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Extended p-adic q-invariant integrals on $\mathbb{Z}_p$ associated with applications of umbral calculus

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Abstract
The fundamental aim of this paper is to consider some applications of umbral calculus by utilizing from the extended p-adic q-invariant integral on $\mathbb{Z}_p$. From those considerations, we derive some new interesting properties on the extended p-adic q-Bernoulli numbers and polynomials. That is, a systemic study of the class of Sheffer sequences in connection with generating function of the p-adic q-Bernoulli polynomials are given in the present work.

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1 Preliminaries

In the complex plane, the Bernoulli polynomials, $B_n(x)$, are defined by

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{tx}, \quad |t| < 2\pi. \tag{1.1}$$

In particular, the case $x = 0$ in (1.1), we have $B_n(0) := B_n$ are called Bernoulli numbers. These numbers are extremely important in number theory and other areas of mathematics and physics. With the help of generating function of Bernoulli numbers, one can easily derive that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = \frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}, \ldots,$ and $B_{2n+1} = 0$ for $n \in \mathbb{N}$ (see [1–8]). As is well known, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } s \in \mathbb{C}. \tag{1.2}$$

We note that the Bernoulli numbers interpolate by the Riemann zeta function, which plays an important role in analytic number theory and has applications in physics, probability theory and applied statistics. Firstly, Leonard Euler studied and introduced the Riemann zeta function in a real argument without using complex analysis. From (1.1) and (1.2), one has

$$\zeta(1-n) = -\frac{B_n}{n} \quad \text{for } n \in \mathbb{N} = \{1, 2, 3, \ldots\}.$$

A link between the zeta function and prime numbers was discovered by Euler, who proved the following identity:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^s}&=\frac{1}{1-2^{-s}}\frac{1}{1-3^{-s}}\cdots\frac{1}{1-p^{-s}}\cdots\\&=\prod_{p\text{ prime}}\frac{1}{1-p^{-s}},\end{aligned}$$

where the both sides of Euler product formula converge for $\Re e(s)>1$ (for details on this subject, see [1–12]).

Let p be a fixed odd prime number. Throughout this work, we use the following notations, where $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^*=\mathbb{N}\cup\{0\}$. The p-adic absolute value is defined by $|p|_p=p^{-1}$. Also, we assume that $|q-1|_p<1$ is an indeterminate. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f\in\mathrm{UD}(\mathbb{Z}_p)$, Kim defined p-adic q-invariant integral on $\mathbb{Z}_p$ by the rule:

$$I_q(f)=\int_{\mathbb{Z}_p}f(\xi)\,d\mu_q(\xi)=\lim_{n\to\infty}\frac{1}{[p^n]_q}\sum_{A=0}^{p^n-1}f(A)q^A,\tag{1.3}$$

where $[x]_q$ is q-analogue of x defined by

$$[x]_q=\frac{q^x-1}{q-1}.$$

We note that $\lim_{q\to1}[x]_q=x$ (for details, see [1, 7, 13–26]).

Let $f_1(\xi):=f(\xi+1)$. By (1.3), we have

$$qI_q(f_1)=I_q(f)+\frac{q-1}{\log q}f'(0),\tag{1.4}$$

where $f'(0)=\frac{df(x)}{dx}|_{x=0}$ (for details, see [14, 15]).

In [14], Kim showed that Carlitz's q-Bernoulli numbers and polynomials can be expressed as an integral by the q-analogue μ_q of the ordinary p-adic invariant measure as follows:

$$B_m(q)=\int_{\mathbb{Z}_p}[\xi]_q^m\,d\mu_q(\xi)=\lim_{n\to\infty}\frac{1}{[p^n]_q}\sum_{A=0}^{p^n-1}[A]_q^n q^A.\tag{1.5}$$

Now also, we consider the extended p-adic q-invariant integral on $\mathbb{Z}_p$ due to Kim [14] in the following form: for $|1-\beta|_p<1$

$$I_q(f:\beta)=\int_{\mathbb{Z}_p}\beta^{\xi}f(\xi)\,d\mu_q(\xi)=\lim_{n\to\infty}\frac{1}{[p^n]_q}\sum_{A=0}^{p^n-1}\beta^A f(A)q^A,\tag{1.6}$$

where $I_q(f:\beta)$ are called extended p-adic q-invariant integral on $\mathbb{Z}_p$.

