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The existence of solutions for a nonlinear mixed problem of singular fractional differential equations

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Abstract

By using fixed point results on cones, we study the existence of solutions for the singular nonlinear fractional boundary value problem

$$
\begin{aligned}
&{}^cD^{\alpha}u(t) = f\bigl(t, u(t), u'(t), {}^cD^{\beta}u(t)\bigr),\\
&u(0) = au(1), \qquad u'(0) = b\,{}^cD^{\beta}u(1), \qquad u''(0) = u'''(0) = u^{(n-1)}(0) = 0,
\end{aligned}
$$

where $n \geq 3$ is an integer, $\alpha \in (n-1, n)$, $0 < \beta < 1$, $0 < a < 1$, $0 < b < \Gamma(2-\beta)$, f is an L^q-Caratheodory function, $q > \frac{1}{\alpha-1}$ and $f(t, x, y, z)$ may be singular at value 0 in one dimension of its space variables x, y, z. Here, ${}^cD$ stands for the Caputo fractional derivative.

Keywords: boundary value problem; fixed point; fractional differential equation; Green function; regularization; singular

1 Introduction

Fractional differential equations (see, for example, [1–6] and references therein) started to play an important role in several branches of science and engineering. There are some works about existence of solutions for the nonlinear mixed problems of singular fractional boundary value problem (see, for example, [7–11] and [12]). Also, there are different methods for solving distinct fractional differential equations (see, for example, [13–18] and [19]). By using fixed point results on cones, we focus on the existence of positive solutions for a nonlinear mixed problem of singular fractional boundary value problem. For the convenience of the reader, we present some necessary definitions from fractional calculus theory (see, for example, [20]). The Caputo derivative of fractional order α for a function $f : [0,\infty) \to \mathbb{R}$ is defined by

$$
{}^cD^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds \quad \bigl(n-1 < \alpha < n, n = [\alpha]+1\bigr).
$$

Let $q \geq 1$. As you know, $L^q[0,1]$ denotes the space of functions, whose qth powers of modulus are integrable on $[0,1]$, equipped with the norm $\|x\|_q = (\int_0^1 |x(t)|^q\,dt)^{\frac{1}{q}}$. We consider the sup norm

$$
\|x\| = \sup\bigl\{\bigl|x(t)\bigr| : t \in [0,1]\bigr\}
$$

on the space $C[0,1]$. Also, $AC[0,1]$ is the set of absolutely continuous functions on $[0,1]$. Let B be a subset of $\mathbb{R}^3$. A function $f:[0,1]\times B\to\mathbb{R}$ is called an L^q-Caratheodory function whenever the real-valued function $f(\cdot,x,y,z)$ on $[0,1]$ is measurable for all $(x,y,z)\in B$, the function $f(t,\cdot,\cdot,\cdot):B\to\mathbb{R}$ is continuous for almost all $t\in(0,1]$, and for each compact set $U\subset B$, there exists a function $\varphi_u\in L^q[0,1]$ such that $|f(t,x,y,z)|\le\varphi_u(t)$ for almost all $t\in[0,1]$ and $(x,y,z)\in U$. Consider the nonlinear fractional boundary value problem

$$\begin{aligned}
&{}^cD^\alpha u(t)=f\bigl(t,u(t),u'(t),{}^cD^\beta u(t)\bigr),\\
&u(0)=au(1),\qquad u'(0)=b\,{}^cD^\beta u(1),\qquad u''(0)=u'''(0)=u^{(n-1)}(0)=0,
\end{aligned}\tag{$*$}$$

where $n\ge 3$ is an integer, $\alpha\in(n-1,n)$, $0<\beta<1$, $0<a<1$, $0<b<\Gamma(2-\beta)$ and $q>\frac{1}{\alpha-1}$. We say that the function $u:[0,1]\to\mathbb{R}$ is a positive solution for the problem whenever $u>0$ on $[0,1]$, ${}^cD^\alpha u$ is a function in $L^q[0,1]$, and u satisfies the boundary conditions almost everywhere on $[0,1]$. In this paper, we suppose that f is an L^q-Caratheodory function on $[0,1]\times B$, where $B=(0,\infty)\times(0,\infty)\times(0,\infty)$, there exists a positive constant m such that $m\le f(t,x,y,z)$ for almost all $t\in[0,1]$ and $(x,y,z)\in B$, f satisfies the estimate

$$f(t,x,y,z)\le h(x)+r\bigl(|y|\bigr)+k\bigl(|z|\bigr)+\gamma(t)w\bigl(x,|y|,|z|\bigr),$$

where $h,r,k\in C(0,\infty)$ are positive and non-increasing, $\gamma\in L^q[0,1]$ and $w\in C([0,\infty)\times[0,\infty)\times[0,\infty))$ are positive, w is non-decreasing in all its variables, $\int_0^1 h^q(s^\alpha)\,ds<\infty$, $\int_0^1 r^q(s^{\alpha-1})\,ds<\infty$, $\int_0^1 k^q(s^{\alpha-\beta})\,ds<\infty$, and $\lim_{x\mapsto\infty}\frac{w(x,x,x)}{x}=0$. Since we suppose that problem $(*)$ is singular, that is, $f(t,x,y,z)$ may be singular at the value 0 of its space variables x, y, z, we use regularization and sequential techniques for the existence of positive solutions of the problem. In this way, for each natural number n define the function f_n by

$$f_n(t,x,y,z)=f\bigl(t,\chi_n^+(x),\chi_n^+(y),\chi_n^+(z)\bigr)$$

for all $t\in[0,1]$ and $(x,y,z)\in\mathbb{R}^3$, where

$$\chi_n^+(u)=\begin{cases}u, & u\ge\frac1n,\\ \frac1n, & u<\frac1n.\end{cases}$$

