{(m+1)(l-i+1)^k} S_1(n,m)(-a_j)^{m+1-l} B_{m+1-l} B_i(x|a_1,\ldots,a_r).
\end{aligned}
$$

Since

$$
\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t) = \left(\frac{1}{2^{k-1}} - \frac{1}{2^k}\right) t + \cdots, \tag{25}
$$

the second term in (24) is

$$
\begin{aligned}
&\sum_{m=0}^{n} S_1(n,m) \frac{\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t)}{t} B_m(x|a_1,\ldots,a_r) \\
&\quad= \sum_{m=0}^{n} S_1(n,m) \bigl(\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t)\bigr) \frac{B_{m+1}(x|a_1,\ldots,a_r)}{m+1} \\
&\quad= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \bigl(\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t)\bigr) B_{m+1}(x|a_1,\ldots,a_r) \\
&\quad= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \left(\sum_{l=0}^{m+1} \frac{t^l}{l!(l+1)^{k-1}} B_{m+1}(x|a_1,\ldots,a_r) - \sum_{l=0}^{m+1} \frac{t^l}{l!(l+1)^k} B_{m+1}(x|a_1,\ldots,a_r) \right) \\
&\quad= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \left(\sum_{l=0}^{m+1} \frac{\binom{m+1}{l}}{(l+1)^{k-1}} B_{m+1-l}(x|a_1,\ldots,a_r) - \sum_{l=0}^{m+1} \frac{\binom{m+1}{l}}{(l+1)^k} B_{m+1-l}(x|a_1,\ldots,a_r) \right) \\
&\quad= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \sum_{l=1}^{m+1} \frac{\binom{m+1}{l} l}{(l+1)^k} B_{m+1-l}(x|a_1,\ldots,a_r) \\
&\quad= \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{m}{l-1} S_1(n,m) \frac{1}{(l+1)^k} B_{m+1-l}(x|a_1,\ldots,a_r) \\
&\quad= \sum_{m=0}^{n} \sum_{l=0}^{m} \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m) B_l(x|a_1,\ldots,a_r).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
D_{n+1}^{(k)}(x|a_1,\ldots,a_r) = xD_n^{(k)}(x-1|a_1,\ldots,a_r) \\
&- \sum_{m=0}^{n} \sum_{j=1}^{r} \sum_{l=1}^{m+1} \sum_{i=0}^{l} \frac{\binom{m+1}{l}\binom{l}{i}}{(m+1)(l-i+1)^k} S_1(n,m) \\
&\times (-a_j)^{m+1-l} B_{m+1-l} B_i(x-1|a_1,\ldots,a_r) \\
&+ \sum_{m=0}^{n} \sum_{l=0}^{m} \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m) B_l(x-1|a_1,\ldots,a_r),
\end{aligned}
$$

which is the identity (22). □

3.5 Differentiation

Theorem 5

$$\frac{d}{dx}D_n^{(k)}(x|a_1,\dots,a_r) = n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}D_l^{(k)}(x|a_1,\dots,a_r). \tag{26}$$

Proof We shall use

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\langle\bar{f}(t)|x^{n-l}\rangle s_l(x)$$

(*cf.* [7, Theorem 2.3.12]). Since

$$\begin{aligned}\langle\bar{f}(t)|x^{n-l}\rangle &= \langle\ln(1+t)|x^{n-l}\rangle \\ &= \left\langle\sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m}\middle|x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}\langle t^m|x^{n-l}\rangle \\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}(n-l)!\delta_{m,n-l} \\ &= (-1)^{n-l-1}(n-l-1)!,\end{aligned}$$

with (12), we have

$$\begin{aligned}\frac{d}{dx}D_n^{(k)}(x|a_1,\dots,a_r) &= \sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!D_l^{(k)}(x|a_1,\dots,a_r) \\ &= n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}D_l^{(k)}(x|a_1,\dots,a_r),\end{aligned}$$

which is the identity (26). □

3.6 One more relation

The classical Cauchy numbers c_n are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty}c_n\frac{t^n}{n!}$$

(see *e.g.* [1, 10]).