Let us now consider $f_1(\xi) := f(\xi + 1)$, then we compute as follows:

$$\begin{aligned} q\beta I_q(f_1 : \beta) &= \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{A=0}^{p^n-1} \beta^{A+1} f(A+1) q^{A+1} \\ &= I_q(f:\beta) + (1-q) \lim_{n\to\infty} \left(\frac{-f(0) + \beta^{p^n} q^{p^n} f(p^n)}{1-q^{p^n}} \right) \\ &= I_q(f:\beta) + \frac{q-1}{\log q} f'(0). \end{aligned}$$

Therefore, we state the following lemma.

Lemma 1 *For* $f \in \mathrm{UD}(\mathbb{Z}_p)$,

$$q\beta I_q(f_1 : \beta) = I_q(f:\beta) + \frac{q-1}{\log q} f'(0).$$

Taking $f(\xi) = e^{t(x+\xi)} \in \mathrm{UD}(\mathbb{Z}_p)$ in Lemma 1, then we consider the following generating function:

$$\begin{aligned} \int_{\mathbb{Z}_p} \beta^{\xi} e^{t(x+\xi)}\, d\mu_q(\xi) &= \frac{q-1}{\log q} \frac{t}{q\beta e^t - 1} e^{tx} \\ &= \sum_{n=0}^{\infty} B_{n,\beta}(x \mid q) \frac{t^n}{n!} \quad \bigl(q\beta \neq 1 \text{ and } \bigl|\log(q\beta) + t\bigr| < 2\pi \bigr), \end{aligned} \tag{1.7}$$

where $B_{n,\beta}(x \mid q)$ are called extended q-Bernoulli polynomials. In the special case, $x = 0$, $B_{n,\beta}(0 \mid q) := B_{n,\beta}(q)$ are called extended q-Bernoulli numbers.

We note that

$$\lim_{\substack{\beta=1 \\ q\to 1}} \left(\frac{q-1}{\log q} \frac{t}{q\beta e^t - 1} e^{tx} \right) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

That is, we have

$$\lim_{\substack{\beta=1 \\ q\to 1}} B_{n,\beta}(x \mid q) = B_n(x).$$

The relation between extended p-adic q-Bernoulli numbers and extended p-adic q-Bernoulli polynomials is given by

$$B_{n,\beta}(x \mid q) = \sum_{l=0}^{n} \binom{n}{l} x^l B_{n-l,\beta}(q) = \bigl(x + B_\beta(q)\bigr)^n, \tag{1.8}$$

with the usual of replacing $(B_\beta(q))^n$ by $B_{n,\beta}(q)$. By (1.7) and (1.8), we easily see that

$$B_{0,\beta}(q) = \frac{q-1}{\log q} \quad \text{and} \quad q\beta\bigl(B_\beta(q) + 1\bigr)^n - B_{n,\beta}(q) = \begin{cases} \frac{q-1}{\log q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

From (1.7), we derive Witt's formulae for extended p-adic q-Bernoulli numbers and polynomials, respectively:

$$B_{n,\beta}(q) = \int_{\mathbb{Z}_p} \beta^{\xi} \xi^n \, d\mu_q(\xi) \quad \text{and} \quad B_{n,\beta}(x \mid q) = \int_{\mathbb{Z}_p} \beta^{\xi} (x+\xi)^n \, d\mu_q(\xi). \tag{1.9}$$

By (1.7), we have

$$-\frac{B_{n,\beta}(x \mid q)}{n} = \frac{q-1}{\log q} \sum_{m=0}^{\infty} q^m (m+x)^{n-1} \beta^m \quad \text{for } n \in \mathbb{N}. \tag{1.10}$$

Let us now consider the following:

$$F_{q,\beta}(x,t) = \frac{q-1}{\log q} \frac{t}{q\beta e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_{n,\beta}(x \mid q) \frac{t^n}{n!}. \tag{1.11}$$

By applying Mellin transformation to (1.11), we derive that for $s \in \mathbb{C}$:

$$\begin{aligned} \zeta(s,x:q:\beta) &= \frac{\log q}{(q-1)\Gamma(s)} \int_0^{\infty} F_{q,\beta}(x,-t) t^{s-2} \, dt \\ &= \sum_{m=0}^{\infty} q^m \beta^m \left(\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-t(m+x)} \, dt \right) \\ &= \sum_{m=0}^{\infty} \frac{q^m \beta^m}{(m+x)^s}. \end{aligned} \tag{1.12}$$

Here, $\Gamma(s)$ is Euler's Gamma function. Thanks to (1.10) and (1.12), we discover the following:

$$\zeta(1-n,x:q:\beta) = -\frac{\log q}{q-1} \frac{B_{n,\beta}(x \mid q)}{n} \quad \text{for any } n \in \mathbb{N}^*. \tag{1.13}$$

Setting $\beta = 1$ and $q \to 1$ in (1.13) reduces to

$$\zeta(1-n,x) = -\frac{B_n(x)}{n},$$

which has a profound effect on number theory and complex analysis.