It is easy to see that each f_n is an L^q-Caratheodory function on $[0,1]\times\mathbb{R}^3$, $m\le f_n(t,x,y,z)$,

$$f_n(t,x,y,z)\le h\left(\frac1n\right)+r\left(\frac1n\right)+k\left(\frac1n\right)+\gamma(t)w\bigl(1+x,1+|y|,1+|z|\bigr)$$

and

$$f_n(t,x,y,z)\le h(x)+r\bigl(|y|\bigr)+k\bigl(|z|\bigr)+\gamma(t)w\bigl(1+x,1+|y|,1+|z|\bigr)$$

for almost all $t\in[0,1]$ and all $(x,y,z)\in B$. In 2012, Agarwal *et al.* proved the following result.

Lemma 1.1 [7] *Let* $\rho \in L^q[0,1]$ *and* $0 \le t_1 < t_2 \le 1$. *Then we have* $|\int_0^t (t-s)^{\alpha-2}\rho(s)\,ds| \le (\frac{t^d}{d})^{1/p}\|\rho\|_q$ *for all* $t \in [0,1]$ *and*

$$\begin{aligned}&\left|\int_0^{t_2}(t_2-s)^{\alpha-2}\rho(s)\,ds - \int_0^{t_1}(t_1-s)^{\alpha-2}\rho(s)\,ds\right| \\ &\quad \le \left(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d}\right)^{1/p}\|\rho\|_q + \left(\frac{(t_2-t_1)^d}{d}\right)^{1/p}\|\rho\|_q,\end{aligned}$$

where $d = (\alpha-2)p+1$.

2 Main results

Now, we are ready to investigate the problem in regular and singular cases. First, we give the following result.

Lemma 2.1 *Let* $y \in C[0,1]$. *Then the boundary value problem*

$$\begin{aligned}&{}^cD^\alpha u(t) = y(t) \quad \bigl(t \in (0,1)\bigr), \\ &u(0) = au(1), \qquad u'(0) = b\,{}^cD^\beta u(1), \qquad u''(0) = u'''(0) = u^{(n-1)}(0) = 0\end{aligned}$$

is equivalent to the fractional integral equation $u(t) = \int_0^1 G(t,s)y(s)\,ds$, *where*

$$\begin{aligned}G(t,s) &= \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad + \frac{a\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)(1-s)^{\alpha-1} + b\Gamma(\alpha)\Gamma(2-\beta)(a+t-at)(1-s)^{\alpha-\beta-1}}{(1-a)\Gamma(\alpha)\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)}\end{aligned}$$

whenever $0 \le s \le t \le 1$ *and*

$$G(t,s) = \frac{a\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)(1-s)^{\alpha-1} + b\Gamma(\alpha)\Gamma(2-\beta)(a+t-at)(1-s)^{\alpha-\beta-1}}{(1-a)\Gamma(\alpha)\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)}$$

whenever $0 \le t \le s \le 1$.

Proof From ${}^cD^\alpha u(t) = y(t)$ and the boundary conditions, we obtain

$$\begin{aligned}u(t) &= I^\alpha y(t) + u(0) + u'(0)t + \frac{u''(0)}{2!}t^2 + \cdots + \frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1} \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + u(0) + u'(0)t.\end{aligned}$$

By properties of the Caputo derivative, we get

$$\begin{aligned}{}^cD^\beta u(t) &= I^{\alpha-\beta}y(t) + {}^cD^\beta\bigl(u(0)+u'(0)t\bigr) \\ &= \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}y(s)\,ds + \frac{u'(0)t^{1-\beta}}{\Gamma(2-\beta)}.\end{aligned}$$

Thus, $u(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\, ds + u(0) + u'(0)$ and

$$^cD^\beta u(1) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds + \frac{u'(0)}{\Gamma(2-\beta)}.$$

By using the boundary conditions $u(0) = au(1)$ and $u'(0) = b\,^cD^\beta u(1)$, we get $u(0) = a(\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\, ds + u(0) + u'(0))$ and

$$u'(0) = b\left(\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds + \frac{u'(0)}{\Gamma(2-\beta)}\right).$$

Hence, $u'(0) = \frac{b\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds$ and

$$\begin{aligned} u(0) &= \frac{a}{(1-a)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\, ds \\ &\quad + \frac{ab\Gamma(2-\beta)}{(1-a)\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds. \end{aligned}$$