Theorem 6

$$\begin{aligned}D_n^{(k)}(x|a_1,\dots,a_r) &= xD_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\ &\quad + \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}c_l D_{n-l}^{(k-1)}(x-1|a_1,\dots,a_r)\end{aligned}$$

$$+\frac{r-1}{n}\sum_{l=0}^{n}\binom{n}{l}c_l D_{n-l}^{(k)}(x-1|a_1,\dots,a_r)$$
$$-\frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_j c_l D_{n-l}^{(k)}(x+a_j-1|a_1,\dots,a_r,a_j). \tag{27}$$

Proof For $n \geq 1$, we have

$$\begin{aligned}
D_n^{(k)}(y|a_1,\dots,a_r) &= \left\langle \sum_{l=0}^{\infty} D_l^{(k)}(y|a_1,\dots,a_r)\frac{t^l}{l!}\,\middle|\, x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)(1+t)^y \,\middle|\, x^n\right\rangle \\
&= \left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)(1+t)^y\right) \,\middle|\, x^{n-1}\right\rangle \\
&= \left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)\operatorname{Lif}_k\big(\ln(1+t)\big)(1+t)^y \,\middle|\, x^{n-1}\right\rangle \\
&\quad+ \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\partial_t\operatorname{Lif}_k\big(\ln(1+t)\big)\big)(1+t)^y \,\middle|\, x^{n-1}\right\rangle \\
&\quad+ \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)\big(\partial_t(1+t)^y\big) \,\middle|\, x^{n-1}\right\rangle.
\end{aligned}$$

The third term is

$$\begin{aligned}
&y\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)(1+t)^{y-1} \,\middle|\, x^{n-1}\right\rangle \\
&\quad= yD_{n-1}^{(k)}(y-1|a_1,\dots,a_r).
\end{aligned}$$

By (25), the second term is

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\frac{\operatorname{Lif}_{k-1}(\ln(1+t))-\operatorname{Lif}_k(\ln(1+t))}{(1+t)\ln(1+t)}(1+t)^y \,\middle|\, x^{n-1}\right\rangle \\
&\quad= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\frac{\operatorname{Lif}_{k-1}(\ln(1+t))-\operatorname{Lif}_k(\ln(1+t))}{t}(1+t)^{y-1} \,\middle|\, \frac{t}{\ln(1+t)}x^{n-1}\right\rangle \\
&\quad= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\frac{\operatorname{Lif}_{k-1}(\ln(1+t))-\operatorname{Lif}_k(\ln(1+t))}{t}(1+t)^{y-1} \,\middle|\, \sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^{n-1}\right\rangle \\
&\quad= \sum_{l=0}^{n-1}\binom{n-1}{l}c_l \\
&\qquad\times\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^{y-1} \,\middle|\, \frac{\operatorname{Lif}_{k-1}(\ln(1+t))-\operatorname{Lif}_k(\ln(1+t))}{t}x^{n-1-l}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{n-1} \binom{n-1}{l} c_l \\
&\quad \times \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) (1+t)^{y-1} \middle| \bigl(\mathrm{Lif}_{k-1}\bigl(\ln(1+t)\bigr) - \mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigr) \frac{x^{n-l}}{n-l} \right\rangle \\
&= \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} c_l \left(\left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \mathrm{Lif}_{k-1}\bigl(\ln(1+t)\bigr)(1+t)^{y-1} \middle| x^{n-l} \right\rangle \right. \\
&\quad \left. - \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \mathrm{Lif}_{k}\bigl(\ln(1+t)\bigr)(1+t)^{y-1} \middle| x^{n-l} \right\rangle \right) \\
&= \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} c_l \bigl(D_{n-l}^{(k-1)}(y-1|a_1,\ldots,a_r) - D_{n-l}^{(k)}(y-1|a_1,\ldots,a_r) \bigr).
\end{aligned}
$$