By (1.6) and (1.7), we develop as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} \beta^{\xi} (x+\xi)^n \, d\mu_q(\xi) &= \lim_{m \to \infty} \frac{1}{[dp^m]_q} \sum_{A=0}^{dp^m-1} \beta^A (x+A)^n q^A \\ &= \frac{d^n}{[d]_q} \sum_{k=0}^{d-1} \beta^k q^k \left(\lim_{m \to \infty} \frac{1}{[p^m]_{q^d}} \sum_{A=0}^{p^m-1} (\beta^d)^A (q^d)^A \left(\frac{x+k}{d} + A \right)^n \right) \\ &= \frac{d^n}{[d]_q} \sum_{k=0}^{d-1} \beta^k q^k \int_{\mathbb{Z}_p} \beta^{d\xi} \left(\frac{x+k}{d} + \xi \right)^n d\mu_{q^d}(\xi), \end{aligned}$$

where d is a natural number. That is,

$$\int_{\mathbb{Z}_p} \beta^{\xi}(x+\xi)^n \, d\mu_q(\xi) = \frac{d^n}{[d]_q} \sum_{k=0}^{d-1} \beta^k q^k \int_{\mathbb{Z}_p} \beta^{d\xi} \left(\frac{x+k}{d} + \xi \right)^n d\mu_{q^d}(\xi). \tag{1.14}$$

By (1.9) and (1.14), we get

$$B_{n,\beta}(dx \mid q) = \frac{d^n}{[d]_q} \sum_{k=0}^{d-1} \beta^k q^k B_{n,\beta^d}\left(x + \frac{k}{d} \,\Big|\, q^d \right). \tag{1.15}$$

Putting $\beta = 1$ and $q \to 1$ in (1.15), then it leads to $B_n(dx) = d^n \sum_{k=0}^{d-1} B_n(x + \frac{k}{d})$, which is well known as Raabe's formula.

Let us now define the following notations, where $\mathbb{C}$ denotes the set of complex numbers, $\mathcal{F}$ denotes the set of all formal power series in the variable t over $\mathbb{C}$ with $\mathcal{F} = \{f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C}\}$, $\mathcal{P} = \mathbb{C}[x]$ and $\mathcal{P}^*$ denotes the vector space of all linear functional on $\mathcal{P}$, $\langle L \mid p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, and it is well known that the vector space operation on $\mathcal{P}^*$ is defined by $\langle L + M \mid p(x)\rangle = \langle L \mid p(x)\rangle + \langle M \mid p(x)\rangle$ and $\langle cL \mid p(x)\rangle = c\langle L \mid p(x)\rangle$ for some constant c in $\mathbb{C}$ (see [27–30]).

The following is well known as a formal power series by the rule:

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F},$$

which describes a linear functional on $\mathcal{P}$ as $\langle f(t) \mid x^n\rangle = a_n$ for all $n \geq 0$ (for details, see [27–30]). Moreover,

$$\langle t^k \mid x^n \rangle = n!\delta_{n,k}, \tag{1.16}$$

where $\delta_{n,k}$ is the Kronecker delta. It is easy to see that

$$f_L(t) = \sum_{k=0}^{\infty} \langle L \mid x^k \rangle \frac{t^k}{k!},$$

therefore we procure

$$\langle f_L(t) \mid x^n \rangle = \langle L \mid x^n \rangle$$

and so as linear functionals $L = f_L(t)$ (see [27–30]). Additionally, the map $L \to f_L(t)$ is a vector space isomorphism from $\mathcal{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of the formal power series in t and the vector space of all linear functionals on $\mathcal{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. $\mathcal{F}$ will be called as umbral algebra (see [27–30]).

Obviously, $\langle e^{yt} \mid x^n\rangle = y^n$. From this, it reduces to

$$\langle e^{yt} \mid p(x) \rangle = p(y)$$

(see [27–31]). We note that for all $f(t)$ in $\mathcal{F}$

$$f(t) = \sum_{k=0}^{\infty} \langle f(t) \mid x^k \rangle \frac{t^k}{k!} \tag{1.17}$$

and for all polynomial $p(x)$,

$$p(x) = \sum_{k=0}^{\infty} \langle t^k \mid p(x) \rangle \frac{x^k}{k!} \tag{1.18}$$