Thus,

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + u(0) + u'(0)t \\ &= \int_0^t \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{a\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)(1-s)^{\alpha-1}}{(1-a)\Gamma(\alpha)\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)} \right. \\ &\quad \left. + \frac{b\Gamma(\alpha)\Gamma(2-\beta)(a+t-at)(1-s)^{\alpha-\beta-1}}{(1-a)\Gamma(\alpha)\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)} \right) y(s)\, ds \\ &\quad + \int_t^1 \left(\frac{a\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)(1-s)^{\alpha-1}}{(1-a)\Gamma(\alpha)\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)} \right. \\ &\quad \left. + \frac{b\Gamma(\alpha)\Gamma(2-\beta)(a+t-at)(1-s)^{\alpha-\beta-1}}{(1-a)\Gamma(\alpha)\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)} \right) y(s)\, ds \\ &= \int_0^1 G(t,s) y(s)\, ds. \end{aligned}$$

This completes the proof. □

Put $k_1 = \frac{\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)+b\Gamma(\alpha)\Gamma(2-\beta)}{(1-a)\Gamma(\alpha)\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)}$ and $k_2 = \frac{ab\Gamma(2-\beta)}{(1-a)\Gamma(\alpha-\beta)(\Gamma(2-\beta)-b)}$. It is easy to check that the Green function G in the last result belongs to $C([0,1] \times [0,1])$, $G(t,s) > 0$ for all $(t,s) \in [0,1) \times [0,1)$,

$$G(t,s) \leq k_1 (1-s)^{\alpha-\beta-1} \leq 1 \quad \text{and} \quad G(t,s) \geq k_2 (1-s)^{\alpha-\beta-1}$$

for all $(t,s) \in [0,1] \times [0,1]$. Consider the Banach space $X = C^1[0,1]$ with the norm $\|x\|_* = \max\{\|x\|, \|x'\|\}$ and the cone

$$P = \{x \in X : x(t) \geq 0 \text{ and } x'(t) \geq 0 \text{ for all } t \in [0,1]\}.$$

For each natural number n, define the operator Q_n on P by

$$(Q_n x)(t) = \int_0^1 G(t,s) f_n\big(s, u(s), u'(s), {}^cD^\beta u(s)\big)\, ds.$$

Now, we prove that Q_n is a completely continuous operator (see [2]).

Lemma 2.2 *The operator Q_n is a completely continuous operator.*

Proof Let $x \in P$. Then, ${}^cD^\beta x \in C[0,1]$ and ${}^cD^\beta x \geq 0$. Now, define $\rho(t) = f_n(t, u(t), u'(t), {}^cD^\beta u(t))$ for almost all $t \in [0,1]$. Then $\rho \in L^q[0,1]$ and $\rho(t) \geq m$ for almost all $t \in [0,1]$. By using the properties of fractional integral I^α, it is easy to see that $Q_n x \in C[0,1]$, $(Q_n x)(t) \geq 0$ and

$$(Q_n x)'(t) = \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t-s)^{\alpha-2} \rho(s)\, ds$$

for all $t \in [0,1]$. This implies that $(Q_n x)' \in C[0,1]$ and $(Q_n x)' \geq 0$ on $[0,1]$. Consequently, Q_n maps P into P. In order to prove that Q_n is a continuous operator, let x_m be a convergent sequence in P and $\lim_{m\mapsto\infty} x_m = x$. Thus, $\lim_{m\mapsto\infty} x_m^{(j)}(t) = x^{(j)}(t)$ uniformly on $[0,1]$ for $j = 0,1$. Since

$$\begin{aligned} {}^cD^\beta x(t) &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} \big(x(s) - x(0)\big)\, ds \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} x'(s)\, ds, \end{aligned}$$

we get $|{}^cD^\beta x_m(t) - {}^cD^\beta x(t)| \leq \frac{\|x'_m - x'\|}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}\, ds \leq \frac{\|x_m - x\|_*}{\Gamma(\beta)}$ and $\lim_{m\mapsto\infty} {}^cD^\beta x_m(t) = {}^cD^\beta x(t)$ uniformly on $[0,1]$. Also, we have $|{}^cD^\beta x_m(t)| \leq \frac{x'_m}{\Gamma(\beta)}$ on $[0,1]$, and so $\|{}^cD^\beta x_m\| \leq \frac{\|x'_m\|}{\Gamma(\beta)}$. Now, put

$$\rho_m(t) = f_n\big(t, x_m(t), x'_m(t), {}^cD^\beta x_m(t)\big) \quad \text{and} \quad \rho(t) = f_n\big(t, x(t), x'(t), {}^cD^\beta x(t)\big).$$

