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Variational iteration method for fractional calculus - a universal approach by Laplace transform

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Abstract

A novel modification of the variational iteration method (VIM) is proposed by means of the Laplace transform. Then the method is successfully extended to fractional differential equations. Several linear fractional differential equations are analytically solved as examples and the methodology is demonstrated.

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Keywords: variational iteration method; fractional calculus; Laplace transform; symbolic computation

1 Introduction

The Lagrange multiplier technique [1] was widely used to solve a number of nonlinear problems which arise in mathematical physics and other related areas, and it was developed into a powerful analytical method, *i.e.*, the variational iteration method [2, 3] for solving differential equations. The method has been applied to initial boundary value problems [4–9], fractal initial value problems [10, 11], q-difference equations [12] and fuzzy equations [13–15], *etc.*

Generally, in applications of VIM to initial value problems of differential equations, one usually follows the following three steps: (a) establishing the correction functional; (b) identifying the Lagrange multipliers; (c) determining the initial iteration. The step (b) is very crucial. Applications of the method to fractional differential equations (FDEs) mainly and directly used the Lagrange multipliers in ordinary differential equations (ODEs) which resulted in poor convergences. This point of view needs some explanations will elucidate the target of the suggested improvement, among them:

(1) When the Riemann-Liouville (RL) integral emerges in the constructed correctional functional, the integration by parts is difficult to apply;
(2) To avoid this problem, the RL integral is replaced by an integer one which allows the integration by parts. This is a very strong simplification but it affects the next steps of the application of the method;
(3) Therefore, the Lagrange multiplier is determined by a simplification not reasonably explained in the literature, so far.

To overcome this drawback, the present article conceives a method how the Lagrange multiplier has to be defined from Laplace transform. The technique can be readily and universally extended to solve both differential equations and FDEs with initial value conditions.

2 Basics of the variation iteration method

In order to illustrate the basic idea of the technique, consider the following general nonlinear system:

$$\frac{d^m u}{dt^m} + R[u] + N[u] = g(t), \tag{1}$$

where $u = u(t)$, R is a linear operator, N is a nonlinear operator and $g(t)$ is a given continuous function and $d^m u/dt^m$ is the term of the highest-order derivative.

The basic character of the method is to construct the following correction functional for Eq. (1):

$$u_{n+1} = u_n + \int_0^t \lambda(t,\tau)\left(\frac{d^m u}{d\tau^m} + R[u_n] + N[u_n] - g(\tau)\right) d\tau, \tag{2}$$

where $\lambda(t,\tau)$ is called the general Lagrange multiplier [1–3] and u_n is the nth order approximate solution.

According to the VIM's rules [2, 3, 16], readers may note that the integration by parts plays an important role in the derivation of the Lagrange multipliers. But in fractional calculus, generally, the following integration by parts cannot hold (as it is mentioned in point (2) of the preceding section):

$${}_0I_t^\alpha v {}_0^C D_t^\alpha u = [uv]|_0^t - {}_0I_t^\alpha u {}_0^C D_t^\alpha v, \tag{3}$$

where $v = v(t)$, ${}_0^C D_t^\alpha$ and ${}_0I_t^\alpha$ are the notations of the Caputo derivative and the RL integration, respectively. That's why the VIM was not so successful as other analytical methods such as the Adomian decomposition method (ADM) [17–19] and the homotopy perturbation method (HPM) [20–22] in fractional calculus. For this reason, we consider the following reconstruction of the method using the Laplace transform.

3 New identification of the Lagrange multipliers

Let us revisit the original idea of the Lagrange multipliers in the case of an algebraic equation. Firstly, an iteration formula for finding the solution of the algebraic equation $f(x) = 0$ can be constructed as

$$x_{n+1} = x_n + \lambda f(x_n). \tag{4}$$

The optimality condition for the extreme $\frac{\delta x_{n+1}}{\delta x_n} = 0$ leads to

$$\lambda = -\frac{1}{f'(x_n)}, \tag{5}$$

where δ is the classical variational operator. From (4) and (5), for a given initial value x_0, we can find the approximate solution x_{n+1} by the iterative scheme for (5)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_0) \neq 0, n = 0, 1, 2, \ldots. \tag{6}$$

This algorithm is well known as the Newton-Raphson method and has quadratic convergence.