(for details, see [27–30]). The order $o(f(t))$ of the power series $f(t) \neq 0$ is the smallest integer k for which a_k does not vanish. It is considered $o(f(t)) = \infty$ if $f(t) = 0$. We see that $o(f(t)g(t)) = o(f(t)) + o(g(t))$ and $o(f(t) + g(t)) \geq \min\{o(f(t)), o(g(t))\}$. The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$, if and only if $o(f(t)) = 0$. Such series is called an invertible series. A series $f(t)$ for which $o(f(t)) = 1$ is called a delta series (see [27–31]). For $f(t), g(t) \in \mathcal{F}$, we have $\langle f(t)g(t) \mid p(x)\rangle = \langle f(t) \mid g(t)p(x)\rangle$. A delta series $f(t)$ has a compositional inverse $\overline{f}(t)$ such that $f(\overline{f}(t)) = \overline{f}(f(t)) = t$.

For $f(t), g(t) \in \mathcal{F}$, we have $\langle f(t)g(t) \mid p(x)\rangle = \langle f(t) \mid g(t)p(x)\rangle$. From (1.17), we have

$$p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{l=k}^{\infty} \frac{\langle t^l \mid p(x)\rangle}{l!} l(l-1)\cdots(l-k+1)x^{l-k}. \tag{1.19}$$

Hence, we get that

$$p^{(k)}(0) = \langle t^k \mid p(x) \rangle = \langle 1 \mid p^{(k)}(x) \rangle. \tag{1.20}$$

By (1.19), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}. \tag{1.21}$$

So from the above

$$e^{yt} p(x) = p(x+y). \tag{1.22}$$

Let $S_n(x)$ be a polynomial with $\deg S_n(x) = n$. Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k \mid S_n(x)\rangle = n!\delta_{n,k}$ for all $n, k \geq 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ or that $S_n(t)$ is Sheffer for $(g(t), f(t))$.

The Sheffer sequence for $(1, f(t))$ is called the associated sequence for $f(t)$ or $S_n(x)$ is associated to $f(t)$. The Sheffer sequence for $(g(t), t)$ is called the Appell sequence for $g(t)$ or $S_n(x)$ is Appell for $g(t)$.

Let $p(x) \in \mathcal{P}$. Then we have

$$\begin{aligned} &\langle f(t) \mid xp(x) \rangle = \langle \partial_t f(t) \mid p(x) \rangle = \langle f'(t) \mid p(x) \rangle, \\ &\langle e^{yt} + 1 \mid p(x) \rangle = p(y) + p(0) \quad \text{(see [30])}. \end{aligned} \tag{1.23}$$

Let $S_n(x)$ be Sheffer for $(g(t),f(t))$. Then

$$\begin{aligned}
&h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \mid S_k(x)\rangle}{k!} g(t)f(t)^k, \quad h(t) \in \mathcal{F}, \\
&p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t)f(t)^k \mid p(x)\rangle}{k!} S_k(x), \quad p(x) \in \mathcal{P}, \\
&\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} S_k(y)\frac{t^k}{k!} \quad \text{for all } y \in \mathbb{C}, \qquad (1.24) \\
&f(t)S_n(x) = nS_{n-1}(x).
\end{aligned}$$

Also, it is well known in [30] that

$$\bigl\langle f_1(t)f_2(t)\cdots f_m(t) \mid x^n\bigr\rangle = \sum \binom{n}{i_1,\ldots,i_m} \bigl\langle f_1(t) \mid x^{i_1}\bigr\rangle \cdots \bigl\langle f_m(t) \mid x^{i_m}\bigr\rangle, \tag{1.25}$$

where $f_1(t),f_2(t),\ldots,f_m(t) \in \mathcal{F}$ and the sum is over all nonnegative integers $i_1,\ldots,i_m$ such that $i_1 + \cdots + i_m = n$ (see [30]).

Dere and Simsek have studied applications of umbral algebra to special functions in [29]. Kim *et al.* also gave some properties of umbral calculus for Frobenius-Euler polynomials [27] and Euler polynomials [28]. Also, they investigated some new applications of umbral calculus associated with p-adic invariants integral on $\mathbb{Z}_p$ and fermionic p-adic integral on $\mathbb{Z}_p$ in [13].

By the same motivation of the above, we also discover both new and interesting applications of umbral calculus by using extended p-adic q-invariant integral on $\mathbb{Z}_p$. By virtue of which, we procure some new interesting equalities on the extended p-adic q-Bernoulli numbers and polynomials and extended p-adic q-Bernoulli polynomials of order k. Recently, several authors have studied the q-Bernoulli numbers and polynomials. Also, we note that our q-extensions of Bernoulli numbers and polynomials in the present paper are different from the q-extensions of Bernoulli numbers and polynomials of several authors in previous papers.

