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A Jacobi operational matrix for solving a fuzzy linear fractional differential equation

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Abstract

This paper reveals a computational method based using a tau method with Jacobi polynomials for the solution of fuzzy linear fractional differential equations of order $0 < v < 1$. A suitable representation of the fuzzy solution via Jacobi polynomials diminishes its numerical results to the solution of a system of algebraic equations. The main advantage of this method is its high robustness and accuracy gained by a small number of Jacobi functions. The efficiency and applicability of the proposed method are proved by several test examples.

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1 Introduction

Recently, the enormous number of applications in the field of fractional calculus and fractional differential equations has been visualized. Fractional differential equations provide an outstanding instrument to describe the complex phenomena in fields of viscoelasticity, electromagnetic waves, diffusion equations and so on [1–5]. Moreover, the fractional order models of real systems are more sufficient in comparison with the integer order cases. Therefore, the field of fractional calculus has motivated the interest of researchers in various fields like physics, chemistry, engineering and even finance [6–10].

Finding a high accurate and efficient numerical method has become a significant research due to except for a few number of these equations, there exists difficulty to find the exact solution of fractional differential equations (FDEs). Consequently, various numerical methods have appeared to approximate reasonably the analytical solutions. These methods are such as the predictor corrector method [11], Adomian decomposition method (ADM) [12–15], variational iteration method (VIM) [16, 17] and homotopy analysis method (HAM) [18, 19].

Orthogonal functions have received noticeable consideration in dealing with various problems. The main advantage behind the approach using this method is that it reduces these problems to those of solving a system of algebraic equations leading to simplify the original problem clearly. Saadatmandi and Dehghan [20] presented a shifted Legendre tau method with an operational matrix for the numerical solution of a multilinear and nonlinear fractional differential equation. Esmaeili *et al.* [21] introduced a direct method using the collocation method and Müntz polynomials for the solution of FDEs. Consequently, the operational matrix of the other orthogonal polynomials has been derived for solving FDEs with boundary conditions and initial conditions, like Chebyshev polynomials [22, 23], Laguerre series [24], fractional Legendre polynomials [25], generalized hat basis functions [26] and Jacobi polynomials [27, 28].

The study of fuzzy differential equations (*e.g.*, in this contribution, we consider fuzzy fractional differential equation) creates a suitable setting for mathematical modeling of real-world problems in which uncertainties or vagueness penetrate. A comprehensive approach to this kind of equations has been considered by Seikkala [29] and Kaleva [30]. Despite the vast applications of the H-derivative introduced by them, due to an important drawback in this kind of derivative, Bede and Gal [31] introduced strongly generalized differentiability and followed up by the authors in [32, 33]. Actually, strongly generalized differentiability can be applied for a more enormous class of fuzzy differential equations than Hukuhara differentiability.

Recently, some attempts have been made for solving fuzzy fractional differential equations (FFDEs) that Agarwal *et al.* was a pioneer [34]. They considered the solution of FFEDs under Riemann-Liouville's differentiability. Also, Salahshour *et al.* [35] studied the existence, uniqueness and approximate solutions of (FFDEs) under Caputo's H-differentiability. Afterward, Mazandarani, Vahidian Kamyad [36] applied the fractional Euler method for FFDEs under Caputo-type differentiability and Salahshour *et al.* [37] extended fuzzy Laplace transforms for solving FFDEs under the Riemann-Liouville H-derivative.

Our main motivation for preparing this paper is to generalize shifted Jacobi function operational matrix for solving fuzzy fractional differential equations of order $0 < \nu < 1$ under Caputo's H-differentiability. We introduce a suitable way to approximate fuzzy solution of linear fuzzy fractional differential equations by means of shifted Jacobi functions based on the fuzzy residual of the problem in which the Jacobi operational matrix is introduced to be applied in the derivation of the proposed method. Another motivation is that the approximate solution based on the shifted Jacobi polynomials, $P_i^{(\alpha,\beta)}(x)$ $(i \geq 0, \alpha, \beta > -1)$, can be obtained in terms of the Jacobi parameters α and β. Therefore, instead of using with particular indexes, the solution can be derived generally to extend for other requests.

The paper organized as follows. In Section 2, we present some relevant properties of fuzzy sets, fuzzy differential equations and Jacobi polynomials with its error bound for approximate function accompanied by some details of JOM based on shifted Jacobi polynomials in crisp concept. Also, Caputo type derivative definition and its properties in the crisp sense is considered in this section. Some basic concepts of fuzzy fractional derivatives are explained in Section 3. Section 4 is devoted to the fuzzy approximation function using shifted Jacobi polynomials. Additionally, the Jacobi operational matrix (JOM) based on shifted Jacobi polynomials is extended for solving FFDEs in this section. Several examples are experienced to depict the effectiveness of the proposed method in Section 5. Finally, some conclusions are drawn in Section 6.

2 Preliminaries

Let us denote by $\mathbb{E}$ the class of fuzzy subsets u of the real axis $\mathbb{R}$ (*i.e.* $u : \mathbb{R} \to [0,1]$ satisfying the following properties):

(i) u is upper semicontinuous,

(ii) u is fuzzy convex, *i.e.*, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$, $\lambda \in [0,1]$,

(iii) u is normal, *i.e.*, $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,
(iv) $\operatorname{supp} u = \{x \in \mathbb{R} | u(x) > 0\}$ is the support of the u, and its closure $\operatorname{cl}(\operatorname{supp} u)$ is compact.

Then $\mathbb{E}$ is called the space of fuzzy real numbers and any $u \in \mathbb{E}$ is called fuzzy real number (see, *e.g.*, [38]).

The α-level set of a fuzzy number $\tilde{u} \in \mathbb{E}$, $0 \le r \le 1$, denoted by $[u]^r$, is defined as

$$[u]_r = \begin{cases} \{x \in \mathbb{R} | u(x) \ge r\} & \text{if } 0 < r \le 1, \\ \operatorname{cl}(\operatorname{supp} u) & \text{if } r = 0. \end{cases}$$

It is clear that the r-level set of a fuzzy number is a closed and bounded interval $[u_-^r, u_+^r]$, where u_-^r denotes the left-hand endpoint of $[u]^r$ and u_+^r denotes the right-hand endpoint of $[u]^r$. Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number $\tilde{y}$ defined by

$$\tilde{y}(t) = \begin{cases} 1 & \text{if } t = y, \\ 0 & \text{if } t \ne y, \end{cases}$$

$\mathbb{R}$ can be embedded in $\mathbb{E}$.

The addition and scaler multiplication of fuzzy number in $\mathbb{E}$ are defined as follows:

$$\begin{aligned} &(1) \quad u \oplus v = (u_- + v_-, u_+ + v_-), \\ &(2) \quad (\lambda \odot \tilde{u}) = \begin{cases} (\lambda u_-^r, \lambda u_+^r), & \lambda \ge 0, \\ (\lambda u_+^r, \lambda u_-^r), & \lambda < 0. \end{cases} \end{aligned}$$

We can define a matrix D on $\mathbb{E}$ ($D : \mathbb{E} \times \mathbb{E} \to \mathbb{R}_+ \cup 0$) by a distance which is so-called Hausdorff distance as follows:

$$D(u, v) = \sup_{r \in [0,1]} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}.$$

Then the following properties are known (see [38, 39]):
(i) $D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = D(\tilde{u}, \tilde{v})$, $\forall \tilde{u}, \tilde{v}, \tilde{w} \in \mathbb{E}$,
(ii) $D(k \odot \tilde{u}, k \odot \tilde{v}) = |k| D(\tilde{u}, \tilde{v})$, $\forall k \in \mathbb{R}$, $\tilde{u}, \tilde{v} \in \mathbb{E}$,
(iii) $D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{e}) \le D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e})$, $\forall \tilde{u}, \tilde{v}, \tilde{w} \in \mathbb{E}$,
(iv) $D(\tilde{u} + \tilde{v}, \tilde{0}) \le D(u, \tilde{0}) + D(v, \tilde{0})$, $\forall u, v \in \mathbb{E}$,
(v) $(\mathbb{E}, D)$ is a complete metric space.

Definition 1 ([40]) Let f and g be the two fuzzy-number-valued functions on the interval $[a, b]$, *i.e.*, $f, g : [a, b] \to \mathbb{E}$. The uniform distance between fuzzy-number-valued functions is defined by

$$D^*(f, g) := \sup_{x \in [a,b]} D(f(x), g(x)). \tag{1}$$

Remark 1 ([39]) Let $f : [a, b] \to \mathbb{E}$ be fuzzy continuous. Then from property (iv) of Hausdorff distance, we can define

$$D(f(x), \tilde{0}) = \sup_{r \in [0,1]} \max\{|f_-^r(x)|, |f_+^r(x)|\}, \quad \forall x \in [a, b].$$

Definition 2 ([30]) Let $x, y \in \mathbb{E}$. If there exists $z \in \mathbb{E}$ such that $x = y \oplus z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.

In this paper, the sign '$\ominus$' always stands for H-difference and note that $x \oplus y \neq x + (-y)$. Also, throughout of paper is assumed the Hukuhara difference and generalized Hukuhara differentiability are existed.

Definition 3 ([41]) The generalized difference (g-difference for short) of two fuzzy numbers $u, v \in \mathbb{E}$ is given by the expression

$$[u \ominus_g v]_\alpha = \Big[\inf_{\beta \geq \alpha} \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \sup_{\beta \geq \alpha} \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}\Big].$$

Proposition 1 ([41]) *For any fuzzy numbers $u, v \in \mathbb{E}$ the g-difference $u \ominus_g v$ exists and it is a fuzzy number.*

In this paper, we consider the definition of fuzzy differentiability presented by Bede and Gal in [32].

Definition 4 ([32]) Let $f : (a, b) \to \mathbb{E}$ and $x_0 \in (a, b)$. We say that f is strongly generalized differential at x_0. If there exists an element $f'(x) \in \mathbb{E}$, such that

(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric d)

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the limits (in the metric d)

$$\lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0),$$

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the limits (in the metric d)

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0),$$

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric d)

$$\lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0).$$

Remark 2 f is so-called (1)-differentiable on (a, b), if f is differentiable in the sense (i) of Definition 4 and also f is (2)-differentiable on (a, b), if f is differentiable in the sense (ii) of Definition 4.

The following theorem was proved by Chalco-Cano and Román-Flores [42] based on Definition 4.

Theorem 1 (see [42]) *Let $F : (a,b) \to \mathbb{E}$ be a function and denote $[F(t)]^r = [f_r(t), g_r(t)]$, for each $r \in [0,1]$. Then*:

(1) *If F is* (1)-*differentiable, then $f_r(t)$ and $g_r(t)$ are differentiable functions and*

$$[F'(t)]^r = [f_r'(t), g_r'(t)].$$

(2) *If F is* (1)-*differentiable, then $f_r(t)$ and $g_r(t)$ are differentiable functions and*

$$[F'(t)]^r = [g_r'(t), f_r'(t)].$$

Theorem 2 (see [43]) *Let $f(x)$ be a fuzzy-valued function on $[a,\infty)$ and it is represented by $(f_-^r(x), f_+^r(x))$. For any fixed $r \in [0,1]$, assume $(f_-^r(x)$ and $f_+^r(x))$ are Riemann-integrable on $[a,b]$ for every $b \geq a$, and assume there are two positive M_-^r and M_+^r such that $\int_a^b |f_-^r(x)|\,dx \leq M_-^r$ and $\int_a^b |f_+^r(x)|\,dx \leq M_+^r$ for every $b \geq a$. Then $f(x)$ is improper fuzzy Riemann-integrable on $[a,\infty)$ and the improper fuzzy Riemann-integral is a fuzzy number. Furthermore, we have*

$$\int_a^\infty f(x)\,dx = \left[\int_a^\infty f_-^r(x)\,dx, \int_a^\infty f_+^r(x)\,dx\right].$$

Definition 5 ([39]) $f(x) : [a,b] \to \mathbb{E}$. We say that f *fuzzy-Riemann integrable* to $I \in \mathbb{E}$, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v];\xi\}$ of $[a,b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_P{}^{*}(u-v) \odot f(\zeta), I\right) < \varepsilon,$$

where $\sum^{*}$ means addition with respect to $\oplus$ in $\mathbb{E}$

$$I := (FR)\int_a^b f(x)\,dx.$$

We also call an f as above (FR)-integrable.