Now, we extend this idea to finding the unknown Lagrange multiplier. The main step is to first take the Laplace transform to Eq. (1). Then the linear part is transformed into an algebraic equation as follows:

$$s^m U(s) - u^{(m-1)}(0) - \cdots - s^{m-1}u(0) + L\big[R[u]\big] + L\big[N[u]\big] - L\big[g(t)\big] = 0, \tag{7}$$

where $U(s) = L[u(t)] = \int_0^\infty e^{-st}u(t)\,dt$.

The iteration formula of (7) can be used to suggest the main iterative scheme involving the Lagrange multiplier as

$$\begin{aligned} U_{n+1}(s) = U_n(s) + \lambda(s)\big[&s^m U_n(s) - u^{(m-1)}(0) - \cdots - s^{m-1}u(0) \\ &+ L\big(R[u_n] + N[u_n] - g(t)\big)\big]. \end{aligned} \tag{8}$$

Considering $L(R[u_n] + N[u_n])$ as restricted terms, one can derive a Lagrange multiplier as

$$\lambda(s) = -\frac{1}{s^m}. \tag{9}$$

With Eq. (9) and the inverse-Laplace transform L^{-1}, the iteration formula (8) can be explicitly given as

$$\begin{aligned} u_{n+1}(t) &= u_n(t) - L^{-1}\left[\frac{1}{s^m}\big[s^m U_n(s) - u^{(m-1)}(0) + \cdots - s^{m-1}u(0) \right. \\ &\quad \left. + L\big(R[u_n] + N[u_n] - g(t)\big)\big]\right] \\ &= L^{-1}\left(\frac{1}{s^m}u^{(m-1)}(0) + \cdots + \frac{u(0)}{s} - \frac{1}{s^m}L\big(R[u_n] + N[u_n] - g(t)\big)\right), \end{aligned} \tag{10}$$

where the initial iteration $u_0(t)$ can be determined by

$$u_0(t) = L^{-1}\left(\frac{1}{s^m}u^{(m-1)}(0) + \cdots + \frac{u(0)}{s}\right) = u(0) + u'(0)t + \cdots + \frac{u^{(m-1)}(0)t^{m-1}}{(m-1)!}. \tag{11}$$

Eq. (11) also explained why the initial iteration in the classical VIM is determined by the Taylor series.

This modified VIM here transfers the problem into the partial differential equation in the Laplace s-domain and removes the differentiation with respect to time. This idea has been used in other analytical methods such as the Laplace ADM [23, 24] and the Laplace HPM [25], respectively.

4 Illustrative examples

We now consider the applications of the modified VIM to both ODEs and FDEs.

4.1 Ordinary differential equations

Example 1 Consider the following simple line differential equation:

$$\frac{du}{dt} + u = 0, \qquad u(0) = u_0, \tag{12}$$

which has the exact solution $u(t) = u_0 e^{-t}$.

We can obtain the successive approximate solutions as

$$\begin{aligned}
u_0(t) &= u(0) = u_0, \\
u_1(t) &= L^{-1}\left(\frac{1}{s} - \frac{1}{s^2}\right) = u_0(1-t), \\
u_2(t) &= L^{-1}\left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right) = u_0\left(1 - t + \frac{t^2}{2}\right), \\
&\vdots \\
u_n(t) &= L^{-1}\left(\frac{1}{s} - \frac{1}{s^2} + \cdots + (-1)^{n+1}\frac{1}{s^{n+1}}\right) = u_0\left[\sum_{k=0}^{n} \frac{(-t)^n}{n!}\right].
\end{aligned} \tag{13}$$

For $n \to \infty$, $u_n(t)$ tends to the exact solution $u_0 e^{-t}$.

Example 2 The logistic differential equation [26]

$$\frac{du}{dt} = u(1-u), \qquad u(0) = \frac{1}{2} \tag{14}$$

has the exact solution $u(t) = \frac{e^t}{1+e^t}$. By the present VIM, we have the following solutions:

$$\begin{aligned}
u_0(t) &= \frac{1}{2}, \\
u_1(t) &= L^{-1}\left(\frac{1}{2s} + \frac{1}{4s^2}\right) = \frac{1}{2} + \frac{1}{4}t, \\
u_2(t) &= L^{-1}\left(\frac{1}{2s} + \frac{1}{4s^2} - \frac{1}{8s^4}\right) = \frac{1}{2} + \frac{1}{4}t - \frac{1}{48}t^3, \\
&\vdots
\end{aligned} \tag{15}$$

The same solutions using the classical VIM can be found in [26].