Then, it is easy to see that $\lim_{m\mapsto\infty} \rho_m(t) = \rho(t)$ for almost all $t \in [0,1]$, and there exists $\beta \in L^q[0,1]$ such that $0 \leq \rho_m(t) \leq \beta(t)$ for almost all $t \in [0,1]$ and all $m \geq 1$. Since f_n is an L^q-Caratheodory function, $\{x_m\}$ is bounded in $C^1[0,1]$, and $\{{}^cD^\beta x_m\}$ is bounded in $C[0,1]$. Therefore, $\lim_{m\mapsto\infty} (Q_n x_m)(t) = (Q_n x)(t)$ uniformly on $[0,1]$. Since $\{\rho_m\}$ is L^q-convergent on $[0,1]$,

$$\lim_{m\mapsto\infty} (Q_n x_m)'(t) = \frac{1}{\Gamma(\alpha-1)} \lim_{m\mapsto\infty} \int_0^t (t-s)^{\alpha-2} \rho_m(s)\, ds = (Q_n x)'(t)$$

uniformly on $[0,1]$. Hence, Q_n is a continuous operator. Now, we have to show that for each bounded sequence $\{x_m\}$ in P, the sequence $\{Q_n x_m\}$ is relatively compact in $C[0,1]$. Choose a positive constant k such that $\|x_m\| \leq k$ and $\|x'_m\| \leq k$ for all m. Note that $\|{}^cD^\beta x_m\| \leq \frac{k}{\Gamma(\beta)}$ and $|\int_0^t (t-s)^{\alpha-2} \rho_m(s)\, ds| \leq (\int_0^t (t-s)^{(\alpha-2)p}\, ds)^{\frac{1}{p}} (\int_0^t |\rho_m(s)|^q ds)^{\frac{1}{q}} \leq (\frac{t^d}{d})^{\frac{1}{p}} \|\rho_m\|_q$ for all m, where $d = (\alpha-2)p + 1$. But we have

$$0 \leq (Q_n x_m)(t) = \int_0^1 G(t,s) \rho_m(s)\, ds \leq \int_0^1 G(t,s) \beta(s)\, ds \leq \frac{\|\beta\|_1}{\Gamma(\alpha)}$$

and

$$\begin{aligned}0 \le (Q_n x_m)'(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \rho_m(s)\,ds \\ &\le \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \beta(s)\,ds \le \frac{1}{\Gamma(\alpha-1)} \left(\frac{1}{(\alpha-2)p+1}\right)^{\frac{1}{p}} \|\beta\|_q\end{aligned}$$

for all $t \in [0,1]$ and m. This implies that $\{Q_n x_m\}$ is bounded in $C^1[0,1]$. Also, we have

$$\begin{aligned}&\bigl|(Q_n x_m)'(t_2) - (Q_n x_m)'(t_1)\bigr| \\ &\quad = \frac{1}{\Gamma(\alpha-1)} \left| \int_0^{t_2} (t_2-s)^{\alpha-2} \rho_m(s)\,ds - \int_0^{t_1} (t_1-s)^{\alpha-2} \rho_m(s)\,ds \right| \\ &\quad \le \frac{\|\rho_m\|_q}{\Gamma(\alpha-1)} \left(\left(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d} \right)^{\frac{1}{p}} + \left(\frac{(t_2-t_1)^d}{d} \right)^{\frac{1}{p}} \right) \\ &\quad \le \frac{\|\beta\|_q}{\Gamma(\alpha-1)} \left(\left(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d} \right)^{\frac{1}{p}} + \left(\frac{(t_2-t_1)^d}{d} \right)^{\frac{1}{p}} \right)\end{aligned}$$

for all $0 \le t_1 \le t_2 \le 1$, where $d = (\alpha-2)p+1$. Hence, $\{(Q_n x_m)'\}$ is equicontinuous on $[0,1]$. Thus, $\{Q_n x_m\}$ is relatively compact in $C^1[0,1]$ by the Arzela-Ascoli theorem. Hence, Q_n is a completely continuous operator. □

We need the following result (see [2] and [21]).

Lemma 2.3 [21] *Let Y be a Banach space, P a cone in Y and Ω_1 and Ω_2 bounded open balls in Y centered at the origin with $\bar{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that $\|Tx\| \ge \|x\|$ for all $x \in P \cap \partial\Omega_1$ and $\|Tx\| \le \|x\|$ for all $x \in P \cap \partial\Omega_2$. Then T has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.*

Theorem 2.4 *For each natural number n, problem $(*)$ has a solution $u_n \in P$ such that $u_n \ge \frac{mk_2}{\alpha-\beta}$, $u_n'(t) \ge \frac{mt^{\alpha-1}}{\Gamma(\alpha)}$ and ${}^cD^\beta u_n(t) \ge \frac{mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$ for all $t \in [0,1]$.*