Definition 6 ([44]) Consider the $n \times n$ linear system of equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2, \\ \qquad\qquad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n. \end{cases} \tag{2}$$

The matrix form of the above equations is

$$AX = Y, \tag{3}$$

where the coefficient matrix $A = (a_{ij})$, $1 \leq i,j \leq n$ is a crisp $n \times n$ matrix and $y_i \in \mathbb{E}$, $1 \leq i \leq n$. This system is called a fuzzy linear system (FLS).

Definition 7 ([44]) A fuzzy number vector $(x_1, x_2, \dots, x_n)^t$ given by $x_i = (x_{i-}^r, x_{i+}^r)$, $1 \le i \le n$, $0 \le r \le 1$ is called a solution of the fuzzy linear system (2) if

$$\left(\sum_{j=1}^{n} a_{ij}x_j\right)_-^r = \sum_{j=1}^{n}(a_{ij}x_j)_-^r = y_{i-}^r, \qquad \left(\sum_{j=1}^{n} a_{ij}x_j\right)_+^r = \sum_{j=1}^{n}(a_{ij}x_j)_+^r = y_{i+}^r.$$

If for a particular k, $a_{kj} > 0$, $1 \le j \le n$, we simply get

$$\sum_{j=1}^{n} a_{kj}x_{j-}^r = y_{k-}^r, \qquad \sum_{j=1}^{n} a_{kj}x_{j+}^r = y_{k+}^r.$$

To solve fuzzy linear systems, one can refer to [44, 45].

Now, we review some basic definitions of fractional integral and derivative, especially Caputo type, with their properties presented in crisp context [6, 46].

Remark 3 The fuzzy fractional derivative, in this paper, is assumed in the Caputo sense. The reason for adopting the Caputo definition, as pointed by Momani and Noor [47], is as follows: to solve differential equations (both classical and fractional), we need to specify additional conditions in order to produce a unique solution. Therefore, for the case of the fuzzy Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical fuzzy differential equations, and are therefore familiar to us. In contrast, for the fuzzy Riemann-Liouville fractional differential equations, these additional conditions constitute certain fuzzy fractional derivatives (and/or integrals) of the unknown solution at the initial point $x = 0$, which are functions of x. These fuzzy initial conditions are not physical like in the crisp concept; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. See more details in [35, 37, 48].

Definition 8 ([46]) The Riemann-Liouville fractional integral operator of order ν, $\nu \ge 0$ is defined as

$$\begin{aligned} &(I^\nu f)(x) = \frac{1}{\Gamma(\nu)}\int_0^x \frac{f(t)}{(x-t)^{1-\nu}}\,dt, \quad \nu > 0, x > 0, \\ &I^0 f(x) = f(x). \end{aligned}$$

Definition 9 ([6]) The Caputo fractional derivatives of order ν is defined as

$$D^\nu f(x) = I^{m-\nu}D^m f(x) = \frac{1}{\Gamma(m-\nu)}\int_0^x (x-t)^{m-\nu-1} f^m(t)\,dt, \quad m-1 < \nu \le m, x > 0,$$

where D^m is the classical differential operator of order m.

For the Caputo derivative, we have:

$$\begin{aligned} &{}^cD^\nu C = 0 \quad (C \text{ is a constant}), \\ &{}^cD^\nu x^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \nu \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} x^{\beta-\nu}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \ge \lceil \nu \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor \nu \rfloor. \end{cases} \end{aligned}$$

The ceiling function $\lceil \nu \rceil$ is used to denote the smallest integer greater than or equal to ν, and the floor function $\lfloor \nu \rfloor$ to denote the largest integer less than or equal to ν. Also $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Definition 10 ([48]) Similar to the differential equation of integer order, the Caputo's fractional differentiation is a linear operation, *i.e.*,

$$^{c}D^{\nu}\big(\lambda f(x) + \mu g(x)\big) = \lambda {}^{c}D^{\nu} f(x) + \mu {}^{c}D^{\nu} g(x),$$

where λ and μ are constants.

Definition 11 ([49]) A classical (crisp) set is normally defined as a collection of elements or objects $x \in X$ which can be finite, accountable, or overcountable. Each single element can either belong to or not belong to a set A, $A \subseteq X$ while in a fuzzy set (subset) elements of the set have a degree of membership in the set.

Remark 4 Throughout the paper, we use the crisp context frequently, regarding to Definition 11.

2.1 Jacobi polynomials

The Jacobi polynomials, denoted by $J_n^{\alpha,\beta}(z)$, are orthogonal with Jacobi weight function: $w(z) = (1-z)^{\alpha}(1+z)^{\beta}$ over $[-1,1]$, namely [50, 51],

$$\int_{-1}^{1} J_n^{\alpha,\beta}(z) J_m^{\alpha,\beta}(z) w(z)\, dz = \gamma_n^{\alpha,\beta} \delta_{nm},$$

where δ_{nm} is the Kronecker function and

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.$$

Also, the Jacobi polynomials can be created by means of the following recurrence formula:

$$\begin{aligned} J_i^{(\alpha,\beta)}(z) = {} & \frac{(\alpha+\beta+2i-1)\{\alpha^2-\beta^2+t(\alpha+\beta+2i)(\alpha+\beta+2i-2)\}}{2i(\alpha+\beta+i)(\alpha+\beta+2i-2)} J_{i-1}^{(\alpha,\beta)}(z) \\ & - \frac{(\alpha+i-1)(\beta+i-1)(\alpha+\beta+2i)}{i(\alpha+\beta+i)(\alpha+\beta+2i-2)} J_{i-2}^{(\alpha,\beta)}(z), \end{aligned}$$

for $i = 2, 3, \ldots$, where $J_0^{(\alpha,\beta)}(z) = 1$, and $J_1^{(\alpha,\beta)}(z) = \frac{\alpha+\beta+2}{2} z + \frac{\alpha-\beta}{2}$. In order to use these polynomials on the interval $x \in [0,1]$, we define the so-called shifted Jacobi polynomials by introducing the change of variable $z = 2x - 1$. Let the shifted Jacobi polynomials $J_n^{(\alpha,\beta)}(2x-1)$ be denoted by $P_i^{(\alpha,\beta)}(x)$. The shifted Jacobi polynomials are orthogonal with respect to the weight function $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha} x^{\beta}$ in the interval $[0,1]$ with the orthogonality property

$$\int_0^1 P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x) w^{(\alpha,\beta)}(x)\, dx = \nu_n^{\alpha,\beta} \delta_{nm},$$

at which $\nu_n^{\alpha,\beta}\delta_{nm} = \gamma_n^{\alpha,\beta}/2^{\alpha+\beta+1}$. The analytic form of the shifted Jacobi polynomials $P_i^{\alpha,\beta}(x)$ of degree i is acquired by

$$P_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!k!} x^k,$$

that

$$P_0^{(\alpha,\beta)}(0) = (-1)^i \frac{\Gamma(i+\beta+1)}{\Gamma(\beta+1)i!}, \qquad P_1^{(\alpha,\beta)}(1) = \frac{\Gamma(i+\alpha+1)}{\Gamma(\alpha+1)i!}.$$

Also, the shifted Jacobi polynomial can be stated by the following concise form.

Lemma 1 ([28]) *The shifted Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be obtained in the form of*

$$P_n^{(\alpha,\beta)}(x) = \sum_{i=0}^{n} P_i^{(n)} x^i,$$

in which $P_i^{(n)}$ are

$$P_i^{(n)} = (-1)^{n-i} \binom{n+\alpha+\beta+i}{i} \binom{n+\alpha}{n-i}, \quad i = 0,1,\ldots,n.$$

Lemma 2 ([28]) *For $\nu > 0$*

$$\int_0^1 x^\nu P_j^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)\,dx = \sum_{l=0}^{j} P_l^{(j)} B(\nu+l+\beta+1, \alpha+1),$$

in which $B(s,t)$ is the Beta function and stated as

$$B(s,t) = \int_0^1 \nu^{s-1}(1-\nu)^{t-1}\,d\nu = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

Let $\Omega = (0,1)$ and $\{P_i^{(\alpha,\beta)}(x)\}_{i=0}^m$ generate the space $\mathbb{P}^{m+1,\alpha,\beta}$. A function f belonging to $L_w^2(\Omega)$, can be expanded in $\mathbb{P}^{m+1,\alpha,\beta}$ by

$$f(x) = \sum_{i=0}^{+\infty} \theta_i P_i^{(\alpha,\beta)}(x),$$

where the coefficients θ_i are gained by

$$\theta_i = \frac{1}{\nu_i^{\alpha,\beta}} \int_0^1 P_i^{(\alpha,\beta)}(x) f(x) w^{(\alpha+\beta)}(x)\,dx, \quad i = 0,1,\ldots.$$

Realistically, only the first $(m+1)$-terms shifted Jacobi polynomials are considered. Then we have

$$f(x) \simeq f_m(x) = \sum_{i=0}^{m} \theta_i P_i^{(\alpha,\beta)}(x) = \Theta^T \Phi(x), \tag{4}$$

that

$$\Theta = [\theta_0, \theta_1, \dots, \theta_m]^T,$$
$$\Phi(x) = \left[P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \dots, P_m^{(\alpha,\beta)}(x)\right]^T.$$

Regarding to $\mathbb{P}^{m+1,\alpha,\beta}$ as a finite dimensional vector space, f has a unique best approximation from $\mathbb{P}^{m+1,\alpha,\beta}$, say $f_m(x) \in \mathbb{P}^{m+1,\alpha,\beta}$, that is,

$$\forall y \in \mathbb{P}^{m+1,\alpha,\beta}, \quad \|f(x) - f_m(x)\|_w \le \|f(x) - y\|_w.$$

So, the following lemma provides the upper bound of approximate function $f_{m+1}(x)$ using shifted Jacobi polynomials. This error bound proves that the approximate function $f_m(x)$ converges to $f(x)$ based on shifted Jacobi polynomials.

Lemma 3 ([28]) *Let the function $f : [x_0, 1] \to \mathbb{R}$ be $m+1$ times continuously differentiable for $x_0 > 0$, $f \in C^{m+1}[x_0, 1]$, and $\mathbb{P}^{m+1,\alpha,\beta} = \operatorname{Span}\{P_i^{(\alpha,\beta)}(x)\}_{i=0}^m$. If $f_m = \Theta^T\Phi(x)$ is the best approximation to f from $\mathbb{P}^{m+1,\alpha,\beta}$, then the error bound is presented as follows*:

$$\|f(x) - f_m(x)\|_w \le \frac{MS^{m+1}}{(m+1)!}\sqrt{B(\alpha+1, \beta+1)},$$

that $M = \max_{x\in[x_0,1]} f^{(m+1)}(x)$ and $S = \max\{1 - x_0, x_0\}$.

2.2 Operational matrix of Caputo's derivative of order *v*

In this section, the Jacobi operational matrix method based on the Caputo-type fractional derivative with using shifted Jacobi polynomials is explained. Afterward, an upper bound for the absolute error between the exact and approximate values of Caputo fractional derivative operator is provided (for more details, see [27, 28]).