Since

$$
\partial_t \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) = \frac{1}{1+t} \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \frac{\sum_{i=1}^{r} \bigl(\frac{t}{\ln(1+t)} - \frac{a_i t (1+t)^{a_i}}{(1+t)^{a_i}-1} \bigr)}{t},
$$

with

$$
\sum_{i=1}^{r} \left(\frac{t}{\ln(1+t)} - \frac{a_i t (1+t)^{a_i}}{(1+t)^{a_i}-1} \right) = -\frac{1}{2} \left(\sum_{i=1}^{r} a_i \right) t + \cdots
$$

a series with order (≥ 1), the first term is

$$
\begin{aligned}
&\left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \mathrm{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{y-1} \middle| \frac{\sum_{i=1}^{r} \bigl(\frac{t}{\ln(1+t)} - \frac{a_i t (1+t)^{a_i}}{(1+t)^{a_i}-1} \bigr)}{t} x^{n-1} \right\rangle \\
&\quad = \frac{1}{n} \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \mathrm{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{y-1} \middle| \sum_{i=1}^{r} \left(\frac{t}{\ln(1+t)} - \frac{a_i t (1+t)^{a_i}}{(1+t)^{a_i}-1} \right) x^{n} \right\rangle \\
&\quad = \frac{r}{n} \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \mathrm{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{y-1} \middle| \frac{t}{\ln(1+t)} x^{n} \right\rangle \\
&\qquad - \frac{1}{n} \sum_{i=1}^{r} a_i \left\langle \frac{\ln(1+t)}{(1+t)^{a_i}-1} \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \mathrm{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{y+a_i-1} \middle| \frac{t}{\ln(1+t)} x^{n} \right\rangle \\
&\quad = \frac{r}{n} \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \mathrm{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{y-1} \middle| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^{n} \right\rangle \\
&\qquad - \frac{1}{n} \sum_{i=1}^{r} a_i \left\langle \frac{\ln(1+t)}{(1+t)^{a_i}-1} \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \mathrm{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{y+a_i-1} \middle| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^{n} \right\rangle \\
&\quad = \frac{r}{n} \sum_{l=0}^{n} \binom{n}{l} c_l \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \mathrm{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{y-1} \middle| x^{n-l} \right\rangle \\
&\qquad - \frac{1}{n} \sum_{i=1}^{r} a_i \sum_{l=0}^{n} \binom{n}{l} c_l
\end{aligned}
$$

$$\times \left\langle \frac{\ln(1+t)}{(1+t)^{a_i}-1} \prod_{j=1}^{r} \left(\frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{y+a_i-1} \middle| x^{n-l} \right\rangle$$

$$= \frac{r}{n} \sum_{l=0}^{n} \binom{n}{l} c_l D_{n-l}^{(k)}(y-1|a_1,\ldots,a_r)$$

$$- \frac{1}{n} \sum_{i=1}^{r} a_i \sum_{l=0}^{n} \binom{n}{l} c_l D_{n-l}^{(k)}(y+a_i-1|a_1,\ldots,a_r,a_i).$$

Therefore, we obtain

$$\begin{aligned} D_n^{(k)}(x|a_1,\ldots,a_r) &= xD_{n-1}^{(k)}(x-1|a_1,\ldots,a_r) \\ &\quad + \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} c_l \bigl(D_{n-l}^{(k-1)}(x-1|a_1,\ldots,a_r) - D_{n-l}^{(k)}(x-1|a_1,\ldots,a_r) \bigr) \\ &\quad + \frac{r}{n} \sum_{l=0}^{n} \binom{n}{l} c_l D_{n-l}^{(k)}(x-1|a_1,\ldots,a_r) \\ &\quad - \frac{1}{n} \sum_{j=1}^{r} \sum_{l=0}^{n} \binom{n}{l} a_j c_l D_{n-l}^{(k)}(x+a_j-1|a_1,\ldots,a_r,a_j) \\ &= xD_{n-1}^{(k)}(x-1|a_1,\ldots,a_r) \\ &\quad + \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} c_l D_{n-l}^{(k-1)}(x-1|a_1,\ldots,a_r) \\ &\quad + \frac{r-1}{n} \sum_{l=0}^{n} \binom{n}{l} c_l D_{n-l}^{(k)}(x-1|a_1,\ldots,a_r) \\ &\quad + \frac{1}{n} c_n - \frac{1}{n} \sum_{j=1}^{r} \sum_{l=0}^{n} \binom{n}{l} a_j c_l D_{n-l}^{(k)}(x+a_j-1|a_1,\ldots,a_r,a_j) \\ &= xD_{n-1}^{(k)}(x-1|a_1,\ldots,a_r) + \frac{1}{n} \sum_{l=0}^{n} \binom{n}{l} c_l D_{n-l}^{(k-1)}(x-1|a_1,\ldots,a_r) \\ &\quad + \frac{r-1}{n} \sum_{l=0}^{n} \binom{n}{l} c_l D_{n-l}^{(k)}(x-1|a_1,\ldots,a_r) \\ &\quad - \frac{1}{n} \sum_{j=1}^{r} \sum_{l=0}^{n} \binom{n}{l} a_j c_l D_{n-l}^{(k)}(x+a_j-1|a_1,\ldots,a_r,a_j), \end{aligned}$$