On the other hand, if we use du/dt and the linear term u when determining the Lagrange multiplier, we can derive a Lagrange multiplier explicitly

$$\delta U_{n+1}(s) = \delta U_n(s) + \delta\lambda\left[sU_n(s) - u(0) - U_n(s)\right] = \delta U_n(s) + \lambda(s)(s-1)\delta U_n(s) \tag{16}$$

and

$$\lambda(s) = -\frac{1}{s-1}. \tag{17}$$

There can be various choices of $u_0(t)$ and $\lambda(s)$ which affect the speed of the convergence. We note that the integration by parts is not used and the calculation of the Lagrange multiplier here is much simpler. Furthermore, the VIM can be easily extended to FDEs and this is the main purpose of our work.

4.2 Fractional differential equations

In the early application of VIM [2] to FDEs, the term ${}_0^C D_t^\alpha u$ is considered as a restricted variation, *i.e.*,

$$\frac{du}{dt} + {}_0^C D_t^\alpha u = g(t,u), \quad 0 < t, 0 < \alpha < 1,$$

and the variational iteration formula is given as

$$u_{n+1} = u_n + \int_0^t \lambda(t,\tau)\left(\frac{du_n}{d\tau} + {}_0^C D_\tau^\alpha u - g(\tau,u)\right) d\tau,$$

where ${}_0^C D_t^\alpha$ is the Caputo derivative [27] and $g(\tau, u_n)$ is a nonlinear term.

But for the following FDEs, the above popular applications of the VIM were not successful:

$$\begin{aligned} &{}_0^C D_t^\alpha u + R[u] + N[u] = g(t), \\ &u^{(k)}(0^+) = a_k, \quad 0 < t, 0 < \alpha, m = [\alpha] + 1, k = 0, \ldots, m-1. \end{aligned} \tag{18}$$

Now, we consider the application of the modified VIM.

The following Laplace transform [27–29] of the term ${}_0^C D_t^\alpha u$ holds:

$$L\left[{}_0^C D_t^\alpha u\right] = s^\alpha U(s) - \sum_{k=0}^{m-1} u^{(k)}(0^+) s^{\alpha-1-k}, \quad m-1 < \alpha \le m. \tag{19}$$

Taking the above Laplace transform to both sides of (18), the iteration formula of Eq. (18) can be constructed as

$$U_{n+1}(s) = U_n(s) + \lambda(s)\left[s^\alpha U_n(s) - \sum_{k=0}^{m-1} u^{(k)}(0) s^{\alpha-k-1} + L\bigl(R[u_n] + N[u_n] - g(t)\bigr)\right].$$

As a result, after the identification of a Lagrange multiplier $\lambda = -\frac{1}{s^\alpha}$, one can derive

$$\begin{aligned} u_{n+1}(t) &= u_n(t) - L^{-1}\left[\frac{1}{s^\alpha}\left[s^\alpha U_n(s) - \sum_{k=0}^{m-1} u^{(k)}(0) s^{\alpha-k-1} + L\bigl(R[u_n] + N[u_n] - g(t)\bigr)\right]\right] \\ &= L^{-1}\left(\sum_{k=0}^{m-1} u^{(k)}(0) s^{-k-1} - \frac{1}{s^\alpha} L\bigl(R[u_n] + N[u_n] - g(t)\bigr)\right), \quad m-1 < \alpha \le m \end{aligned} \tag{20}$$

and

$$u_0(t) = L^{-1}\left(\sum_{k=0}^{m-1} u^{(k)}(0) s^{-k-1}\right) = u(0) + u'(0)t + \cdots + \frac{u^{(m-1)}(0) t^{m-1}}{(m-1)!}. \tag{21}$$

Let us apply the above VIM to solve FDEs of Caputo type.

Example 3 Consider the relaxation oscillator equation

$$ {}_0^C D_t u + \omega^\alpha u = 0, \qquad u(0) = 1, \qquad u'(0) = 0, \quad t > 0, 0 < \alpha < 2, \omega > 0, \tag{22} $$

with the exact solution $E_\alpha((-\omega t)^\alpha)$ [30], where $E_\alpha((-\omega t)^\alpha)$ denotes the Mittag-Leffler function.