Lemma 4 ([27]) *Let $\Phi(x)$ be shifted Jacobi vector defined in Eq.* (4) *and also let $v > 0$. Then*

$$D^v\Phi(x) \simeq D^{(v)}\Phi(x), \tag{5}$$

where $D^{(v)}$ is $(m+1) \times (m+1)$ operational matrix of derivatives of order v in the Caputo sense and is defined by:

$$D^{(v)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \Delta_v(\lceil v\rceil, 0) & \Delta_v(\lceil v\rceil, 1) & \Delta_v(\lceil v\rceil, 0) & \cdots & \Delta_v(\lceil v\rceil, N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \Delta_v(i, 0) & \Delta_v(i, 1) & \Delta_v(i, 2) & \cdots & \Delta_v(i, N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \Delta_v(m, 0) & \Delta_v(m, 1) & \Delta_v(m, 2) & \cdots & \Delta_v(m, m) \end{pmatrix},$$

where

$$\Delta_\nu(i,j) = \sum_{k=\lceil \nu \rceil}^{i} \delta_{ijk},$$

and δ_{ijk} *is given by*

$$\delta_{ijk} = \frac{(-1)^{i-k}\Gamma(j+\beta+1)\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\nu_j\Gamma(j+k+\alpha+\beta+1)\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)\Gamma(k-\nu+1)(i-k)!} \times \sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(j+l+\alpha+\beta+1)\Gamma(\alpha+1)\Gamma(l+k+\beta-\nu+1)}{\Gamma(l+\beta+1)\Gamma(l+k+\alpha+\beta-\nu+2)(j-l)!l!}.$$

Note that in $D^{(\nu)}$, *the first* $\lceil \nu \rceil$ *rows, are all zeros.*

Now, the following lemma gives us an upper bound for error estimation of Caputo derivative operator mentioned in Lemma 4. But initially, we define the error vector E_ν as:

$$E_\nu = D^\nu\Phi - D^{(\nu)}\Phi = [E_{0,\nu}, E_{1,\nu}, \dots, E_{m,\nu}]^T,$$

where

$$E_{k,\nu} = D^\nu P_k^{(\alpha,\beta)}(x) - \sum_{j=0}^{m} D_{kj}^{(\nu)} P_j^{(\alpha,\beta)}(x), \quad k = 0,1,\dots,m.$$

Lemma 5 *If the error function of Caputo fractional derivative operator for Jacobi polynomials* $E_{k,\nu} : [x_0, 1] \to \mathbb{R}$ *is* $m+1$ *times continuously differentiable for* $0 < x_0 \le x$, $x \in (0,1]$. *Also* $E_{k,\nu} \in C^{m+1}[x_0, 1]$ *and* $\nu < m+1$ *then the error bound is gained as follows*:

$$\|E_{k,\nu}\|_w \le \frac{x_0^{-\nu}}{(m+1)!|\Gamma(1-\nu)|}(S)^{m+1}\binom{k+\beta}{k}\sqrt{B(\alpha+1,\beta+1)}. \tag{6}$$

Proof Analogously to the demonstration of Lemma 5 in [28], we can prove this lemma. □

Therefore, the maximum norm of error vector E_ν is achieved as

$$\|E_\nu\|_\infty \le \begin{cases} \frac{x_0^{-\nu}}{(m+1)!|\Gamma(1-\nu)|}(S)^{m+1}\binom{m+1+\beta}{m+1}\sqrt{B(\alpha+1,\beta+1)}, & \beta \ge 0, \\ \frac{x_0^{-\nu}}{(m+1)!|\Gamma(1-\nu)|}(S)^{m+1}\sqrt{B(\alpha+1,\beta+1)}, & \beta < 0. \end{cases}$$

3 Fuzzy Caputo-type fractional differentiability

The fuzzy fractional differentiability of order $0 < \nu < 1$, particularly Caputo type, is considered in this section. Some basic definitions and theorems are given and introduced the necessary notation, which will be used in the rest of paper. See, for example, [34, 35, 37].

At first, some notations are presented which are put to use throughout the remaining sections. It is easy to find these notations in the crisp sense. See [46, 48].

- $L_p^{\mathbb{E}}(a,b)$, $1 \leq p \leq \infty$ is the set of all fuzzy-valued measurable functions f on $[a,b]$ where $\|f\|_p = (\int_0^1 (d(f(t),0))^p\,dt)^{\frac{1}{p}}$.
- $C^{\mathbb{E}}[a,b]$ is a space of fuzzy-valued functions, which are continuous on $[a,b]$.
- $C_n^{\mathbb{E}}[a,b]$ indicates the set of all fuzzy-valued functions, which are continuous up to order n.
- $AC^{\mathbb{E}}[a,b]$ denotes the set of all fuzzy-valued functions, which are absolutely continuous.

Definition 12 ([37]) Let $f \in C^{\mathbb{E}} \cap L_p^{\mathbb{E}}$, the Riemann-Liouville integral of fuzzy-valued function f is as follows:

$$\left(I_{a+}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\nu}}, \quad x > a. \tag{7}$$

To specify the fuzzy Riemann-Liouville integral of fuzzy-valued function f based on the lower and upper functions, the following definition is determined.

Definition 13 ([37]) Let $f \in C^{\mathbb{E}} \cap L_p^{\mathbb{E}}$, the parametric form of the Riemann-Liouville integral of fuzzy-valued function f can be expressed by

$$\left(I_{a^+}^{\nu} f\right)^r(x) = \left[\frac{1}{\Gamma(1-\nu)} \int_a^x \frac{f_-^r(t)}{(x-t)^{\nu}}\,dt, \frac{1}{\Gamma(1-\nu)} \int_a^x \frac{f_+^r(t)}{(x-t)^{\nu}}\,dt\right],$$

in which $f^r(x) = [f_-^r(x), f_+^r(x)]$ and $0 < \nu < 1$.

Definition 14 ([35]) Let $f : L^{\mathbb{E}} \cap C^{\mathbb{E}}$ be a fuzzy set-value function. Then f is said to be Caputo's fuzzy differentiable at x when

$$\left({}^C D_{a^+}^{\nu} f\right)(x) = \frac{1}{\Gamma(1-\nu)} \int_a^x \frac{f'(t)}{(x-t)^{\nu}}\,dt, \tag{8}$$

where $0 < \alpha < 1$.

Definition 15 ([35]) Let $f : L^{\mathbb{E}} \cap C^{\mathbb{E}}$ and $x_0 \in (a,b)$ and $\Phi(x) = \frac{1}{\Gamma(1-\nu)} \int_a^x \frac{f(t)}{(x-t)^{\nu}}\,dt$. We say that $f(x)$ is fuzzy Caputo fractional differentiable of order $0 < \nu < 1$ at x_0, if there exists an element $({}^c D_{a^+}^{\nu} f)(x_0) \in C^E$ such that for all $0 \leq r \leq 1$, $h > 0$,

$$\text{(i)} \quad \left({}^c D_{a^+}^{\nu} f\right)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0 + h) \ominus \Phi(x_0)}{h} = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 - h)}{h},$$

or

$$\text{(ii)} \quad \left({}^c D_{a^+}^{\nu} f\right)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{\Phi(x_0 - h) \ominus \Phi(x_0)}{-h},$$

or

$$\text{(iii)} \quad \left({}^c D_{a^+}^{\nu} f\right)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0 + h) \ominus \Phi(x_0)}{h} = \lim_{h \to 0^+} \frac{\Phi(x_0 - h) \ominus \Phi(x_0)}{-h},$$

or

$$\text{(iv)} \quad \left({}^c D_{a^+}^{\nu} f\right)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 - h)}{h}.$$

Remark 5 A fuzzy-valued function f is ${}^c[1-\nu]$-differentiable, if it is differentiable as in the Definition 15, case (i), and it is ${}^c[2-\nu]$-differentiable, if it is differentiable as in the Definition 15, case (ii).

Theorem 3 ([35]) *Let us assume that $f \in C^{\mathbb{E}}[a,b]$, then we have the following*:

$$\left(I^{\nu}_{a^+}\,{}^cD^{\nu}_{a^+}f\right)(x) = f(x) \ominus f(a), \quad 0 < \nu < 1,$$

when f is ${}^c[1-\nu]$-differentiable and

$$\left(I^{\nu}_{a^+}\,{}^cD^{\nu}_{a^+}f\right)(x) = -f(a) \ominus \left(-f(x)\right), \quad 0 < \nu < 1,$$

when f is ${}^c[2-\nu]$-differentiable.

Lemma 6 ([35]) *Let $0 < \alpha < 1$ and $f \in AC^{\mathbb{E}}[a,b]$, then the fuzzy Caputo derivative can be stated using the fuzzy fractional Riemann-Liouville integral operator as follows*:

$$\left({}^cD^{\nu}_{a^+}f\right)(x;r) = \left(I^{1-\nu}_{a^+}Df\right)(x;r) = \left[\left(I^{1-\nu}_{a^+}Df\right)^r_-(x), \left(I^{1-\nu}_{a^+}Df\right)^r_+(x)\right],$$

when f is ${}^c[1-\nu]$-differentiable, and

$$\left({}^CD^{\nu}_{a^+}f\right)(x;r) = \left(I^{1-\nu}_{a^+}Df\right)(x;r) = \left[\left(I^{1-\nu}_{a^+}Df\right)^r_+(x), \left(I^{1-\alpha}_{a^+}Df\right)^r_-(x)\right],$$

when f is ${}^c[2-\nu]$-differentiable.

4 Extension of JOM method for FFEDs

In this section, fuzzy approximation function by means of shifted Jacobi polynomials is derived. Moreover, the Jacobi operational matrix based of fuzzy shifted Jacobi polynomials is introduced with details and provided the application of the method for solving fuzzy linear fractional differential equations of order $0 < \nu < 1$. It should be mentioned that this method is the extension of the researches implemented in the crisp sense by Doha *et al.* [27] and Kazem [28].

In [52–54], the authors established the concepts of the best approximation of fuzzy function and as an application, Lowen introduced fuzzy approximation of fuzzy function by means of Lagrange interpolation [55]. Firstly, we define the approximate fuzzy function using shifted Jacobi polynomials.

Definition 16 For $y \in L^{\mathbb{E}}_p[0,1] \cap C^{\mathbb{E}}[0,1]$ and shifted Jacobi polynomial $P^{(\alpha,\beta)}_n(x)$ a real valued function over $[0,1]$, the fuzzy function is approximated by

$$y(x) = \sum_{i=0}^{+\infty}{}^{*}\theta_i \odot P^{(\alpha,\beta)}_i(x),$$

where the fuzzy coefficients θ_i are gained by

$$\theta_i = \frac{1}{\nu^{\alpha,\beta}_i}\int_0^1 P^{(\alpha,\beta)}_i(x) \odot y(x) \odot w^{(\alpha+\beta)}(x)\,dx, \quad i = 0,1,\ldots \qquad (9)$$

in which $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha} \odot x^{\beta}$, $P_i^{(\alpha,\beta)}(x)$ is as the same as the shifted Jacobi polynomials described in Section 2.1, and $\sum^*$ means addition with respect to $\oplus$ in $\mathbb{E}$.

Remark 6 In practice, only the first $(m+1)$-terms shifted Jacobi polynomials are taken into consideration. So, we have

$$y(x) \simeq \tilde{y}_{m+1}(x) = \sum_{i=0}^{m}{}^{*} \theta_i \odot P_i^{(\alpha,\beta)}(x) = \Theta_{m+1}^T \odot \Phi_{m+1}(x), \tag{10}$$

that the fuzzy shifted Jacobi coefficient vector Θ_{m+1}^T and shifted Jacobi function vector $\Phi_{m+1}(x)$ are explained by:

$$\Theta_{m+1}^T = [\theta_0, \theta_2, \dots, \theta_m], \tag{11}$$

$$\Phi_{m+1}(x) = \left[P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \dots, P_m^{(\alpha,\beta)}(x)\right]^T. \tag{12}$$

Since $\theta_i^r = [\theta_{i,-}^r, \theta_{i,+}^r]$, for all $0 \le r \le 1$, then we can point out the fuzzy approximation function $y_m(x)$, according to the lower and upper functions as follows.

Definition 17 Let $y(x) \in L_p^{\mathbb{E}}[0,1] \cap C^{\mathbb{E}}[0,1]$, the approximation of fuzzy-valued function $y(x)$ in the parametric form is

$$y^r(x) \simeq \tilde{y}_{m+1}^r(x) = \left[\sum_{i=0}^{m} \theta_{i,-}^r P_i^{(\alpha,\beta)}(x), \sum_{i=0}^{m} \theta_{i,+}^r P_i^{(\alpha,\beta)}(x),\right], \quad 0 \le r \le 1. \tag{13}$$

Theorem 4 *The best approximation of a fuzzy function based on the Jacobi points exists and is unique.*

Proof The proof is an immediate result of Theorem 4.2.1 in [54]. □

Now, in the following theorem, we will achieve the error bound for the fuzzy approximate function based on shifted Jacobi polynomials. Actually, this error bound depicts that the approximation converges to the fuzzy function $y(x)$.