2 Identities involving extended p-adic q-invariant integrals on $\mathbb{Z}_p$ related to applications of umbral calculus

Suppose that $S_n(x)$ is an Appell sequence for $g(t)$. Then, by (1.24), we have

$$\frac{1}{g(t)}x^n = S_n(x) \quad \Leftrightarrow \quad x^n = g(t)S_n(x) \quad (n \geq 0). \tag{2.1}$$

We now consider that

$$g_{q,\beta}(t) = \frac{\log q}{q-1}\frac{q\beta e^t - 1}{t} \in \mathcal{F}.$$

Therefore, we easily notice that $g(t)$ is an invertible series. By (2.1), we have

$$\sum_{n=0}^{\infty} B_{n,\beta}(x \mid q)\frac{t^n}{n!} = \frac{1}{g_{q,\beta}(t)} e^{xt}, \tag{2.2}$$

which means

$$\frac{1}{g_{q,\beta}(t)}x^n = B_{n,\beta}(x \mid q). \tag{2.3}$$

Also, by (1.24), we have

$$tB_{n,\beta}(x \mid q) = \big(B_{n,\beta}(x \mid q)\big)' = nB_{n-1,\beta}(x \mid q). \tag{2.4}$$

Because of (2.3) and (2.4), we have the following proposition.

Proposition 1 *For $n \geq 0$, $B_{n,\beta}(x \mid q)$ is an Appell sequence for $g_{q,\beta}(t) = \frac{\log q}{q-1}\frac{q\beta e^t-1}{t}$.*

By (1.9), we have

$$\begin{aligned}\sum_{n=1}^{\infty} B_{n,\beta}(x \mid q)\frac{t^n}{n!} &= \frac{xg_{q,\beta}(t)e^{xt} - g'_{q,\beta}(t)e^{xt}}{g_{q,\beta}(t)^2} \\ &= \sum_{n=0}^{\infty}\left(x\frac{1}{g_{q,\beta}(t)}x^n - \frac{g'_{q,\beta}(t)}{g_{q,\beta}(t)}\frac{1}{g_{q,\beta}(t)}x^n\right)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

Because of (2.3) and (2.5), we discover the following:

$$B_{n+1,\beta}(x \mid q) = xB_{n,\beta}(x \mid q) - \frac{g'_{q,\beta}(t)}{g_{q,\beta}(t)}B_{n,\beta}(x \mid q).$$

Therefore, we arrive at the following theorem.

Theorem 1 *Let $g_{q,\beta}(t) = \frac{\log q}{q-1}\frac{q\beta e^t-1}{t} \in \mathcal{F}$. Then we have for $n \geq 0$:*

$$B_{n+1,\beta}(x \mid q) = \left(x - \frac{g'_{q,\beta}(t)}{g_{q,\beta}(t)}\right)B_{n,\beta}(x \mid q). \tag{2.6}$$

Also,

$$\zeta(1-n, x : q : \beta) = \frac{n}{n+1}\left(x - \frac{g'_{q,\beta}(t)}{g_{q,\beta}(t)}\right)\zeta(1-n, x : q : \beta),$$

where $g'_{q,\beta}(t) = \frac{dg_{q,\beta}(t)}{dt}$.

By (1.9), it is not difficult to see that

$$\sum_{n=0}^{\infty}\big(q\beta B_{n,\beta}(x+1 \mid q) - B_{n,\beta}(x \mid q)\big)\frac{t^n}{n!} = \frac{q-1}{\log q}\sum_{n=0}^{\infty}x^n\frac{t^{n+1}}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the above, we have the following:

$$q\beta B_{n,\beta}(x+1 \mid q) - B_{n,\beta}(x \mid q) = \frac{q-1}{\log q}nx^{n-1}. \tag{2.7}$$

By Theorem 1, we derive

$$g_{q,\beta}(t)B_{n+1,\beta}(x \mid q) = g_{q,\beta}(t)xB_{n,\beta}(x \mid q) - g'_{q,\beta}(t)B_{n,\beta}(x \mid q). \tag{2.8}$$

So from above

$$\bigl(q\beta e^t - 1\bigr)B_{n+1,\beta}(x \mid q) = \bigl(q\beta e^t - 1\bigr)xB_{n,\beta}(x \mid q) - \left(\frac{\log q}{q-1}q\beta e^t - g_{q,\beta}(t)\right)B_{n,\beta}(x \mid q).$$

Thus, we have

$$\begin{aligned} &q\beta B_{n+1,\beta}(x+1 \mid q) - B_{n+1,\beta}(x \mid q) \\ &\quad = q\beta(x+1)B_{n,\beta}(x+1 \mid q) - xB_{n,\beta}(x \mid q) - \frac{q\beta \log q}{q-1}B_{n,\beta}(x+1 \mid q) + x^n. \end{aligned} \tag{2.9}$$

From (2.7), (2.8) and (2.9), we have the following theorem.