Proof Let $n \ge 1$. It is sufficient to show that Q_n has a fixed point u_n in P with the desired conditions. In this way, note that

$$\begin{aligned}(Q_n x)(t) &= \int_0^1 G(t,s) f_n\bigl(s, x(s), x'(s), {}^cD^\beta x(s)\bigr)\,ds \\ &\ge m \int_0^1 G(t,s)\,ds \ge m \int_0^1 k_2 (1-s)^{\alpha-\beta-1}\,ds = \frac{mk_2}{\alpha-\beta},\end{aligned}$$

and so $\|Q_n x\|_* \ge \|Q_n(x)\| \ge \frac{mk_2}{\alpha-\beta}$. Put $\Omega_1 = \{x \in X : \|x\|_* < \frac{mk_2}{\alpha-\beta}\}$. Then $\|Q_n x\|_* \ge \|x\|_*$ for all $x \in P \cap \partial\Omega_1$. If $v_n = h(\frac{1}{n}) + r(\frac{1}{n}) + k(\frac{1}{n})$, then

$$\begin{aligned}\bigl|(Q_n x)(t)\bigr| &\le \left| \int_0^1 G(t,s) f_n\bigl(s, u(s), u'(s), {}^cD^\beta u(s)\bigr)\,ds \right| \\ &\le \int_0^1 \bigl|G(t,s)\bigr| \bigl(v_n + \gamma(s) w\bigl(1 + |x(s)|, 1 + |x'(s)|, 1 + |{}^cD^\beta x(s)|\bigr)\bigr)\,ds \\ &\le k_1 \bigl(v_n + w\bigl(1 + \|x\|, 1 + \|x'\|, 1 + \|{}^cD^\beta x\|\bigr) \|\gamma\|_1\bigr)\end{aligned}$$

and

$$\begin{aligned}
&\left|(Q_n x)'(t)\right| \\
&\quad = \left|\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f_n\bigl(s, x(s), x'(s), {}^cD^\beta x(s)\bigr)\,ds\right| \\
&\quad \le \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\bigl(\nu_n + \gamma(s) w\bigl(1+|x(s)|, 1+|x'(s)|, 1+|{}^cD^\beta x(s)|\bigr)\bigr)\,ds \\
&\quad \le \frac{1}{\Gamma(\alpha-1)}\left(\frac{\nu_n t^{\alpha-1}}{\alpha-1} + w\bigl(1+\|x\|, 1+\|x'\|, 1+\|{}^cD^\beta x\|\bigr)\int_0^t (t-s)^{\alpha-2}\gamma(s)\,ds\right)
\end{aligned}$$

for all $x \in P$ and $t \in [0,1]$, because w is non-decreasing in all its variables. Since $\|x\| \le \|x\|_*$, $\|x'\| \le \|x\|_*$, $\|{}^cD^\beta x\| \le \frac{\|x'\|}{\Gamma(\beta)} \le \frac{\|x\|_*}{\Gamma(\beta)}$ and

$$\int_0^t (t-s)^{\alpha-2}\gamma(s)\,ds \le \left(\frac{1}{d}\right)^{1/p}\|\gamma\|_q,$$

where $d = (\alpha-2)p+1$, we have

$$\|Q_n(x)\| \le k_1\left(\nu_n + w\left(1+\|x\|_*, 1+\|x\|_*, 1+\frac{\|x\|_*}{\Gamma(\beta)}\right)\|\gamma\|_1\right)$$

and

$$\begin{aligned}
&\|(Q_n x)'\| \\
&\quad \le \frac{1}{\Gamma(\alpha-1)}\left(\frac{\nu_n}{\alpha-1} + w\left(1+\|x\|_*, 1+\|x\|_*, 1+\frac{\|x\|_*}{\Gamma(\beta)}\right)(1/d)^{1/p}\|\gamma\|_q\right).
\end{aligned}$$

Hence, $\|Q_n x\|_* \le M(\frac{\nu_n}{\alpha-1} + Nw(1+\|x\|_*, 1+\|x\|_*, 1+\frac{\|x\|_*}{\Gamma(\beta)}))$, where $N = \max\{\|\gamma\|_1, (1/d)^{1/p} \times \|\gamma\|_q\}$ and $M = \max\{\frac{1}{\Gamma(\alpha-1)}, k_1\}$. Since

$$\lim_{\nu\mapsto\infty}\frac{1}{\nu}w(1+\nu, 1+\nu, 1+\nu) = 0,$$

there exists a positive constant L such that

$$M\left(\frac{\nu_n}{\alpha-1} + Nw\left(1+\nu, 1+\nu, \frac{\nu}{\Gamma(\beta)}\right)\right) < \nu$$

for all $\nu \ge L$. Thus, $\|Q_n x\|_* < \|x\|_*$ for all $x \in P$ with $\|x\|_* \ge L$. Put $\Omega_2 = \{x \in X : \|x\|_* < L\}$. Then $\|Q_n x\|_* < \|x\|_*$ for all $x \in P \cap \partial\Omega_2$. By using last result, Q_n has a fixed point u_n in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$. But $u_n = (Q_n u_n)(t) \ge \frac{mk_2}{\alpha-\beta}$ and

$$\begin{aligned}
(Q_n x)'(t) &= \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f_n\bigl(s, x(s), x'(s), {}^cD^\beta x(s)\bigr)\,ds \\
&\ge \frac{m}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\,ds = \frac{mt^{\alpha-1}}{\Gamma(\alpha)}
\end{aligned}$$

for all $t \in [0,1]$ and $x \in P$. Since $\int_0^t (t-s)^{-\beta} s^{\alpha-1}\, ds = \frac{\Gamma(\alpha)\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta}$,

$$\begin{aligned} {}^cD^\beta u_n(t) &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} u_n'(s)\, ds \\ &\geq \frac{m}{\Gamma(\alpha)\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} s^{\alpha-1}\, ds = \frac{mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \end{aligned}$$

for all $t \in [0,1]$. This completes the proof. □

Now, we give our last result.