Theorem 5 *Consider the function $y(x) : [x_0, 1] \to L_p^{\mathbb{E}}[0,1] \cap C^{\mathbb{E}}[0,1]$ is $m+1$ times continuously fuzzy differentiable for $x_0 > 0$, $y(x) \in C_{m+1}^{\mathbb{E}}[x_0, 1]$, and $\mathbb{P}^{m+1,\alpha,\beta} = \mathrm{Span}\{P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \dots, P_m^{(\alpha,\beta)}(x)\}$. If $y_m = \Theta^T \Phi(x)$ is the best fuzzy approximation to $y(x)$ from $\mathbb{P}^{m+1,\alpha,\beta}$, then the error bound is presented as follows*:

$$D^*\big(y(x), y_m(x)\big) \le \frac{MS^{m+1}}{(m+1)!} \sqrt{B(\alpha+1, \beta+1)},$$

that $M^r = \max_{x \in [x_0,1]}\{M_-^r, M_+^r\}$ and $S = \max\{1 - x_0, x_0\}$.

Proof It follows from Definition 1 of D^* that

$$\begin{aligned} D^*\big(y(x), y_m(x)\big) &= \sup_{x\in[0,1]} D\big(y(x), y_m(x)\big) \\ &= \sup_{x\in[0,1]} \sup_{r\in[0,1]} \max\big\{\big|y_-^r(x) - y_{m,-}^r(x)\big|, \big|y_+^r(x) - y_{m,+}^r(x)\big|\big\} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{r\in[0,1]} \max\left\{\left\|y_-^r(x) - y_{m,-}^r(x)\right\|_\infty, \left\|y_+^r(x) - y_{m,+}^r(x)\right\|_\infty\right\} \\
&\overset{\text{Lemma 3}}{\le} \sup_{r\in[0,1]} \max\left\{\frac{M_-^r S^{m+1}}{(m+1)!}\sqrt{B(\alpha+1,\beta+1)}, \frac{M_+^r S^{m+1}}{(m+1)!}\sqrt{B(\alpha+1,\beta+1)}\right\} \\
&\le \frac{MS^{m+1}}{(m+1)!}\sqrt{B(\alpha+1,\beta+1)},
\end{aligned}
$$

in which $M^r = \max_{x\in[x_0,1]}\{M_-^r, M_+^r\}$ and $S = \max\{1-x_0, x_0\}$. This completes the proof of theorem. □

4.1 Jacobi operational matrix

This part is devoted to the operational matrix of shifted Jacobi polynomials regarding to fuzzy Caputo's derivative. The operational matrix play an important role in solving fractional differential equations by means of orthogonal functions [22, 23, 25, 27, 28]. Our aim in this section is to generalize this method for solving fuzzy linear fractional differential equations.

Lemma 7 *The fuzzy Caputo fractional derivative of order* $0 < \nu < 1$ *over the shifted Jacobi functions can be acquired in the form of*

$$
{}^cD^\nu P_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{i} P_k^{\prime(i)} \frac{\Gamma(k+1)}{\Gamma(k-\nu+1)} x^{k-\nu},
$$

where $P_k^{\prime(i)} = 0$ *for* $i < \lceil\nu\rceil$ *and for* $i \ge \lceil\nu\rceil$, *we have* $P_k^{\prime(i)} = P_k^{(i)}$.

Proof It is straightforward from Section 2.1 and the Caputo derivative of x^k. □

The fuzzy Caputo operational matrix based on the shifted Jacobi polynomials is expressed as well as relation (5). So, we have

$$
{}^{\mathbb{E}}D^\nu \Phi(x) \simeq {}^{\mathbb{E}}D^{(\nu)}\Phi(x), \tag{14}
$$

where $D^{(\nu)}$ is the $(m+1)$-square operational matrix of fuzzy fractional Caputo's derivative of shifted Jacobi polynomials and ${}^{\mathbb{E}}D^\nu\Phi(x) \in C^{\mathbb{E}}[a,b]$. So, using (10) and (14), we can approximate the fuzzy fractional Caputo's derivative as

$$
{}^{\mathbb{E}}D^\nu y(x) \simeq {}^{\mathbb{E}}D^{(\nu)}\tilde{y}_{m+1}(x) = \sum_{i=0}^{m}{}^{*}\theta_i \odot {}^{\mathbb{E}}D^{(\nu)}P_i^{(\alpha,\beta)}(x) = \Theta_{m+1}^T \odot {}^{\mathbb{E}}D^{(\nu)}\Phi_{m+1}(x). \tag{15}
$$

The subsequent property of the product of two fuzzy Jacobi function vectors will also be utilized

$$
\Phi\Phi^T\Theta \simeq {}^{\sim}\Theta\Phi, \tag{16}
$$

that ${}^{\sim}\Theta$ is a $m+1$ product operational matrix for the vector Θ, which its elements $\{{}^{\sim}\Theta\}_{i,j=0}^m$ can be calculated from:

$$
{}^{\sim}\Theta_{ij} = \frac{1}{\nu_j^{\alpha,\beta}} \sum_{k=0}^{m}{}^{*}\theta_k \odot g_{ijk}, \tag{17}
$$

where g_{ijk} is acquired by

$$g_{ijk} = \int_0^1 P_i^{\alpha,\beta}(x) P_j^{\alpha,\beta}(x) w^{\alpha,\beta}(x)\,dx.$$

The error bound of fuzzy Caputo fractional differential operator is taken into consideration in the next theorem for $0 < \nu < 1$. Therefore, we clarify the ${}^{\mathbb{E}}E_\nu$ as

$${}^{\mathbb{E}}E_\nu = {}^{\mathbb{E}}D^\nu \Phi(x) - {}^{\mathbb{E}}D^{(\nu)}\Phi(x) = [E_{0,\nu}, E_{1,\nu}, \ldots, E_{m,\nu}]^T,$$

where

$$E_{k,\nu} = {}^{\mathbb{E}}D^\nu P_k^{(\alpha,\beta)}(x) - \sum_{j=0}^{m} {}^{\mathbb{E}}D_{kj}^{(\nu)} P_j^{(\alpha,\beta)}(x), \quad k = 0, 1, \ldots, m.$$

Theorem 6 *Assume that the error function of fuzzy Caputo fractional derivative operator for shifted Jacobi polynomials $E_{k,\nu}$ be $m+1$ times continuously fuzzy differentiable for $0 < x_0 \le x$, $x \in (0,1]$. Additionally, $E_{k,\nu} \in C_{m+1}^{\mathbb{E}}[x_0, 1]$ and $0 < \nu < 1$ then the error bound is given by*

$$D^*\big({}^{\mathbb{E}}D^\nu \Phi(x), {}^{\mathbb{E}}D^{(\nu)}\Phi(x)\big) \le \frac{x_0^{-\nu}}{(m+1)!|\Gamma(1-\nu)|} (S)^{m+1} \binom{m+1+\beta}{m+1} \sqrt{B(\alpha+1, \beta+1)}. \tag{18}$$

Proof Again using Definition 1 of D^* and Lemma 5, we have

$$\begin{aligned}
&D^*\big({}^{\mathbb{E}}D^\nu \Phi(x), {}^{\mathbb{E}}D^{(\nu)}\Phi(x)\big) \\
&\quad = \sup_{x\in[0,1]} D\big({}^{\mathbb{E}}D^\nu \Phi(x), {}^{\mathbb{E}}D^{(\nu)}\Phi(x)\big) \\
&\quad = \sup_{x\in[0,1]} \sup_{r\in[0,1]} \max\Big\{\big|\big({}^{\mathbb{E}}D^\nu \Phi(x)\big)_-^r - \big({}^{\mathbb{E}}D^{(\nu)}\Phi(x)\big)_-^r\big|, \big|\big({}^{\mathbb{E}}D^\nu \Phi(x)\big)_+^r - \big({}^{\mathbb{E}}D^{(\nu)}\Phi(x)\big)_+^r\big|\Big\} \\
&\quad = \sup_{x\in[0,1]} \sup_{r\in[0,1]} \max\big\{\big|{}^{\mathbb{E}}D^\nu\big(\Phi_-^r(x)\big) - {}^{\mathbb{E}}D^{(\nu)}\big(\Phi_-^r(x)\big)\big|, \big|\big({}^{\mathbb{E}}D^\nu \Phi_+^r(x)\big) - \big({}^{\mathbb{E}}D^{(\nu)}\Phi_+^r(x)\big)\big|\big\} \\
&\quad = \sup_{r\in[0,1]} \max\big\{\big\|{}^{\mathbb{E}}D^\nu\big(\Phi_-^r(x)\big) - {}^{\mathbb{E}}D^{(\nu)}\big(\Phi_-^r(x)\big)\big\|_\infty, \big\|\big({}^{\mathbb{E}}D^\nu \Phi_+^r(x)\big) - \big({}^{\mathbb{E}}D^{(\nu)}\Phi_+^r(x)\big)\big\|_\infty\big\} \\
&\quad = \sup_{r\in[0,1]} \max\big\{\big\|{}^{\mathbb{E}}E_{\nu,-}^r\big\|_\infty, \big\|{}^{\mathbb{E}}E_{\nu,+}^r\big\|_\infty\big\} \\
&\quad \le \frac{x_0^{-\nu}}{(m+1)!|\Gamma(1-\nu)|} (S)^{m+1} \binom{m+1+\beta}{m+1} \sqrt{B(\alpha+1, \beta+1)}.
\end{aligned}$$

□

4.2 Application of the JOM of the fractional Caputo derivative

In this section, the Jacobi operational matrix derived from the previous sections is applied for solving linear FFDEs of order $0 < \nu < 1$ based on the shifted Jacobi polynomials. The *fuzzy residual* of the general single-term FFDEs is obtained and then using the orthogonal property of the Jacobi polynomials, a fuzzy algebraic system is extracted, which is solved easily to find the unknown fuzzy coefficient of the approximate solution of the problem.

Let us consider the general linear fuzzy fractional differential equation

$$\begin{cases} ({}^cD^{\nu}_{0^+}y)(x) = y(x) \oplus f(x), \quad 0 < \nu \le 1, \\ y(0) = y_0 \in \mathbb{E}, \end{cases} \tag{19}$$

in which $y(x) : L^{\mathbb{E}} \cap C^{\mathbb{E}}$ is a continuous fuzzy-valued function, ${}^cD^{\nu}_{0^+}$ denotes the fuzzy Caputo fractional derivative of order ν and $f(x) : [0,1] \to \mathbb{E}$.

In the following theorem, we clarify the way to find the fuzzy unknown coefficient of the fuzzy approximate function $\tilde{y}_{m+1}(x)$, using the fuzzy residual function of the problem easily.