which is the identity (27). □

3.7 A relation including the Stirling numbers of the first kind

Theorem 7 *For $n \geq m \geq 1$, we have*

$$m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) D_l^{(k)}(a_1,\ldots,a_r)$$

$$= \frac{mr}{n} \sum_{l=0}^{n-m} \sum_{i=0}^{l} \binom{n}{l} \binom{l}{i} S_1(n-l,m) c_{l-i} D_i^{(k)}(-1|a_1,\ldots,a_r)$$

$$-\frac{m}{n}\sum_{l=0}^{n-m}\sum_{i=0}^{l}\sum_{j=1}^{r}\binom{n}{l}\binom{l}{i}S_1(n-l,m)a_jc_{l-i}D_i^{(k)}(a_j-1|a_1,\dots,a_r,a_j)$$
$$+\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_l^{(k-1)}(-1|a_1,\dots,a_r)$$
$$+(m-1)\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_l^{(k)}(-1|a_1,\dots,a_r). \tag{28}$$

Proof We shall compute

$$\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m\middle|x^n\right\rangle$$

in two different ways. On the one hand,

$$\begin{aligned}
&\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m\middle|x^n\right\rangle\\
&\quad=\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)\middle|\big(\ln(1+t)\big)^m x^n\right\rangle\\
&\quad=\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)\middle|\sum_{l=0}^{\infty}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^n\right\rangle\\
&\quad=\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)\middle|x^{n-l-m}\right\rangle\\
&\quad=\sum_{l=0}^{n-m}m!\binom{n}{l+m}S_1(l+m,m)D_{n-l-m}^{(k)}(a_1,\dots,a_r)\\
&\quad=\sum_{l=0}^{n-m}m!\binom{n}{l}S_1(n-l,m)D_l^{(k)}(a_1,\dots,a_r).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m\middle|x^n\right\rangle\\
&\quad=\left\langle\partial_t\left(\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m\right)\middle|x^{n-1}\right\rangle\\
&\quad=\left\langle\left(\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)\operatorname{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m\middle|x^{n-1}\right\rangle\\
&\qquad+\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\partial_t\operatorname{Lif}_k\big(\ln(1+t)\big)\big)\big(\ln(1+t)\big)^m\middle|x^{n-1}\right\rangle\\
&\qquad+\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(\ln(1+t)\big)\big(\partial_t\big(\ln(1+t)\big)^m\big)\middle|x^{n-1}\right\rangle.
\end{aligned}\tag{29}$$

The third term of (29) is equal to

$$
\begin{aligned}
&m\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1}\middle|\bigl(\ln(1+t)\bigr)^{m-1}x^{n-1}\right\rangle \\
&\quad= m\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1}\right| \\
&\qquad \left.\sum_{l=0}^{n-m}\frac{(m-1)!}{(l+m-1)!}S_1(l+m-1,m-1)t^{l+m-1}x^{n-1}\right\rangle \\
&\quad= m\sum_{l=0}^{n-m}\frac{(m-1)!}{(l+m-1)!}S_1(l+m-1,m-1)(n-1)_{l+m-1} \\
&\qquad\times\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1}\middle|x^{n-l-m}\right\rangle \\
&\quad= m!\sum_{l=0}^{n-m}\binom{n-1}{l+m-1}S_1(l+m-1,m-1)D_{n-l-m}^{(k)}(-1|a_1,\ldots,a_r) \\
&\quad= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_{l}^{(k)}(-1|a_1,\ldots,a_r).
\end{aligned}
$$