Theorem 2 *For $n \geq 0$, then we have*

$$q\beta B_{n,\beta}(x+1 \mid q) - B_{n,\beta}(x \mid q) = \frac{q-1}{\log q}nx^{n-1}. \tag{2.10}$$

Suppose that $S_n(x)$ is Sheffer sequence for $(g(t), f(t))$. Then the following is introduced as Sheffer identity by the rule:

$$S_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}P_k(y)S_{n-k}(x) = \sum_{k=0}^{n}\binom{n}{k}P_k(x)S_{n-k}(y), \tag{2.11}$$

where $P_k(y) = S_k(y)g(t)$ is associated to $f(t)$ (for details, see [28–30]).

Thanks to (1.7) and (2.11), we have

$$\begin{aligned} B_{n,\beta}(x+y \mid q) &= \sum_{k=0}^{n}\binom{n}{k}P_k(y)S_{n-k}(x) \\ &= \sum_{k=0}^{n}\binom{n}{k}B_{n,\beta}(y \mid q)x^k. \end{aligned}$$

From the above, we readily see that

$$B_{n,\beta}(x+y \mid q) = \sum_{k=0}^{n}\binom{n}{k}B_{n,\beta}(y \mid q)x^k.$$

By (1.7), we easily get for $\alpha\ (\neq 0) \in \mathbb{C}$:

$$B_{n,\beta}(\alpha x \mid q) = \frac{g_{q,\beta}(t)}{g_{q,\beta}(\frac{t}{\alpha})}B_{n,\beta}(x \mid q). \tag{2.12}$$

By virtue of (1.15) and (2.12), we see that

$$\frac{g_{q,\beta}(t)}{g_{q,\beta}(\frac{t}{\alpha})}B_{n,\beta}(x \mid q) = \frac{\alpha^n}{[\alpha]_q}\sum_{k=0}^{\alpha-1}(q\beta)^k B_{n,\beta^\alpha}\left(x + \frac{k}{\alpha} \;\middle|\; q^\alpha\right).$$

Let us now contemplate the linear functional $f(t)$ by the following expression:

$$\langle f(t) \mid p(x)\rangle = \int_{\mathbb{Z}_p} \beta^{\xi} p(\xi)\, d\mu_q(\xi) \tag{2.13}$$

for all polynomials $p(x)$. From (2.13), we readily derive that

$$f(t) = \sum_{n=0}^{\infty} \frac{\langle f(t) \mid x^n\rangle}{n!} t^n = \sum_{n=1}^{\infty} \left(\int_{\mathbb{Z}_p} \beta^{\xi} \xi^n \, d\mu_q(\xi)\right) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \beta^{\xi} e^{\xi t}\, d\mu_q(\xi). \tag{2.14}$$

Thus, we have

$$f(t) = \int_{\mathbb{Z}_p} \beta^{\xi} e^{\xi t}\, d\mu_q(\xi) = \frac{q-1}{\log q} \frac{t}{\beta q e^t - 1}. \tag{2.15}$$

Therefore, by (2.13) and (2.15), we arrive at the following theorem.

Theorem 3 *For $n \geq 0$, we have*

$$\langle f(t) \mid p(x)\rangle = \int_{\mathbb{Z}_p} \beta^{\xi} p(\xi)\, d\mu_q(\xi). \tag{2.16}$$

Also,

$$\left\langle \frac{q-1}{\log q} \frac{t}{q\beta e^t - 1} \,\middle|\, p(x) \right\rangle = \int_{\mathbb{Z}_p} \beta^{\xi} p(\xi)\, d\mu_q(\xi). \tag{2.17}$$

Obviously that

$$B_{n,\beta}(q) = \left\langle \int_{\mathbb{Z}_p} \beta^{\xi} e^{\xi t}\, d\mu_q(\xi) \,\middle|\, x^n \right\rangle. \tag{2.18}$$

In view of (1.9) and (2.18), we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \beta^{\xi} (x+\xi)^n \, d\mu_q(\xi)\right) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \beta^{\xi} e^{(x+\xi)t}\, d\mu_q(\xi) \\ &= \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \beta^{\xi} e^{\xi t}\, d\mu_q(\xi) x^n \right) \frac{t^n}{n!}. \end{aligned} \tag{2.19}$$

By (1.9) and (2.20), we see that for $n \in \mathbb{N}^*$:

$$B_{n,\beta}(x \mid q) = \int_{\mathbb{Z}_p} \beta^{\xi} (x+\xi)^n \, d\mu_q(\xi) = \int_{\mathbb{Z}_p} \beta^{\xi} e^{\xi t}\, d\mu_q(\xi) x^n. \tag{2.20}$$

Consequently, we get the following theorem.