Theorem 7 *Let $y^r \in C^{\mathbb{E}}[0,1]$ and $0 < \nu \le 1$, then*

$$\bigl[\bigl({}^cD^{(\nu)}\tilde{y}_m\bigr)(x)\bigr]^{(r)} = \bigl[R_m(x) \oplus \tilde{f}_m(x) \oplus \tilde{y}_m(x)\bigr]^{(r)}.$$

Proof Let $r \in [0,1]$. At first, from Eqs. (14) and (15) we can replace solution $y(x)$ with $y_m(x)$,

$$\bigl[\bigl({}^cD^{(\nu)}_{0^+}\tilde{y}_m\bigr)(x)\bigr]^{(r)} = \bigl[\bigl({}^cD^{(\nu)}_{0^+}\tilde{y}_m\bigr)^r_-(x), \bigl({}^cD^{(\nu)}_{0^+}\tilde{y}_m\bigr)^r_+(x)\bigr], \qquad \bigl[\tilde{y}_m(x)\bigr]^{(r)} = \bigl[\tilde{y}^{(r)}_{m-}(x), \tilde{y}^{(r)}_{m+}(x)\bigr].$$

Then the fuzzy residual function of the problem is expressed by

$$\bigl({}^cD^{(\nu)}\tilde{y}^{(r)}_{m\pm}\bigr)(x) = R^{(r)}_{m\pm}(x) - \tilde{y}_m(x)^r_{\pm} - \tilde{f}_m(x)^r_{\pm},$$

hence,

$$\begin{aligned} \bigl[\bigl({}^cD^{(\nu)}\tilde{y}_m\bigr)(x)\bigr]^{(r)} &= \bigl[\bigl({}^cD^{(\nu)}\tilde{y}^{(r)}_{m-}\bigr)(x), \bigl({}^cD^{\nu}\tilde{y}^{(r)}_{m+}\bigr)(x)\bigr] \\ &= \bigl[R^{(r)}_{m-}(x), R^{(r)}_{m+}(x)\bigr] - \bigl[\tilde{y}_m(x)^r_-, \tilde{y}_m(x)^r_+\bigr] - \bigl[\tilde{f}_m(x)^r_-, \tilde{f}_m(x)^r_+\bigr] \\ &= \bigl[R_m(x)\bigr]^{(r)} \oplus \bigl[\tilde{y}_m(x)\bigr]^{(r)} \oplus \bigl[\tilde{f}_m(x)\bigr]^{(r)}. \end{aligned}$$

□

Regarding to Definition 3 of g-difference, we have

$$\bigl[R_m(x)\bigr]^{(r)} = \bigl[\bigl({}^cD^{(\nu)}\tilde{y}_m\bigr)(x)\bigr]^{(r)} \ominus_g \bigl[\tilde{y}_m(x)\bigr]^{(r)} \ominus_g \bigl[\tilde{f}_m(x)\bigr]^{(r)},$$

or in the form of fuzzy operator, we can state

$$[R_m]^{(r)} = \bigl(I \ominus_g {}^cD^{(\nu)}\bigr)y^{(r)} \ominus_g f^{(r)}. \tag{20}$$

Let $\langle \cdot, \cdot \rangle_{\mathbb{E}}$ denotes the fuzzy inner product over $X_{\mathbb{E}} = L^2_{\mathbb{E}}([0,1])$. It is required like a typical tau method in the crisp concept (see [56, 57]) that $R^{(r)}_m$ satisfy

$$\bigl\langle R^{(r)}_m(x), P^{(\alpha,\beta)}_i(x)\bigr\rangle_E = \tilde{0}, \quad i = 0, 1, \ldots, m-1, r \in [0,1], \tag{21}$$

where $\langle R^{(r)}_m(x), P^{(\alpha,\beta)}_i(x)\rangle^r_E = [(FR)\int_0^1 R_m(x) \odot (x), P^{(\alpha,\beta)}_i(x) \odot w^{\alpha,\beta}(x)\,dx]^{(r)}$.

From Eq. (21), we gain (m)-fuzzy linear algebraic equations which are as follows in the expanded form:

$$\sum_{j=0}^{m}{}^{*}\theta^{(r)}_j \odot \bigl\{\bigl\langle D^{(\nu)}P^{(\alpha,\beta)}_j, P^{(\alpha,\beta)}_i\bigr\rangle - \bigl\langle P^{(\alpha,\beta)}_j, P^{(\alpha,\beta)}_i\bigr\rangle\bigr\} = \sum_{j=0}^{m}{}^{*}f^{(r)}_j \odot \bigl\langle P^{(\alpha,\beta)}_j, P^{(\alpha,\beta)}_i\bigr\rangle, \tag{22}$$

for $i = 0, 1, \ldots, m-1$. Also the fuzzy coefficients f_j are defined as

$$f(x) \simeq \tilde{f}_{m+1}(x) = \sum_{j=0}^{m}{}^{*} f_j \odot P_j^{(\alpha,\beta)}(x) = F_{m+1}^T \odot \Phi_{m+1}, \tag{23}$$

where $F_{m+1} = [f_0, f_1, \ldots, f_m]^T$ is acquired as

$$f_i = \frac{1}{v_i^{\alpha,\beta}} \int_0^1 P_i^{(\alpha,\beta)}(x) \odot f(x) \odot w^{(\alpha+\beta)}(x)\,dx, \quad i = 0, 1, \ldots, m. \tag{24}$$

Subsequently, replacing Eq. (10) in the initial condition of the problem (19)

$$y(0) = \sum_{j=0}^{m}{}^{*} \theta_j^{(r)} \odot P_j^{(\alpha,\beta)}(0) = y_0, \tag{25}$$

from the above equation with Eq. (22), $(m+1)$-fuzzy linear algebraic equations are generated. It is obvious that the unknown fuzzy coefficients are obtained with solving this fuzzy system using the method presented for example in [44].

5 Test problems

In this part, different examples are considered to depict the feasibility of the proposed method for solving FFDEs with a suitable accuracy.

Example 1 Consider the following FFDE:

$$\begin{cases} ({}^cD_{0^+}^{\nu} y)(x) = \lambda \odot y(x) \oplus (x+1), & 0 < \nu \leq 1, \\ y^r(0) = [y_0{}_-^r, y_0{}_+^r], & 0 < r \leq 1. \end{cases} \tag{26}$$

Here, suppose that $\lambda = -1$, then using ${}^c[2-\nu]$-differentiability and Theorem 1 we have the following parametric form:

$$\begin{cases} ({}^cD_{0^+}^{\nu} y_-^r)(x) = -y_-^r(x) + x + 1, & 0 < \nu \leq 1, \\ y_-^r(0) = y_0{}_-^r, & 0 < r \leq 1, \end{cases} \tag{27}$$

and

$$\begin{cases} ({}^cD_{0^+}^{\nu} y_+^r)(x; r) = -y_+^r(x) + x + 1, & 0 < \alpha \leq 1, \\ y_+^r(0) = y_0{}_+^r(r), & 0 < r \leq 1, \end{cases} \tag{28}$$

where $y^r(0) = [0.5 + 0.5r, 1.5 - 0.5r]$. The analytical solution of the problem (26) can be acquired using Eqs. (27) and (28) as

$$\begin{cases} \underline{Y}(x; r) = (0.5 + 0.5r)E_{\nu,1}[x^\nu] + \int_0^x (x-t)^{\nu-1} E_{\nu,\nu}[-(x-t)^\nu](x+1)\,dt, & 0 < \nu \leq 1, \\ \overline{Y}(x; r) = (1.5 - 0.5r)E_{\nu,1}[x^\nu] + \int_0^x (x-t)^{\nu-1} E_{\nu,\nu}[-(x-t)^\nu](x+1)\,dt, & 0 < r \leq 1. \end{cases} \tag{29}$$

By applying the technique explained in Section 4, the equation is gained in the matrix form as

$$\begin{cases} \Theta_{m+1,-}^T [D^{(\nu)} + I]\Phi(x) = F_{m+1,-}^T \Phi(x), \\ \Theta_{m+1,+}^T [D^{(\nu)} + I]\Phi(x) = F_{m+1,+}^T \Phi(x), \end{cases} \tag{30}$$

where the values of vector F^T is obtained as Eq. (23). Deriving the fuzzy residual function and multiplying it by $P_j^{(\alpha,\beta)}(x) \odot w^{\alpha,\beta}$, $j = 0,1,\dots,m-1$ to generate (m)-fuzzy algebraic equations

$$\begin{cases} \Theta^T_{m+1,-}[D^{(\nu)} + I] = F^T_{m+1,-}, \\ \Theta^T_{m+1,+}[D^{(\nu)} + I] = F^T_{m+1,+}. \end{cases} \tag{31}$$

Also for $y(0;r) = [{y_0}^r_-, {y_0}^r_+]$, one has

$$\begin{cases} y^r_-(0) \simeq \Theta^T_{m+1,-}\Phi_{m+1} = (0.5 + 0.5r), \\ y^r_+(0) \simeq \Theta^T_{m+1,+}\Phi_{m+1} = (1.5 - 0.5r). \end{cases} \tag{32}$$

Finally, Eqs. (31) and (32) create $(m + 1)$-fuzzy linear equations to give us the unknown fuzzy coefficients θ_j after solving this system.

With $m = 2$, $\alpha = \beta = 0.5$ and $\nu = 0.75$, we have

$$\begin{aligned} D^{(0.75)} &= \begin{pmatrix} 0 & 0 & 0 \\ \Delta_{(0.75)}(1,0) & \Delta_{(0.75)}(1,1) & \Delta_{(0.75)}(1,2) \\ \Delta_{(0.75)}(2,0) & \Delta_{(0.75)}(2,1) & \Delta_{(0.75)}(2,2) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 2.6929 & 0.5524 & -0.1755 \\ -1.2429 & 4.2241 & 1.1048 \end{pmatrix}, \end{aligned}$$

and with the assumption that r-cut = 1,

$$F_3 = \begin{pmatrix} 1.5 \\ 0.25 \\ 1 \end{pmatrix}.$$

So, considering these two matrices and substituting them into Eqs. (30) and (32), we can obtain the fuzzy coefficients as

$$\Theta_3 = [1.1\tilde{5}50, 0.1\tilde{3}84, 0.0\tilde{2}81].$$

From Table 1, we can obtain a good approximation with the exact solution by making use of the proposed method. In this table, the results are gained at $x = 1$. Also, the method is tested with different values of α, β which are depicted in Figure 1. The results is more accurate with $\alpha = 0$, $\beta = 0.5$. We may also see from Figure 2, the absolute error is smaller and smaller when m grows. Finally, the approximate fuzzy solution is illustrated in Figure 3 for different values of ν that shows this approach can solve FFDEs of different fractional order effectively.

Example 2 Consider the inhomogeneous linear fractional relaxation equation in [58] in the sense of fuzzy context, so we have

$$\begin{cases} {}^cD^\nu_{0+}y(x) + y(x) = x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\nu)}x^{(3-\nu)} + \frac{24}{\Gamma(5-\nu)}x^{(4-\nu)}, \\ y^r(0) = [-1 + r, 1 - r], \quad 0 < \nu \le 1, 0 \le x \le 1, \end{cases} \tag{33}$$

Table 1 The absolute error of the proposed method for Example 1 with different values of α, β and $m = 5$

(α,β)	(0,0)	(0.5,0)	(0,0.5)	(0.5,0.5)	(0,0)	(0.5,0)	(0,0.5)	(0.5,0.5)
r	$E^1_{0.75}$	$E^2_{0.75}$	$E^3_{0.75}$	$E^4_{0.75}$	$E^1_{0.75}$	$E^2_{0.75}$	$E^3_{0.75}$	$E^4_{0.75}$
0	1.2389e–3	6.6971e–4	5.3528e–4	6.4838e–5	1.4035e–3	6.2014e–4	6.7068e–4	1.4583e–5
0.1	1.1068e–3	6.0522e–4	4.7498e–4	6.2325e–5	1.2714e–3	5.5565e–4	6.1038e–4	1.7096e–5
0.2	9.7471e–4	5.4073e–4	4.1468e–4	5.9812e–5	1.1393e–3	4.9115e–4	5.5008e–4	1.9608e–5
0.3	8.4258e–4	4.7623e–4	3.5438e–4	5.7299e–5	1.0072e–3	4.2666e–4	4.8978e–4	2.2121e–5
0.4	7.1045e–4	4.1174e–4	2.9408e–4	5.4787e–5	8.7507e–4	3.6217e–4	4.2948e–4	2.4634e–5
0.5	5.7832e–4	3.4725e–4	2.3379e–4	5.2274e–5	7.4295e–4	2.9767e–4	3.6918e–4	2.7146e–5
0.6	4.4619e–4	2.8275e–4	1.7349e–4	4.9761e–5	6.1082e–4	2.3318e–4	3.0889e–4	2.9659e–5
0.7	3.1407e–4	2.1826e–4	1.1319e–4	4.7248e–5	4.7869e–4	1.6869e–4	2.4859e–4	3.2172e–5
0.8	1.8194e–4	1.5377e–4	5.2897e–5	4.4736e–5	3.4656e–4	1.0419e–4	1.8829e–4	3.4685e–5
0.9	4.9815e–5	8.9279e–5	7.4009e–6	4.2223e–5	2.1444e–4	3.9706e–5	1.2799e–4	3.7197e–5
1	8.2312e–5	2.4786e–5	6.7699e–5	3.9710e–5	8.2312e–5	2.4786e–5	6.7699e–5	3.9710e–5