The second term of (29) is equal to

$$
\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\frac{\operatorname{Lif}_{k-1}(\ln(1+t))-\operatorname{Lif}_k(\ln(1+t))}{(1+t)\ln(1+t)}\right)\bigl(\ln(1+t)\bigr)^{m}\middle|x^{n-1}\right\rangle \\
&\quad= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_{k-1}\bigl(\ln(1+t)\bigr)(1+t)^{-1}\middle|\bigl(\ln(1+t)\bigr)^{m-1}x^{n-1}\right\rangle \\
&\qquad- \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_{k}\bigl(\ln(1+t)\bigr)(1+t)^{-1}\middle|\bigl(\ln(1+t)\bigr)^{m-1}x^{n-1}\right\rangle \\
&\quad= (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_{l}^{(k-1)}(-1|a_1,\ldots,a_r) \\
&\qquad- (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_{l}^{(k)}(-1|a_1,\ldots,a_r).
\end{aligned}
$$

The first term of (29) is equal to

$$
\begin{aligned}
&\left\langle \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\frac{\sum_{j=1}^{r}(\frac{t}{\ln(1+t)}-\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1})}{t}\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^{m}\middle|x^{n-1}\right\rangle \\
&\quad= \left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1}\bigl(\ln(1+t)\bigr)^{m}\right| \\
&\qquad\left.\frac{\sum_{j=1}^{r}(\frac{t}{\ln(1+t)}-\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1})}{t}x^{n-1}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1}\bigl(\ln(1+t)\bigr)^{m}\right| \\
&\quad \left.\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)x^n\right\rangle \\
&= \frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1}\right. \\
&\quad \times \left.\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\right|\left.\bigl(\ln(1+t)\bigr)^{m}x^n\right\rangle \\
&= \frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1}\right. \\
&\quad \times \left.\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\right|\left.\sum_{l=0}^{\infty}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^n\right\rangle \\
&= \frac{1}{n}\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1}\right. \\
&\quad \times \left.\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\right|\left.x^{n-l-m}\right\rangle \\
&= \frac{m!}{n}\sum_{l=0}^{n-m}\binom{n}{l+m}S_1(l+m,m) \\
&\quad \times \left(r\left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1}\right|\left.\frac{t}{\ln(1+t)}x^{n-l-m}\right\rangle\right. \\
&\quad - \sum_{j=1}^{r}a_j\left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{a_j-1}\right| \\
&\quad \left.\left.\frac{t}{\ln(1+t)}x^{n-l-m}\right\rangle\right) \\
&= \frac{m!}{n}\sum_{l=0}^{n-m}\binom{n}{l+m}S_1(l+m,m) \\
&\quad \times \left(r\left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1}\right|\left.\sum_{\nu=0}^{\infty}c_\nu\frac{t^\nu}{\nu!}x^{n-l-m}\right\rangle\right. \\
&\quad - \sum_{j=1}^{r}a_j\left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{a_j-1}\right| \\
&\quad \left.\left.\sum_{\nu=0}^{\infty}c_\nu\frac{t^\nu}{\nu!}x^{n-l-m}\right\rangle\right) \\
&= \frac{m!}{n}\sum_{l=0}^{n-m}\binom{n}{l+m}S_1(l+m,m)
\end{aligned}
$$