Theorem 2.5 *Problem* $(*)$ *has a solution* u *such that* $u(t) \geq \frac{mk_2}{\alpha-\beta}$, $u'(t) \geq \frac{mt^{\alpha-1}}{\Gamma(\alpha)}$ *and* ${}^cD^\beta u(t) \geq \frac{mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$ *for all* $t \in [0,1]$.

Proof By using Theorem 2.4, one gets that for each natural number n, problem $(*)$ has a solution $u_n \in P$ with the desired conditions. Thus, $h(u_n(t)) \leq h(\frac{mk_2}{\alpha-\beta})$, $r(|u_n'(t)|) \leq r(\frac{mt^{\alpha-1}}{\Gamma(\alpha)})$ and $k(|{}^cD^\beta u_n(t)|) \leq k(\frac{mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)})$ for all $t \in [0,1]$ and n. Also, we have $\|{}^cD^\beta u_n\| \leq \frac{\|u_n'\|}{\Gamma(\beta)}$. Suppose that

$$S(t) = h\left(\frac{mk_2}{\alpha-\beta}\right) + r\left(\frac{mt^{\alpha-1}}{\Gamma(\alpha)}\right) + k\left(\frac{mt^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right).$$

Then

$$\begin{aligned} m &\leq f_n\big(t, u_n(t), u_n'(t), {}^cD^\beta u_n(t)\big) \\ &\leq S(t) + \gamma(t) w\big(1 + \|u_n\|, 1 + \|u_n'\|, 1 + \|{}^cD^\beta u_n\|\big) \\ &\leq S(t) + \gamma(t) w\left(1 + \|u_n\|_*, 1 + \|u_n'\|_*, 1 + \frac{\|u_n\|_*}{\Gamma(\beta)}\right) \end{aligned}$$

for almost all $t \in [0,1]$ and n. Since $0 \leq G(t,s) \leq k_1$, we get

$$\begin{aligned} 0 &\leq u_n(t) \\ &= \int_0^1 G(t,s) f_n\big(s, u_n(s), u_n'(s), {}^cD^\beta u_n(s)\big)\, ds \\ &\leq k_1 \left(\int_0^1 S(s)\, ds + w\left(1 + \|u_n\|_*, 1 + \|u_n\|_*, 1 + \frac{\|u_n\|_*}{\Gamma(\beta)}\right) \|\gamma\|_1 \right) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq u_n'(t) \\ &\leq \frac{1}{\Gamma(\alpha-1)} \left(\int_0^t (t-s)^{\alpha-2} S(s)\, ds \right. \\ &\quad \left. + w\left(1 + \|u_n\|_*, 1 + \|u_n\|_*, 1 + \frac{\|u_n\|_*}{\Gamma(\beta)}\right) \int_0^t (t-s)^{\alpha-2} \gamma(s)\, ds \right). \end{aligned}$$

We show that $\int_0^t (t-s)^{\alpha-2} S(s)\,ds$ is bounded on $[0,1]$. Let $d = (\alpha-2)p+1$. Note that

$$\begin{aligned}
&\int_0^1 (t-s)^{\alpha-2} h\left(\frac{mk_2}{\alpha-\beta}\right) ds \\
&\quad = h\left(\frac{mk_2}{\alpha-\beta}\right) \int_0^1 (t-s)^{\alpha-2}\,ds = \frac{1}{\alpha-1} h\left(\frac{mk_2}{\alpha-\beta}\right) =: \eta_1 < \infty, \\
&\int_0^t (t-s)^{\alpha-2} r\left(\frac{ms^{\alpha-1}}{\Gamma(\alpha)}\right) ds \\
&\quad = \left(\frac{1}{d}\right)^{1/p} \left(\frac{\Gamma(\alpha)}{m}\right)^{\frac{1}{(\alpha-1)q}} \left(\int_0^{(\frac{m}{\Gamma(\alpha)})^{\frac{1}{\alpha-1}}} r^q\left(s^{\alpha-1}\right) ds\right)^{1/q} =: \eta_2 < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha-2} k\left(\frac{ms^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right) ds \\
&\quad = \left(\frac{1}{d}\right)^{1/p} \left(\frac{\Gamma(\alpha-\beta+1)}{m}\right)^{\frac{1}{(\alpha-\beta)q}} \left(\int_0^{(\frac{m}{\Gamma(\alpha-\beta+1)})^{\frac{1}{\alpha-\beta}}} k^q\left(s^{\alpha-\beta}\right) ds\right)^{1/q} =: \eta_3 < \infty.
\end{aligned}$$