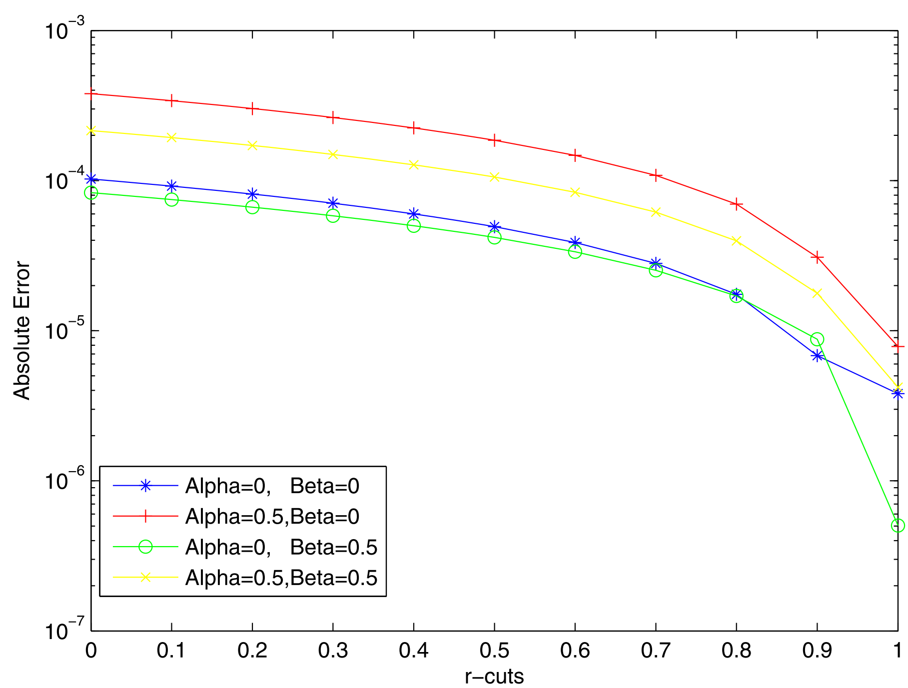

Figure 1 The absolute error for different values (α,β) of Example 1, $\nu = 0.85$, $m = 7$.

in which $y(x) : L^{\mathbb{E}}[0,1] \cap C^{\mathbb{E}}[0,1]$ is a continuous fuzzy function and ${}^cD^{\nu}_{0+}$ indicates the fuzzy Caputo fractional derivative of order ν.

Now, utilizing the definition of $[1-\nu]$-differentiability and Theorem 1, we have

$$\begin{cases} ({}^cD^{\nu}_{0+}y^r_-)(x) + y^r_-(x) = x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\nu)}x^{(3-\nu)} + \frac{24}{\Gamma(5-\nu)}x^{(4-\nu)}, \\ y^r_-(0) = -1 + r, \quad 0 < \nu \le 1, 0 \le x \le 1, \end{cases} \tag{34}$$

and

$$\begin{cases} ({}^cD^{\nu}_{0+}y^r_+)(x) + y^r_+(x) = x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\nu)}x^{(3-\nu)} + \frac{24}{\Gamma(5-\nu)}x^{(4-\nu)}, \\ y^r_+(0) = 1 - r, \quad 0 < \nu \le 1, 0 \le x \le 1. \end{cases} \tag{35}$$

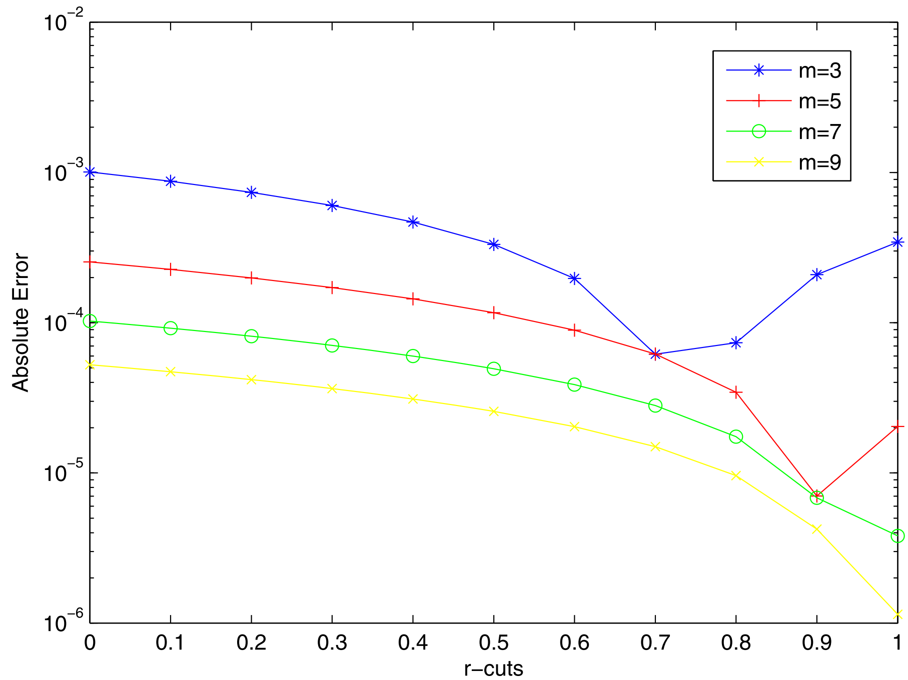

Figure 2 The absolute error for different values *m* of Example 1, $v = 0.85$, $\alpha = \beta = 0$.

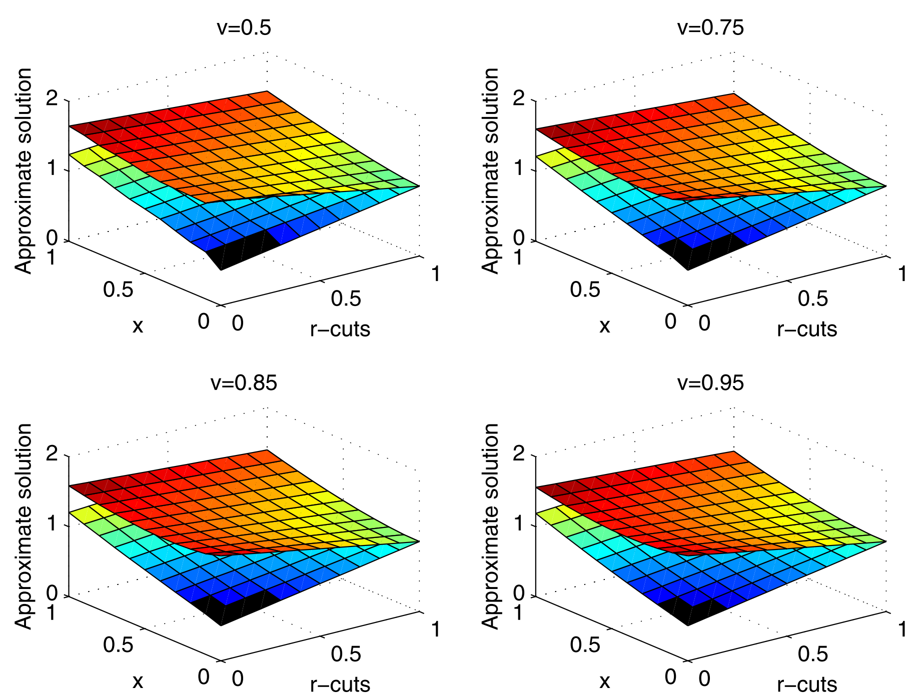

Figure 3 The fuzzy approximate solution of Example 1, for different fractional orders *v*, $\alpha = \beta = 0$, $m = 7$.

Solving Eqs. (34)-(35) causes to specify the solution of FFDE (33) as follows:

$$\begin{cases} y_-^r(x) = (-1+r)E_{\nu,1}[-x^\nu] + \int_0^x (x-t)^{\nu-1}E_{\nu,\nu}[-(x-t)^\nu] \\ \qquad \times (x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\nu)}x^{(3-\nu)} + \frac{24}{\Gamma(5-\nu)}x^{(4-\nu)})\,dt, \quad 0 \le r \le 1, \\ y_+^r(x) = (1-r)E_{\nu,1}[-x^\nu] + \int_0^x (x-t)^{\nu-1}E_{\nu,\nu}[-(x-t)^\nu] \\ \qquad \times (x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\nu)}x^{(3-\nu)} + \frac{24}{\Gamma(5-\nu)}x^{(4-\nu)})\,dt, \quad 0 \le r \le 1. \end{cases}$$

Now, if we apply the technique explained in Section 4 in Eqs. (34) and (35) with $m = 2$ and $\nu = 0.85$, then the 3 unknown fuzzy coefficients with the choices of $\alpha = 0.5$ and $\beta = 0$ are as

$$D^{(0.85)} = \begin{pmatrix} 0 & 0 & 0 \\ 2.2377 & 0.4433 & -0.2028 \\ -0.9457 & 4.6250 & 0.9395 \end{pmatrix},$$

so if we consider r-cut = 0 then the approximate fuzzy function is obtained as

$$\begin{cases} y_-^0(x) = -1 + 0.7681x + 0.9454x^2, \\ y_+^0(x) = 1 - 1.5531x + 1.3842x^2. \end{cases}$$

The absolute error for some various α, β at $x = 1$ are shown in Table 2 and Figure 4. From Table 2, it can be seen that with the few number of Jacobi polynomials, the approximate solution with high accuracy is achievable. Furthermore, Figure 5 shows that the error with increasing the number of Jacobi terms changes slowly. The approximate fuzzy solution has been derived for $\nu = 0.5, 0.75, 0.85, 0.95$ in Figure 6, which shows the feasibility of the proposed method for this kind of problems.

Remark 7 Figure 2 depicts that the approximate solution has a little bit oscillation when the number of Jacobi functions assumed $m = 3, 4$. Actually, it is related to the oscillatory behavior of fuzzy exact solution. Although the approximate solution using the proposed method has a smooth behavior, it can not respond appropriately to match the exact solution, especially when the r-cuts tend to 1. This has lead to the growing of the absolute error which is not significant. This defect is removed with the increasing of the number of Jacobi functions which is obvious according to Figures 1 and 2.