$$
\begin{aligned}
&\times \Biggl(r \sum_{\nu=0}^{n-l-m} \binom{n-l-m}{\nu} c_\nu \Biggl\langle \prod_{i=1}^{r} \biggl(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \biggr) \operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{-1} \Bigg| x^{n-l-m-\nu} \Biggr\rangle \\
&- \sum_{j=1}^{r} a_j \sum_{\nu=0}^{n-l-m} \binom{n-l-m}{\nu} c_\nu \\
&\times \Biggl\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1} \prod_{i=1}^{r} \biggl(\frac{\ln(1+t)}{(1+t)^{a_i}-1} \biggr) \operatorname{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{a_j-1} \Bigg| x^{n-l-m-\nu} \Biggr\rangle \Biggr) \\
&= \frac{m!}{n} \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m) \\
&\times \Biggl(r \sum_{\nu=0}^{n-l-m} \binom{n-l-m}{\nu} c_\nu D_{n-l-m-\nu}^{(k)}(-1|a_1,\ldots,a_r) \\
&- \sum_{j=1}^{r} \sum_{\nu=0}^{n-l-m} \binom{n-l-m}{\nu} a_j c_\nu D_{n-l-m-\nu}^{(k)}(a_j-1|a_1,\ldots,a_r,a_j) \Biggr) \\
&= \frac{m!}{n} \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \\
&\times \Biggl(r \sum_{i=0}^{l} \binom{l}{i} c_i D_{n-i}^{(k)}(-1|a_1,\ldots,a_r) - \sum_{j=1}^{r} \sum_{i=0}^{l} \binom{l}{i} a_j c_i D_{l-i}^{(k)}(a_j-1|a_1,\ldots,a_r,a_j) \Biggr).
\end{aligned}
$$

Therefore, we get for $n \geq m \geq 1$

$$
\begin{aligned}
&m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) D_l^{(k)}(a_1,\ldots,a_r) \\
&= m! \frac{r}{n} \sum_{l=0}^{n-m} \sum_{i=0}^{l} \binom{n}{l} \binom{l}{i} S_1(n-l,m) c_i D_{l-i}^{(k)}(-1|a_1,\ldots,a_r) \\
&- m! \frac{1}{n} \sum_{l=0}^{n-m} \sum_{i=0}^{l} \sum_{j=1}^{r} \binom{n}{l} \binom{l}{i} S_1(n-l,m) a_j c_i D_{l-i}^{(k)}(a_j-1|a_1,\ldots,a_r,a_j) \\
&+ (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) D_l^{(k-1)}(-1|a_1,\ldots,a_r) \\
&- (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) D_l^{(k)}(-1|a_1,\ldots,a_r) \\
&+ m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) D_l^{(k)}(-1|a_1,\ldots,a_r).
\end{aligned}
$$

Dividing both sides by $(m-1)!$, we obtain for $n \geq m \geq 1$

$$
\begin{aligned}
&m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) D_l^{(k)}(a_1,\ldots,a_r) \\
&= \frac{mr}{n} \sum_{l=0}^{n-m} \sum_{i=0}^{l} \binom{n}{l} \binom{l}{i} S_1(n-l,m) c_{l-i} D_i^{(k)}(-1|a_1,\ldots,a_r)
\end{aligned}
$$

$$-\frac{m}{n}\sum_{l=0}^{n-m}\sum_{i=0}^{l}\sum_{j=1}^{r}\binom{n}{l}\binom{l}{i}S_1(n-l,m)a_j c_{l-i}D_i^{(k)}(a_j-1|a_1,\ldots,a_r,a_j)$$
$$+\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_l^{(k-1)}(-1|a_1,\ldots,a_r)$$
$$+(m-1)\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_l^{(k)}(-1|a_1,\ldots,a_r).$$

Thus, we get (28). □

3.8 A relation with the falling factorials

Theorem 8

$$D_n^{(k)}(x|a_1,\ldots,a_r)=\sum_{m=0}^{n}\binom{n}{m}D_{n-m}^{(k)}(a_1,\ldots,a_r)(x)_m. \tag{30}$$

Proof For (12) and (19), assume that $D_n^{(k)}(x|a_1,\ldots,a_r)=\sum_{m=0}^{n}C_{n,m}(x)_m$. By (11), we have