Theorem 4 *For $p(x) \in \mathcal{P}$, we have*

$$\begin{aligned} \int_{\mathbb{Z}_p} \beta^{\xi} p(x+\xi)\, d\mu_q(\xi) &= \int_{\mathbb{Z}_p} \beta^{\xi} e^{\xi t}\, d\mu_q(\xi) p(x) \\ &= \frac{q-1}{\log q} \frac{t}{\beta q e^t - 1} p(x). \end{aligned} \tag{2.21}$$

That is,

$$B_{n,\beta}(x \mid q) = \int_{\mathbb{Z}_p} \beta^{\xi} e^{\xi t}\, d\mu_q(\xi) x^n = \frac{q-1}{\log q} \frac{t}{\beta q e^t - 1} x^n. \tag{2.22}$$

For $|1-\beta|_p < 1$, we introduce extended p-adic q-Bernoulli polynomials of order k as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k+x)t}\, d^*\mu_q(\xi) &= \left(\frac{q-1}{\log q} \frac{t}{q\beta e^t - 1}\right)^k e^{xt} \\ &= \sum_{n=0}^{\infty} B_{n,\beta}^{(k)}(x \mid q) \frac{t^n}{n!}, \end{aligned} \tag{2.23}$$

which we have used the following equality:

$$\int_{\mathbb{Z}_p^k} d^*\mu_q(\xi) := \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} d\mu_q(\xi_1)\, d\mu_q(\xi_2) \cdots d\mu_q(\xi_k).$$

In the special case, for $x = 0$ in (2.23), we have $B_{n,\beta}^{(k)}(0 \mid q) := B_{n,\beta}^{(k)}(q)$, which are called extended p-adic q-Bernoulli numbers of order k.

From (2.23), we have

$$\begin{aligned} &\int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} (\xi_1 + \cdots + \xi_k + x)^n\, d^*\mu_q(\xi) \\ &\quad = \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\ldots,i_m} \int_{\mathbb{Z}_p} \beta^{\xi_1} \xi_1^{i_1}\, d\mu_q(\xi_1) \cdots \int_{\mathbb{Z}_p} \beta^{\xi_k} \xi_k^{i_k}\, d\mu_q(\xi_k) \\ &\quad = \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\ldots,i_m} B_{i_1,\beta}(q) \cdots B_{i_k,\beta}(q) = B_{n,\beta}^{(k)}(x \mid q). \end{aligned} \tag{2.24}$$

Equating (2.23) and (2.24), we have

$$B_{n,\beta}^{(k)}(x \mid q) = \sum_{l=0}^{n} \binom{n}{l} x^l B_{n,\beta}^{(k)}(q). \tag{2.25}$$

From (2.24) and (2.25), we want to note that $B_{n,\beta}^{(k)}(x \mid q)$ is a monic polynomial of degree n with coefficients in $\mathbb{Q}$. For $k \in \mathbb{N}$, let us consider that

$$g_{q,\beta}^{(k)}(t) = \frac{1}{\int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k)t}\, d^*\mu_q(\xi)} = \left(\frac{\log q}{q-1} \frac{q\beta e^t - 1}{t}\right)^k. \tag{2.26}$$

From (2.26), we easily see that $g_{q,\beta}^{(k)}(t)$ is an invertible series. On account of (2.23) and (2.26), we derive that

$$\frac{1}{g_{q,\beta}^{(k)}(t)} e^{xt} = \int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k+x)t}\, d^*\mu_q(\xi) = \sum_{n=0}^{\infty} B_{n,\beta}^{(k)}(x \mid q) \frac{t^n}{n!}. \tag{2.27}$$

Also, we have

$$tB_{n,\beta}^{(k)}(x \mid q) = nB_{n-1,\beta}^{(k)}(x \mid q). \tag{2.28}$$

By virtue of (2.27) and (2.28), we easily see that $B_{n,\beta}^{(k)}(x \mid q)$ is an Appell sequence for $g_{q,\beta}^{(k)}(t)$. Then, by (2.27) and (2.28), we get the following theorem.