Table 2 The absolute error of the proposed method for Example 2 with different values of α, β and $m = 7$

(α,β) r	(0, 0) $E^1_{0.85}$	(0.5, 0) $E^2_{0.85}$	(0, 0.5) $E^3_{0.85}$	(0.5, 0.5) $E^4_{0.85}$	(0, 0) $E^1_{0.85}$	(0.5, 0) $E^2_{0.85}$	(0, 0.5) $E^3_{0.85}$	(0.5, 0.5) $E^4_{0.85}$
0	2.1261e–4	7.7483e–4	1.6513e–4	4.3894e–4	2.1261e–4	7.7483e–4	1.6513e–4	4.3894e–4
0.1	1.9135e–4	6.9735e–4	1.4862e–4	3.9505e–4	1.9135e–4	6.9735e–4	1.4862e–4	3.9505e–4
0.2	1.7009e–4	6.1986e–4	1.3210e–4	3.5115e–4	1.7009e–4	6.1986e–4	1.3210e–4	3.5115e–4
0.3	1.4883e–4	5.4238e–4	1.1559e–4	3.0726e–4	1.4883e–4	5.4238e–4	1.1559e–4	3.0726e–4
0.4	1.2757e–4	4.6490e–4	9.9081e–5	2.6336e–4	1.2757e–4	4.6490e–4	9.9081e–5	2.6336e–4
0.5	1.0630e–4	3.8741e–4	8.2568e–5	2.1947e–4	1.0630e–4	3.8741e–4	8.2568e–5	2.1947e–4
0.6	8.5046e–5	3.0993e–4	6.6054e–5	1.7557e–4	8.5046e–5	3.0993e–4	6.6054e–5	1.7557e–4
0.7	6.3785e–5	2.3245e–4	4.9540e–5	1.3168e–4	6.3785e–5	2.3245e–4	4.9540e–5	1.3168e–4
0.8	4.2523e–5	2.0864e–4	3.3027e–5	8.7789e–5	4.2523e–5	2.0864e–4	3.3027e–5	8.7789e–5
0.9	2.1261e–5	7.7483e–5	1.6513e–5	4.3894e–5	2.1261e–5	7.7483e–5	1.6513e–5	4.3894e–5
1	8.1051e–12	2.2339e–9	1.7246e–9	4.9312e–9	7.9941e–12	2.2338e–9	1.7245e–9	4.9312e–9

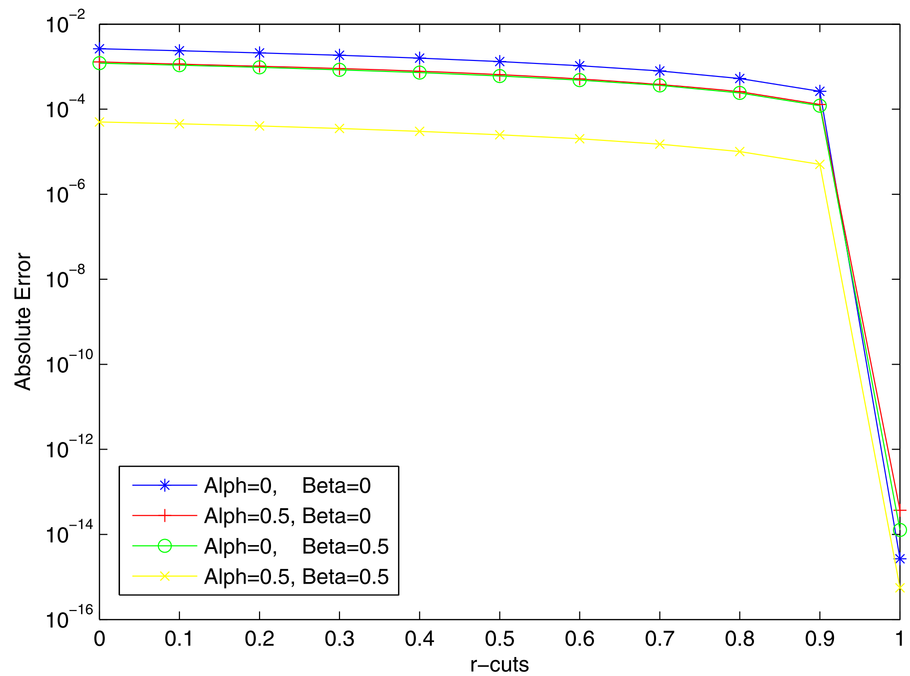

Figure 4 The absolute error for different values (α, β) of Example 2, $v = 0.75$, $m = 5$.

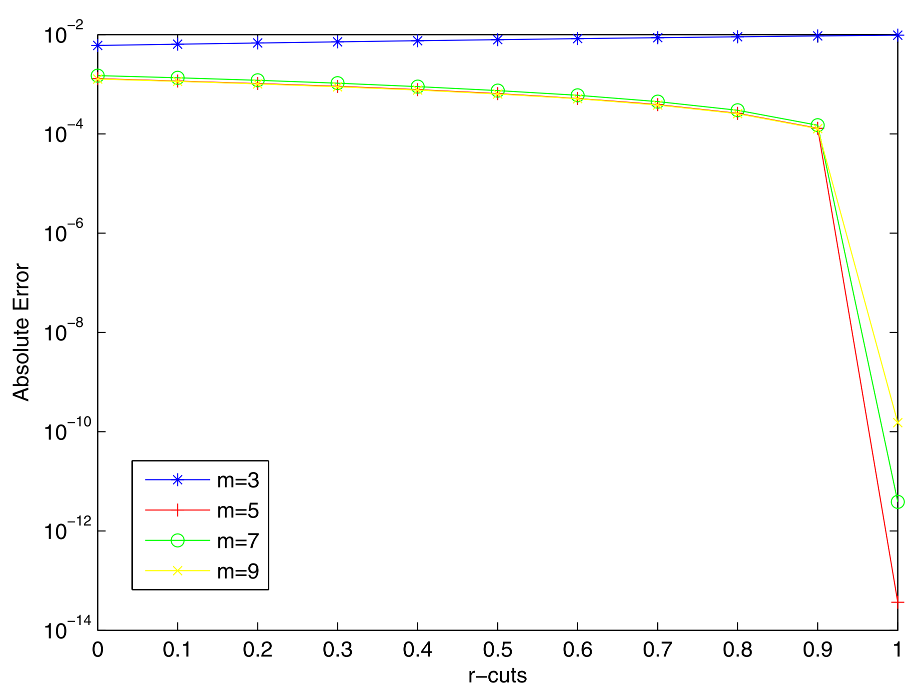

Figure 5 The absolute error for different values m of Example 2, $v = 0.75$, $\alpha = 0.5$, $\beta = 0$.

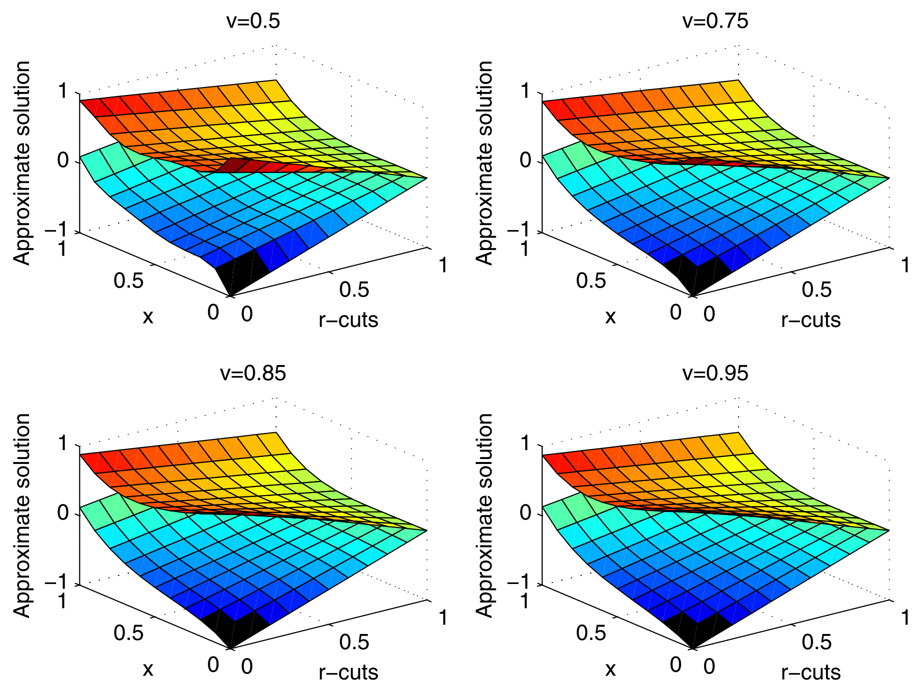

Figure 6 The fuzzy approximate solution of Example 2, for different fractional orders v, $\alpha = 0.5$, $\beta = 0$, $m = 5$.

On the other hand, taking into account Figures 4 and 5. The approximate of lower fuzzy function using JOM method for r-cut = 1 at $x = 1$ is as follow with $m = 5$:

$$\begin{aligned} y_-^1(x) &= (-2.3015\text{e}-16) + (-1.3147\text{e}-13)x + (1.3644\text{e}-12)x^2 \\ &\quad - 0.5x^3 + x^4 - (2.04119\text{e}-12)x^5, \end{aligned}$$

as it can be seen, the approximate solution for lower bound has the negotiable coefficients for x, x^2, x^5 terms and constant value. Therefore, in this condition, the approximate solution, with initial value $y(0) = 0$, approaches the analytical fuzzy solution rapidly with the increasing the number of Jacobi functions.

Example 3 Let us consider the fractional oscillation equation [59] with fuzzy initial conditions as

$$\begin{cases} {}^cD_{0+}^{\nu} y(x) + y(x) = xe^{-x}, \\ y^r(0) = [-1+r, 1-r], \quad 0 < \nu \le 1, 0 \le x \le 1, \end{cases} \tag{36}$$

where $y(x) : L^{\mathbb{E}}[0,1] \cap C^{\mathbb{E}}[0,1]$ is a continuous fuzzy set-value function and ${}^cD_{0^+}^{\nu}$ points out the fuzzy fractional derivative order of Caputo type.

Again, regarding to the case (i) of Definition 15 and Theorem 1, one can determine the parametric form of (36) as

$$\begin{cases} ({}^cD_{0+}^{\nu} y_-^r)(x) + y_-^r(x) = xe^{-x}, \\ y_-^r(0) = -1 + r, \quad 0 < \nu \le 1, 0 \le x \le 1, \end{cases} \tag{37}$$

and

$$\begin{cases} ({}^cD^{\nu}_{0+}y^r_+)(x) + y^r_+(x) = xe^{-x}, \\ y^r_+(0) = 1 - r, \quad 0 < \nu \le 1, 0 \le x \le 1, \end{cases} \tag{38}$$

with the exact solution as

$$\begin{cases} y^r_-(x) = (-1 + r)E_{\nu,1}[-x^{\nu}] + \int_0^x (x-t)^{\nu-1}E_{\nu,\nu}[-(x-t)^{\nu}]xe^{-x}\,dt, \quad 0 \le r \le 1, \\ y^r_+(x) = (1 - r)E_{\nu,1}[-x^{\nu}] + \int_0^x (x-t)^{\nu-1}E_{\nu,\nu}[-(x-t)^{\nu}]xe^{-x}\,dt, \quad 0 \le r \le 1. \end{cases}$$

With exploiting of the presented method in Section 5, we can obtain following fuzzy equations system:

$$\begin{cases} \sum_{j=0}^{m} \theta^r_{j,-}[\Delta_\nu(i,j) + I]P_j^{(\alpha,\beta)}(x) = \sum_{j=0}^{m} f^r_{j,-}P_j^{(\alpha,\beta)}(x), \\ \sum_{j=0}^{m} \theta^r_{j,+}[\Delta_\nu(i,j) + I]P_j^{(\alpha,\beta)}(x) = \sum_{j=0}^{m} f^r_{j,+}P_j^{(\alpha,\beta)}(x), \end{cases}$$

then multiplying this system by $P_i^{(\alpha,\beta)}(x)$ for $i = 0, 1, \ldots, m-1$ using the fuzzy inner product and orthogonal property explained in Section 5, give us (m)-fuzzy linear algebraic equations:

$$\begin{aligned} &\sum_{j=0}^{m-1}{}^{*}\theta_j^{(r)} \odot \biggl\{ (FR)\int_0^1 D^{(\nu)}P_j^{(\alpha,\beta)}(x)P_i^{(\alpha,\beta)}(x) \odot (1-x)^{\alpha}x^{\beta}\,dt \\ &\qquad + (FR)\int_0^1 P_j^{(\alpha,\beta)}(x)P_i^{(\alpha,\beta)}(x) \odot (1-x)^{\alpha}x^{\beta}\,dt \biggr\} \\ &\quad = \sum_{j=0}^{m-1}{}^{*} f_j^{(r)} \odot (FR)\int_0^1 P_j^{(\alpha,\beta)}(x)P_i^{(\alpha,\beta)}(x) \odot (1-x)^{\alpha}x^{\beta}\,dt, \end{aligned} \tag{39}$$

in which $f_i = \frac{1}{\nu_i^{\alpha,\beta}} \int_0^1 P_i^{(\alpha,\beta)}(x) \odot xe^{-x} \odot w^{(\alpha+\beta)}(x)\,dt$.

Afterward, substituting Eq. (10) in the initial condition of Eq. (36) yields

$$y^r(0) = [-1 + r, 1 - r] = \sum_{j=0}^{m}{}^{*}\theta_j^{(r)} \odot P_j^{(\alpha,\beta)}(0). \tag{40}$$

Ultimately, from Eqs. (39) and (40), $(m+1)$-fuzzy linear equations are produced which lead to discover the unknown fuzzy coefficients of the fuzzy approximate solution of problem (36), instantaneously.