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\frac{1}{\prod_{j=1}^{r}(\frac{e^{a_j\ln(1+t)}-1}{\ln(1+t)})\frac{1}{\operatorname{Lif}_k(\ln(1+t))}}t^m\middle|x^n\right\rangle\\&=\frac{1}{m!}\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)\middle|t^m x^n\right\rangle\\&=\binom{n}{m}\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(\ln(1+t)\bigr)\middle|x^{n-m}\right\rangle\\&=\binom{n}{m}D_{n-m}^{(k)}(a_1,\ldots,a_r).\end{aligned}$$

Thus, we get the identity (30). □

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\lambda\in\mathbb{C}$ with $\lambda\neq1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\lambda)$ are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}(x|\lambda)\frac{t^n}{n!}$$

(see *e.g.* [11]).

Theorem 9

$$\begin{aligned}D_n^{(k)}(x|a_1,\ldots,a_r)=\sum_{m=0}^{n}&\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j\right.\\&\left.\times(1-\lambda)^{-j}S_1(n-j-l,m)D_l^{(k)}(a_1,\ldots,a_r)\right)H_m^{(s)}(x|\lambda).\end{aligned} \tag{31}$$

Proof For (12) and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right), \tag{32}$$

assume that $D_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^n C_{n,m}H_m^{(s)}(x|\lambda)$. By (11), similarly to the proof of (28), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{(\frac{e^{\ln(1+t)}-\lambda}{1-\lambda})^s}{\prod_{j=1}^r(\frac{e^{a_j\ln(1+t)}-1}{\ln(1+t)})\frac{1}{\mathrm{Lif}_k(\ln(1+t))}}\big(\ln(1+t)\big)^m \middle| x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^r\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m(1-\lambda+t)^s \middle| x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s} \\
&\quad\times\left\langle \prod_{j=1}^r\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m \middle| \sum_{i=0}^{\min\{s,n\}}\binom{s}{i}(1-\lambda)^{s-i}t^ix^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i \\
&\quad\times\left\langle \prod_{j=1}^r\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m \middle| x^{n-i}\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)D_l^{(k)}(a_1,\dots,a_r) \\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{s}{i}\binom{n-i}{l}(n)_i(1-\lambda)^{-i}S_1(n-i-l,m)D_l^{(k)}(a_1,\dots,a_r).
\end{aligned}$$

Thus, we get the identity (31). □

3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^\infty \frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see *e.g.* [7, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r = \sum_{n=0}^\infty \frac{\mathfrak{C}_n^{(r)}}{n!}t^n$$

(see *e.g.* [12, (2.1)], [13, (6)]).

Theorem 10

$$\begin{aligned} &D_n^{(k)}(x|a_1,\ldots,a_r) \\ &\quad= \sum_{m=0}^{n}\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)D_l^{(k)}(a_1,\ldots,a_r)\right)\mathfrak{B}_m^{(s)}(x). \end{aligned} \tag{33}$$

Proof For (12) and

$$\mathfrak{B}_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^s, t\right), \tag{34}$$

assume that $D_n^{(k)}(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of (28), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \frac{(\frac{e^{\ln(1+t)}-1}{\ln(1+t)})^s}{\prod_{j=1}^{r}(\frac{e^{a_j\ln(1+t)}-1}{\ln(1+t)})\frac{1}{\mathrm{Lif}_k(\ln(1+t))}}\bigl(\ln(1+t)\bigr)^m \middle| x^n\right\rangle \\ &= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m \middle| \left(\frac{t}{\ln(1+t)}\right)^s x^n\right\rangle \\ &= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m \middle| \sum_{i=0}^{\infty}\mathfrak{C}_i^{(s)}\frac{t^i}{i!}x^n\right\rangle \\ &= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m \middle| x^{n-i}\right\rangle \\ &= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)D_l^{(k)}(a_1,\ldots,a_r) \\ &= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)D_l^{(k)}(a_1,\ldots,a_r). \end{aligned}$$

Thus, we get the identity (33). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Author details
[1]Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea. [2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea. [3]Graduate School of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan. [4]Division of General Education, Kwangwoon University, Seoul, 139-701, Republic of Korea.

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