Thus, $\int_0^t (t-s)^{\alpha-2} S(s)\,ds \le \eta$ for all $t \in [0,1]$, where $\eta = \eta_1 + \eta_2 + \eta_3$. Also, we have

$$\begin{aligned}
&\int_0^1 S(s)\,ds \\
&\quad \le \frac{1}{\alpha-1} h\left(\frac{mk_2}{\alpha-\beta}\right) + \left(\frac{\Gamma(\alpha)}{m}\right)^{\frac{1}{\alpha-1}} \int_0^{(\frac{m}{\Gamma(\alpha)})^{\frac{1}{\alpha-1}}} r\left(s^{\alpha-1}\right) ds \\
&\qquad + \left(\frac{\Gamma(\alpha-\beta+1)}{m}\right)^{\frac{1}{\alpha-\beta}} \int_0^{(\frac{m}{\Gamma(\alpha-\beta+1)})^{\frac{1}{\alpha-\beta}}} k\left(s^{\alpha-\beta}\right) ds < \infty.
\end{aligned}$$

Since

$$\|u_n\| = k_1 \left(\int_0^1 S(s)\,ds + w\left(1 + \|u_n\|_*, 1 + \|u_n\|_*, 1 + \frac{\|u_n\|_*}{\Gamma(\beta)}\right) \|\gamma\|_1 \right)$$

and

$$\|u_n'\| \le \frac{1}{\Gamma(\alpha-1)} \left(\eta + w\left(1 + \|u_n\|_*, 1 + \|u_n\|_*, 1 + \frac{\|u_n\|_*}{\Gamma(\beta)}\right) \left(\frac{1}{d}\right)^{1/p} \|\gamma\|_q \right),$$

we get $\|u_n\|_* \le M(\Phi + Kw(1 + \|u_n\|_*, 1 + \|u_n\|_*, 1 + \frac{\|u_n\|_*}{\Gamma(\beta)}))$ for all n, where $\Phi = \max\{\eta, \int_0^1 S(s)\,ds\}$, $K = \max\{\|\gamma\|_1, (\frac{1}{d})^{1/p}\|\gamma\|_q\}$ and also $M = \max\{k_1, \frac{1}{\Gamma(\alpha-1)}\}$. On the other hand, there exists a positive constant L such that $M(\Phi + Kw(1+\nu, 1+\nu, 1+\frac{\nu}{\Gamma(\beta)})) < \nu$ for all $\nu \ge L$, and so $\|u_n\|_* < L$ for all n. Thus, for almost all $t \in [0,1]$ and all n, we have $f_n(t, u_n(t), u_n'(t), {}^cD^\beta u_n(t)) \le R(t)$, where

$$R(t) = S(t) + \gamma(t) w\left(1 + L, 1 + L, 1 + \frac{L}{\Gamma(\beta)}\right).$$

Note that $R \in L^q[0,1]$. We show that $\{u'_n\}$ is equicontinuous on $[0,1]$. Let $\rho_n(t) = f_n(t, u_n(t), u'_n(t), {}^cD^\beta u_n(t))$ and $0 \le t_1 < t_2 \le T$. Then

$$\begin{aligned}
&\left|u'_n(t_2) - u'_n(t_1)\right| \\
&\quad= \frac{1}{\Gamma(\alpha-1)}\left|\int_0^{t_2}(t_2-s)^{\alpha-2}\rho_n(s)\,ds - \int_0^{t_1}(t_1-s)^{\alpha-2}\rho_n(s)\,ds\right| \\
&\quad\le \frac{1}{\Gamma(\alpha-1)}\left(\int_0^{t_1}\left((t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2}\right)\rho_n(s)\,ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-2}\rho_n(s)\,ds\right) \\
&\quad\le \frac{1}{\Gamma(\alpha-1)}\left(\int_0^{t_1}\left((t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2}\right)R(s)\,ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-2}R(s)\,ds\right),
\end{aligned}$$

and so

$$\begin{aligned}
&\left|u'_n(t_2) - u'_n(t_1)\right| \\
&\quad\le \frac{\|R\|_q}{\Gamma(\alpha-1)}\left(\left(\frac{t_1^d + (t_2-t_1)^d - t_2^d}{d}\right)^{1/p} + \left(\frac{(t_2-t_1)^d}{d}\right)^{1/p}\right).
\end{aligned}$$

Hence, $\{u'_n\}$ is equicontinuous on $[0,1]$. Since $\{u_n\}$ is a bounded sequence in $C[0,1]$, by using the Arzela-Ascoli theorem, without loss of generality, we can assume that $\{u_n\}$ is convergent in $C[0,1]$. Let $\lim_{n\mapsto\infty} u_n = u$. Then, it is easy to see that ${}^cD^\beta u_n(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{-\beta}u'_n(s)\,ds$, and ${}^cD^\beta u_n(t)$ uniformly converges to $\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{-\beta}u'(s)\,ds$ on $[0,1]$. Thus, ${}^cD^\beta u_n$ converges to ${}^cD^\beta u$ in $C[0,1]$. Hence,

$$\lim_{n\mapsto\infty} f_n\left(t, u_n(t), u'_n(t), {}^cD^\beta u_n(t)\right) = f\left(t, u(t), u'(t), {}^cD^\beta u(t)\right)$$

for almost all $t \in [0,1]$. Since $R \in L^q[0,1]$, by using the dominated convergence theorem on the relation

$$u_n(t) = \int_0^1 G(t,s) f_n\left(s, u_n(s), u'_n(s), {}^cD^\beta u_n(s)\right)ds,$$

we get $u(t) = \int_0^1 G(t,s) f(s, u(s), u'(s), {}^cD^\beta u(s))$ for all $t \in [0,1]$. This completes the proof. □