Taking $m = 2$, $\nu = 0.95$, $\alpha = 0$, $\beta = 0.5$ and applying the method, we can get

$$D^{(0.95)} = \begin{pmatrix} 0 & 0 & 0 \\ 2.4852 & 0.1137 & -0.0478 \\ 0.3655 & 5.8573 & 0.2754 \end{pmatrix},$$

Table 3 The absolute error of the proposed method for Example 3 with different values of α, β and $m = 9$

(α,β) r	(0, 0) $E^1_{0.95}$	(0.5, 0) $E^2_{0.95}$	(0, 0.5) $E^3_{0.95}$	(0.5, 0.5) $E^4_{0.95}$	(0, 0) $E^1_{0.95}$	(0.5, 0) $E^2_{0.95}$	(0, 0.5) $E^3_{0.95}$	(0.5, 0.5) $E^4_{0.95}$
0	5.4570e–6	1.1057e–4	7.6066e–5	1.0236e–4	5.5968e–6	1.1264e–4	7.7085e–5	1.0411e–4
0.1	4.9043e–6	9.9417e–5	6.8409e–5	9.2044e–5	5.0441e–6	1.0148e–4	6.9428e–5	9.3793e–5
0.2	4.3516e–6	8.8256e–5	6.0751e–5	8.1720e–5	4.4914e–6	9.0325e–5	6.1770e–5	8.3469e–5
0.3	3.7989e–6	7.7094e–5	5.3093e–5	7.1396e–5	3.9387e–6	7.9164e–5	5.4112e–5	7.3144e–5
0.4	3.2462e–6	6.5933e–5	4.5436e–5	6.1071e–5	3.3860e–6	6.8002e–5	4.6455e–5	6.2820e–5
0.5	2.6935e–6	5.4772e–5	3.7778e–5	5.0747e–5	2.8333e–6	5.6841e–5	3.8797e–5	5.2496e–5
0.6	2.1408e–6	4.3610e–5	3.0120e–5	4.0423e–5	2.2806e–6	4.5679e–5	3.1140e–5	4.2171e–5
0.7	1.5882e–6	3.2449e–5	2.2463e–5	3.0098e–5	1.7279e–6	3.4518e–5	2.3482e–5	3.1847e–5
0.8	1.0355e–6	2.1288e–5	1.4805e–5	1.9774e–5	1.1752e–6	2.3357e–5	1.5824e–5	2.1523e–5
0.9	4.8280e–7	1.0126e–5	7.1480e–6	9.4499e–6	6.2258e–7	1.2195e–5	8.1671e–5	1.1198e–5
1	6.9888e–8	1.0345e–6	5.0957e–7	8.7434e–7	6.9888e–8	1.0345e–6	5.0957e–7	8.7434e–7

if we take into account problem (36) with r-cut = 0.5, then the shifted Jacobi polynomials are as follows:

$$P_i^{(\alpha,\beta)}(x) = \begin{cases} 1, \\ -3/2 + 5/2x, \\ 15/8 - 35/4x + 63/8x^2, \end{cases}$$

eventually, putting $D^{(0.95)}$ and $P_i^{(\alpha,\beta)}(x)$ in Eqs. (39) and (40), one can get the fuzzy unknown coefficients as

$$\Theta_{3,-} = [-0.1783, 0.1947, -0.0158],$$

$$\Theta_{3,+} = [0.3806, -0.0385, 0.0329].$$

A comparison between the absolute errors of Example 3 for different values of α, β is explained at $x = 1$ in Table 3. From Table 3, it is obvious that the proposed method is achieved better accuracy with $\alpha = \beta = 0$, which actually with this assumption the method is in agreement with the Legendre tau method proposed in [20]. These results are confirmed again from Figure 7. Although the problem is a fuzzy fractional oscillation equation, this method is successful to attain suitable approximation that its preciseness is risen progressively with the increasing of the number of Jacobi functions in Figure 8. Additionally, the approximate fuzzy solution is described in Figure 9 for different fractional order ν. Ultimately, the CPU time is estimated in Table 4 using MATLAB verion 7.6 (R2008a). From Table 4, one can conclude that the time consumption and number of Jacobi polynomials are increased simultaneously.

6 Conclusion

This article adopted the operational Jacobi operational matrix based on the fuzzy Caputo fractional derivative using shifted Jacobi polynomials. The clear advantage of the usage of this method is that the matrix operators have the main role to find the approximate fuzzy solution of FFDEs instead of considering the methods required the complicated fractional derivatives and their calculations, which consume more time and cost in comparison with this method.

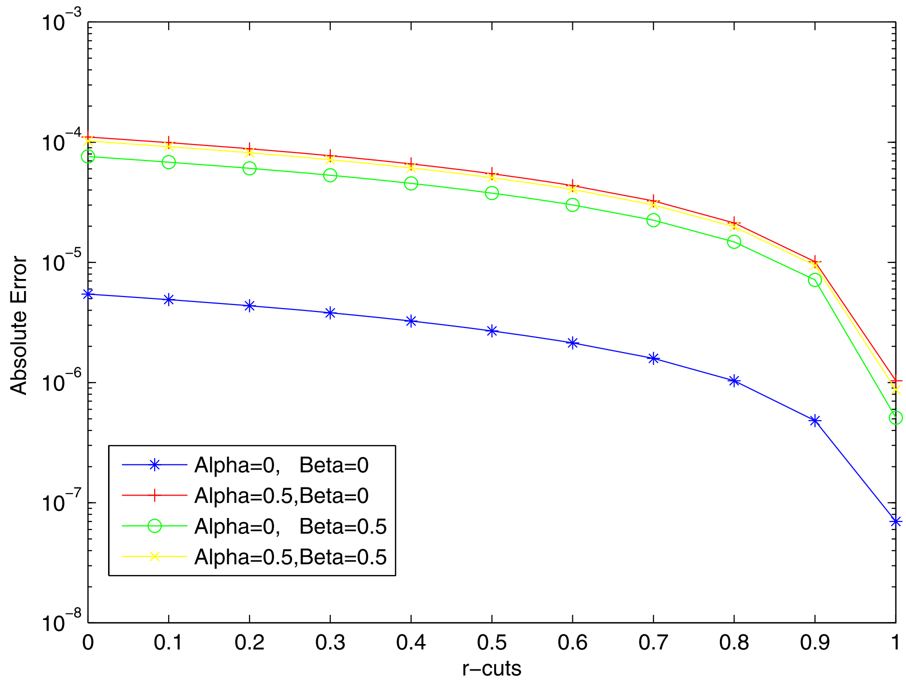

Figure 7 The absolute error for different values (α, β) of Example 3, $v = 0.95$, $m = 9$.

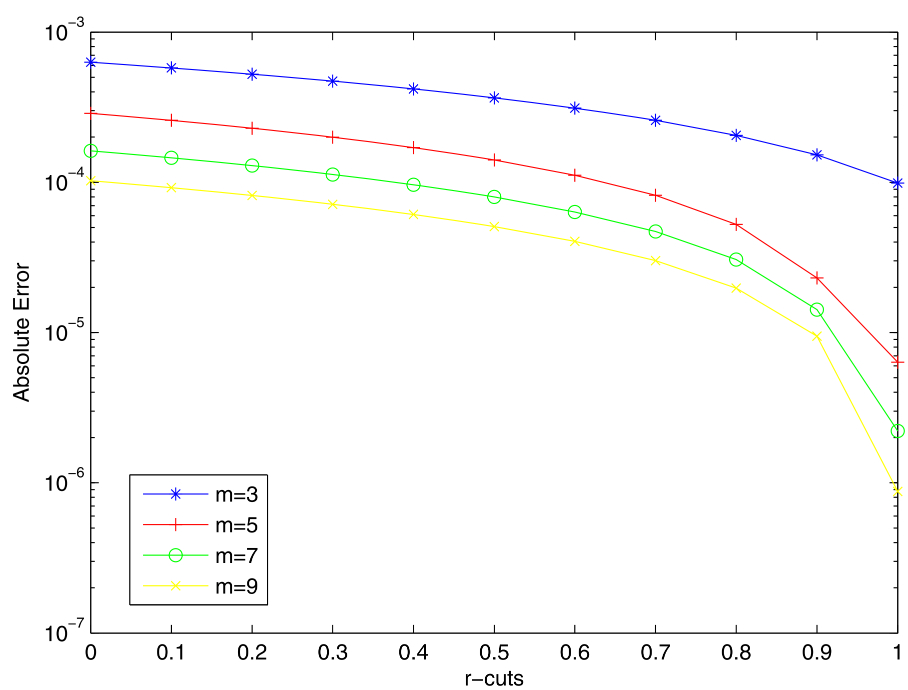

Figure 8 The absolute error for different values m of Example 3, $v = 0.95$, $\alpha = 0.5$, $\beta = 0.5$.

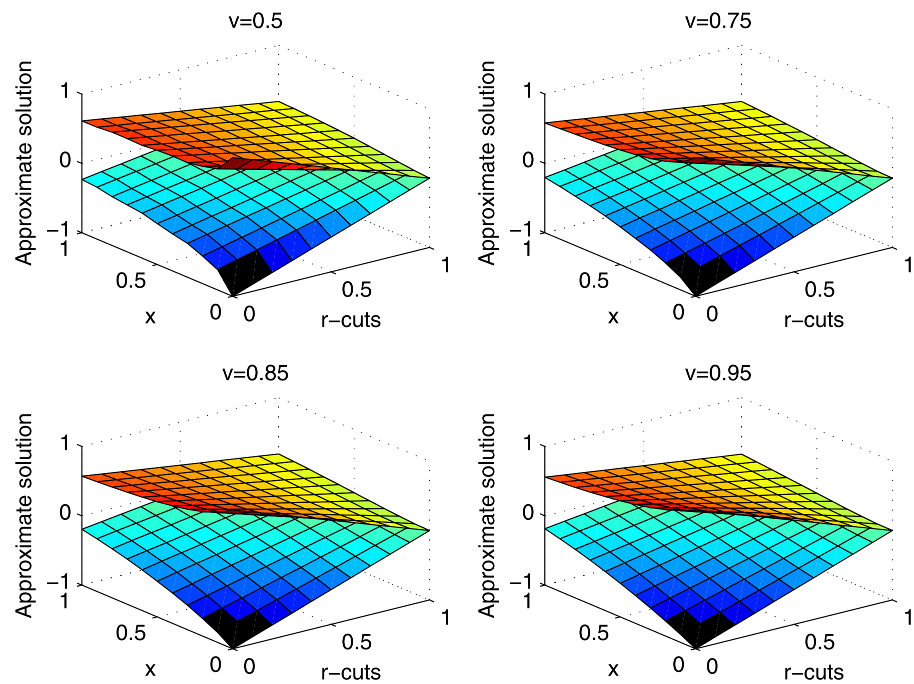

Figure 9 The fuzzy approximate solution of Example 3, for different fractional orders v, $\alpha = 0.5$, $\beta = 0.5$, $m = 9$.

Table 4 CPU time (in seconds) on MATLAB-7.6 (R2008a) for all of the examples, with different m and $v = 0.75$

m	Example 1	Example 2	Example 3
3	1.9843	2.3125	2.5781
5	3.0625	3.5312	3.8437
7	4.9062	5.6718	6.2656
9	7.4218	8.2656	9.1718

Theorems 5 and 6 were proved to demonstrate the error bound of the fuzzy approximate solution and fuzzy fractional Caputo derivative of order $0 < \nu < 1$. Also, various kinds of problems are solved to illustrate the effectiveness and strength of the method, which can be reached to the suitable accuracy with a lower number of Jacobi functions. In addition, the problems were tested for different values of α and β to show that the method is adaptable in dealing with the various issues.

For future researches, we will try to extend this method for solving multilinear and nonlinear problems as well as solving FFDEs of the order $1 < \nu < 2$. Furthermore, the generalization of the other orthogonal polynomials for solving FFDEs is the another scope of our attempts. Finally, we will attempt to expand the proposed method under other kinds of fuzzy derivatives like Riemann-Liouville differentiability.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

AA wrote the first draft, MS and SS corrected and improved it and DB prepared the final version.

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