After taking the Laplace transform to both sides of Eq. (22), we get the following iteration formula:

$$ U_{n+1}(s) = U_n(s) + \lambda(s)\big[s^\alpha U_n(s) - u(0)s^{\alpha-1} - u^{(')}(0^+)s^{\alpha-2} + \omega^\alpha L[u_n]\big]. \tag{23} $$

Setting $L[u_n(t)]$ as a restricted variation, $\lambda(s)$ can be identified as

$$ \lambda(s) = -\frac{1}{s^\alpha}. \tag{24} $$

The approximate solution of Eq. (22) can be given as

$$ \begin{aligned} u_{n+1}(t) &= u_n(t) - L^{-1}\left[\frac{1}{s^\alpha}\big[s^\alpha U_n(s) - u(0)s^{\alpha-1} - u^{(')}(0^+)s^{\alpha-2} + \omega^\alpha L[u_n]\big]\right] \\ &= L^{-1}\left[\frac{1}{s^\alpha}\big(u(0)s^{\alpha-1} + u^{(')}(0^+)s^{\alpha-2} - \omega^\alpha L[u_n]\big)\right], \end{aligned} $$

which reads

$$ \begin{aligned} u_0(t) &= 1, \\ u_1(t) &= 1 - \frac{\omega^\alpha t^\alpha}{\Gamma(1+\alpha)}, \\ u_2(t) &= 1 - \frac{\omega^\alpha t^\alpha}{\Gamma(1+\alpha)} + \frac{\omega^{2\alpha} t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ &\vdots \end{aligned} $$

$u_n(t)$ rapidly tends to the exact solution of Eq. (24) for $n \to \infty$.

Example 4 Consider the fourth example, the time-fractional diffusion equation

$$ {}_0^C D_t^\alpha u = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial (x u(x,t))}{\partial x}, \quad 0 < \alpha < 1, \qquad u(x,0) = x^2. \tag{25} $$

The VIM solution of the fractional semi-derivative equation was developed by Das [31]. Other methods applied to this equation are available in [32] and the monographs [33, 34] in the fractional calculus.

We can have the following iteration formula for Eq. (25):

$$ \begin{cases} u_{n+1}(t) = L^{-1}\big(\frac{x^2}{s} + \frac{1}{s^\alpha} L\big(\frac{\partial^2 u_n(x,t)}{\partial x^2} + \frac{\partial (x u_n(x,t))}{\partial x}\big)\big), \\ u_0(t) = x^2 \end{cases} \tag{26} $$

and $\lambda(s) = -\frac{1}{s^\alpha}$ is used as earlier.

As a result, the successive approximation can be obtained as follows:

$$\begin{aligned} u_0(t) &= x^2, \\ u_1(t) &= L^{-1}\left(\frac{x^2}{s} + \frac{(2+3x^2)}{s^{\alpha+1}}\right) = x^2 t + \frac{(2+3x^2)t^\alpha}{\Gamma(1+\alpha)}, \\ u_2(t) &= x^2 t + \frac{(2+3x^2)t^\alpha}{\Gamma(1+\alpha)} + \frac{(8+9x^2)t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ u_3(t) &= x^2 t + \frac{(2+3x^2)t^\alpha}{\Gamma(1+\alpha)} + \frac{(8+9x^2)t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(26+27x^2)t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ &\vdots \end{aligned} \tag{27}$$

The exact solution can be given in a compact form

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{k^i t^{i\alpha}}{\Gamma(1+i\alpha)} = E_\alpha\left(kt^\alpha\right), \tag{28}$$

where $k^i = x^2 + (1+x^2)(3^i - 1)$.

The method's efficiency for a nonlinear differential equation with variable coefficients is illustrated in [35]. For other applications of a new modified VIM to ODEs and FDEs, readers are also referred to [36–38].

Remarks

(a) The conceived modification of the VIM is a universal approach to both ODEs and FDEs. As a result, it becomes possible to design a 'universal' software package in future work.
(b) Now one can consider implementing other linearized techniques, *i.e.*, the Adomian series and the homotopy series to handle the nonlinear terms and improve the accuracy of the approximate solutions.
(c) This modified VIM can also be used to solve the FDEs of RL type.

5 Conclusions

A new approach is proposed to identify the Lagrange multipliers of the VIM and a concept of the Laplace-Lagrange multipliers is proposed from the Laplace transform. Especially for the FDEs, to the best of our knowledge, there is no effective method to identify the Lagrange multipliers. With the approach given in this paper, we can easily derive Lagrange multipliers without tedious calculation and new variational iteration formulae can be derived. Some FDEs with the Caputo derivatives are illustrated. The results show the modified method's efficiency compared with other versions of the VIM in fractional calculus.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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