2.1 Examples for the problem

Example 2.1 Let $\rho_1, \rho_2 \in L^q[0,1]$, $\rho_1(t) \ge m > 0$ for almost all t in $[0,1]$. Suppose that

$$f(t,x,y,z) = \rho_1(t) + \frac{1}{x^{\frac{1}{3}} - \lambda} + \frac{1}{y^{\frac{1}{4}}} + \frac{1}{z^{\frac{1}{4}}} + \left|\rho_2(t)\right|\left(x^{\frac{1}{3}} + y^{\frac{1}{4}} + z^{\frac{1}{4}}\right)$$

on $[0,1] \times B$, $\lambda = (au(1))^{\frac{1}{3}}$, $h(x) = \frac{1}{x^{\frac{1}{3}} - \lambda}$ whenever $x^{\frac{1}{3}} - \lambda \ge 0$ and $h(x) = 0$ whenever $x^{\frac{1}{3}} - \lambda < 0$, $r(x) = \frac{1}{x^{\frac{1}{4}}}$, $k(x) = \frac{1}{x^{\frac{1}{4}}}$, $w(x,y,z) = (x^{\frac{1}{3}} + y^{\frac{1}{4}} + z^{\frac{1}{4}} + 1)$ and $\gamma(t) = \rho_1(t) + |\rho_2(t)|$. Then Theorem 2.5 guarantees that problem $(*)$ has a positive solution.

Example 2.2 Consider the nonlinear mixed problem of singular fractional boundary value problem

$$\begin{aligned} {}^cD^{\frac{7}{3}}u(t) &= t+1+\frac{1}{u(t)^{\frac{1}{3}}-\rho}+\frac{1}{u'(t)^{\frac{1}{4}}}+\frac{1}{({}^cD^{\frac{1}{4}}u(t))^{\frac{1}{4}}} \\ &\quad + 2\bigl(u(t)^{\frac{1}{3}}+u'(t)^{\frac{1}{4}}+\bigl({}^cD^{\frac{1}{4}}u(t)\bigr)^{\frac{1}{4}}+1\bigr) \end{aligned}$$

via boundary value conditions $u(0)=\frac{1}{4}u(1)$, $u'(0)=\frac{1}{3}({}^cD^{\frac{1}{4}})u(1)$ and $u''(0)=u'''(0)=\cdots=u^{(n-1)}(0)=0$, where $\rho=((\frac{1}{4})u(1))^{\frac{1}{3}}$. Let

$$\begin{aligned} f\bigl(t,u(t),u'(t),{}^cD^{\frac{1}{4}}\bigr) &= t+1+\frac{1}{u(t)^{\frac{1}{3}}-\rho}+\frac{1}{u'(t)^{\frac{1}{4}}}+\frac{1}{({}^cD^{\frac{1}{4}}u(t))^{\frac{1}{4}}} \\ &\quad + 2\bigl(u(t)^{\frac{1}{3}}+u'(t)^{\frac{1}{4}}+\bigl({}^cD^{\frac{1}{4}}u(t)\bigr)^{\frac{1}{4}}+1\bigr). \end{aligned}$$

Then the map f is singular at $t=0$, and f satisfies the desired conditions, where $h(x)=\frac{1}{x^{\frac{1}{3}}-\rho}$ whenever $x^{\frac{1}{3}}-\rho\geq 0$ and $h(x)=0$ whenever $x^{\frac{1}{3}}-\rho<0$, $r(x)=\frac{1}{x^{\frac{1}{4}}}$, $k(x)=\frac{1}{x^{\frac{1}{4}}}$, $w(x,y,z)=x^{\frac{1}{3}}+y^{\frac{1}{4}}+z^{\frac{1}{4}}+1$, $\rho_1(t)=t+1>1=m$, $\rho_2(t)=2$ and $\gamma(t)=\rho_1(t)+|\rho_2(t)|$. Then Theorem 2.5 guarantees that this problem has a positive solution.

3 Conclusions

One of the most interesting branches is obtaining solutions of singular fractional differential via boundary value problems. Having these things in mind, we study the existence of solutions for a singular nonlinear fractional boundary value problem. Two illustrative examples illustrate the applicability of the proposed method. It seems that the obtained results could be extended to more general functional spaces. Finally, note that all calculations in proofs of the results depend on the definition of the fractional derivative.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equal contributions. All authors read and approved the final manuscript.

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