Theorem 5 *For $p(x) \in \mathcal{P}$ and $k \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} p(\xi_1 + \cdots + \xi_k + x)\, d^*\mu_q(\xi) = \left(\frac{q-1}{\log q}\frac{t}{q\beta e^t - 1}\right)^k p(x). \tag{2.29}$$

In the special case, the extended p-adic q-Bernoulli polynomials of degree k are given by

$$B_{n,\beta}^{(k)}(x \mid q) = \left(\frac{q-1}{\log q}\frac{t}{q\beta e^t - 1}\right)^k x^n = \int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k)t}\, d^*\mu_q(\xi) x^n.$$

Thus, we get

$$B_{n,\beta}^{(k)}(x \mid q) \sim \left(\left(\frac{\log q}{q-1}\frac{\beta q e^t - 1}{t}\right)^k, t\right).$$

Let us take the linear functional $f^{(k)}(t)$ that satisfies

$$\langle f^{(k)}(t) \mid p(x)\rangle = \int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} p(\xi_1 + \cdots + \xi_k)\, d^*\mu_q(\xi) \tag{2.30}$$

for all polynomials $p(x)$. Therefore, we develop as follows:

$$\begin{aligned} f^{(k)}(t) &= \sum_{n=0}^{\infty} \frac{\langle f^{(k)}(t) \mid x^n\rangle}{n!} t^n \\ &= \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} (\xi_1 + \cdots + \xi_k)^n\, d^*\mu_q(\xi)\right) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k)t}\, d^*\mu_q(\xi) = \left(\frac{q-1}{\log q}\frac{t}{q\beta e^t - 1}\right)^k. \end{aligned}$$

Therefore, the following theorem can be stated.

Theorem 6 *For $p(x) \in \mathcal{P}$, we have*

$$\left\langle \int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k)t}\, d^*\mu_q(\xi) \,\middle|\, p(x)\right\rangle = \int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} p(\xi_1 + \cdots + \xi_k)\, d^*\mu_q(\xi).$$

Moreover,

$$\left\langle \left(\frac{q-1}{\log q}\frac{t}{q\beta e^t - 1}\right)^k \,\middle|\, p(x)\right\rangle = \int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} p(\xi_1 + \cdots + \xi_k)\, d^*\mu_q(\xi).$$

That is,

$$B_{n,\beta}^{(k)}(q) = \left\langle \int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k)t}\, d^*\mu_q(\xi) \,\middle|\, x^n \right\rangle.$$

From (1.25), we see that

$$\begin{aligned} &\left\langle \int_{\mathbb{Z}_p^k} \beta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k)t}\, d^*\mu_q(\xi) \,\middle|\, x^n \right\rangle \\ &\quad = \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\ldots,i_m} \left\langle \int_{\mathbb{Z}_p} \beta^{\xi_1} e^{\xi_1 t}\, d\mu_q(\xi_1) \,\middle|\, x^{i_1} \right\rangle \cdots \left\langle \int_{\mathbb{Z}_p} \beta^{\xi_k} e^{\xi_k t}\, d\mu_q(\xi_k) \,\middle|\, x^{i_k} \right\rangle. \end{aligned}$$

Therefore, we get

$$B_{n,\beta}^{(k)}(q) = \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\ldots,i_m} B_{i_1,\beta}(q)\cdots B_{i_k,\beta}(q).$$

Remark 1 Our applications for extended p-adic q-Bernoulli polynomials, extended p-adic q-Bernoulli numbers and extended p-adic q-Bernoulli polynomials of order k seem to be interesting for evaluating at $\beta = 1$ and $q \to 1$, which reduce to Bernoulli polynomials and Bernoulli polynomials of order k, are defined respectively by

$$\begin{aligned} &\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}, \\ &\sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^k e^{xt}. \end{aligned}$$

Also, it is known that these polynomials are expressed by the rule:

$$\begin{aligned} &B_n(x) = \lim_{n\to\infty} \frac{1}{p^n} \sum_{A=0}^{p^n-1} (x+A)^n, \\ &B_n^{(k)}(x) = \lim_{n_1,\ldots,n_k\to\infty} p^{-(n_1+n_2+\cdots+n_k)} \sum_{A_1=0}^{p^{n_1}-1} \sum_{A_2=0}^{p^{n_2}-1} \cdots \sum_{A_k=0}^{p^{n_k}-1} (x+A_1+A_2+\cdots+A_k)^n, \end{aligned}$$

where the limits are taken in $\mathbb{Q}_p$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All of the authors contributed equally to the manuscript and read and approved the final draft.

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