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Dynamics of a fourth-order system of rational difference equations

Q Din*, MN Qureshi and A Qadeer Khan

*Correspondence:
qamar.sms@gmail.com
University of Azad Jammu and Kasmir, Muzaffarabad, Pakistan

Abstract

In this paper, we study the equilibrium points, local asymptotic stability of an equilibrium point, instability of equilibrium points, periodicity behavior of positive solutions, and global character of an equilibrium point of a fourth-order system of rational difference equations of the form

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \qquad y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}},$$

$n = 0, 1, \ldots$, where the parameters α, β, γ, α_1, β_1, γ_1 and initial conditions x_0, x_{-1}, x_{-2}, x_{-3}, y_0, y_{-1}, y_{-2}, y_{-3} are positive real numbers. Some numerical examples are given to verify our theoretical results.

MSC: 39A10; 40A05

Keywords: system of rational difference equations; stability; global character

1 Introduction and preliminaries

The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of difference equations also experienced enormous growth in many areas. Many applications of discrete dynamical systems and difference equations have appeared recently in the areas of biology, economics, physics, resource management, and others. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics. It is very interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the local asymptotic stability of their equilibrium points.

Cinar [1] investigated the periodicity of the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{1}{y_n}, \qquad y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}.$$

Stević [2] studied the system of two nonlinear difference equations

$$x_{n+1} = \frac{u_n}{1+v_n}, \qquad y_{n+1} = \frac{w_n}{1+s_n},$$

where u_n, v_n, w_n, s_n are some sequences x_n or y_n.

Kurbanli [3] studied the behavior of positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{1}{y_n z_n}.$$

Bajo and Liz [4] investigated the global behavior of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a + b x_{n-1} x_n}$$

for all values of real parameters a, b.

Kalabušić, Kulenović, and Pilav [5] investigated the global dynamics of the following systems of difference equations:

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{A_1 + y_n}, \qquad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n}.$$

Kurbanli, Çinar, and Yalçinkaya [6] studied the behavior of positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}.$$

Touafek and Elsayed [7] studied the periodic nature and got the form of the solutions of the following systems of rational difference equations:

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-1}}, \qquad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} x_{n-1}}.$$

Similarly, Touafek, and Elsayed [8] studied the periodicity nature of the following systems of rational difference equations:

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)}, \qquad y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}.$$

Recently, Zhang, Yang, and Liu [9] studied the dynamics of a system of the rational third-order difference equation

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \qquad y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots.$$

Our aim in this paper is to investigate the dynamics of a system of fourth-order rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \qquad y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}, \tag{1.1}$$

$n = 0, 1, \ldots$, where the parameters α, β, γ, α_1, β_1, γ_1 and initial conditions x_0, x_{-1}, x_{-2}, x_{-3}, y_0, y_{-1}, y_{-2}, y_{-3} are positive real numbers. This paper is a natural extension of [9, 10].

Let us consider an eight-dimensional discrete dynamical system of the form

$$\begin{aligned} x_{n+1} &= f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}), \\ y_{n+1} &= g(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}), \end{aligned} \tag{1.2}$$

$n = 0, 1, \ldots$, where $f : I^4 \times J^4 \to I$ and $g : I^4 \times J^4 \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=-3}^{\infty}$ of the system (1.2) is uniquely determined by initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-3, -2, -1, 0\}$. Along with the system (1.2), we consider the corresponding vector map $F = (f, x_n, x_{n-1}, x_{n-2}, x_{n-3}, g, y_n, y_{n-1}, y_{n-2}, y_{n-3})$. An equilibrium point of (1.2) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}, \bar{y}), \\ \bar{y} &= g(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}, \bar{y}). \end{aligned}$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map F.

Definition 1.1 Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (1.2).

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every initial condition (x_i, y_i), $i \in \{-3, -2, -1, 0\}$ if $\| \sum_{i=-3}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where $\| \cdot \|$ is the usual Euclidean norm in $\mathbb{R}^2$.

(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.

(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\| \sum_{i=-3}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called a global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(v) An equilibrium point $(\bar{x}, \bar{y})$ is called an asymptotic global attractor if it is a global attractor and stable.

Definition 1.2 Let $(\bar{x}, \bar{y})$ be an equilibrium point of the map

$$F = (f, x_n, x_{n-1}, x_{n-2}, x_{n-3}, g, y_n, y_{n-1}, y_{n-2}, y_{n-3}),$$

where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (1.2) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = F_J X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{pmatrix}$$

and F_J is a Jacobian matrix of the system (1.2) about the equilibrium point $(\bar{x}, \bar{y})$.

To construct the corresponding linearized form of the system (1.1), we consider the following transformation:

$$(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}) \mapsto (f, f_1, f_2, f_3, f_4, g, g_1, g_2, g_3, g_4), \tag{1.3}$$

where $f = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}$, $g = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}$, $f_1 = x_n$, $f_2 = x_{n-1}$, $f_3 = x_{n-2}$, $f_4 = x_{n-3}$, $g_1 = y_n$, $g_2 = y_{n-1}$, $g_3 = y_{n-2}$ and $g_4 = y_{n-3}$. The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (1.3) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & 0 & 0 & A & B & B & B & B \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ C & C & C & C & 0 & 0 & 0 & D \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where $A = \frac{\alpha}{\beta + \gamma \bar{y}^4}$, $B = -\frac{\alpha \gamma \bar{x} \bar{y}^3}{(\beta + \gamma \bar{y}^4)^2}$, $C = -\frac{\alpha_1 \gamma_1 \bar{y} \bar{x}^3}{(\beta_1 + \gamma_1 \bar{x}^4)^2}$ and $D = \frac{\alpha_1}{\beta_1 + \gamma_1 \bar{x}^4}$.

Theorem 1.3 *For the system $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$, of difference equations such that $\bar{X}$ is a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\bar{X}$ is unstable.*

Theorem 1.4 (Routh-Hurwitz criterion)

For real numbers $a_1, a_2, \ldots, a_n$, let

$$P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n. \tag{1.4}$$

Consider the polynomial equation

$$P(\lambda) = 0. \tag{1.5}$$

We define the n matrices as follows:

$$H_1 = (a_1), \qquad H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}, \qquad \ldots,$$

$$H_j = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ a_5 & a_4 & a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2j-1} & a_{2j-2} & a_{2j-3} & \cdots & a_j \end{pmatrix},$$

where (l, m) *element in the matrix* H_j, *for* $0 < 2l - m < k$ *is*

$$a_{2l-m} = \begin{cases} 1 & \textit{for } 2l = m, \\ 0 & \textit{for } 2l < m \textit{ or } 2l > m + k, \end{cases}$$

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ a_5 & a_4 & a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}.$$

The following statements are true:

(i) *A necessary and sufficient condition for all of the roots of* (1.5) *to have a negative real part is* $\det(H_j) > 0$ *for* $j = 1, 2, \ldots, n$.

(ii) *A necessary and sufficient condition for the existence of a root of* (1.5) *with a positive real part is* $\det(H_j) < 0$ *for some* $j \in \{1, 2, \ldots, n\}$.

2 Main results

Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (1.1), then for $\alpha > \beta$ and $\alpha_1 > \beta_1$, the system (1.1) has the following five equilibrium points:

$$P_0 = (0, 0), \qquad P_1 = (A, B), \qquad P_2 = (-A, B), \qquad P_3 = (A, -B), \qquad P_4 = (-A, -B),$$

where $A = (\frac{\alpha_1 - \beta_1}{\gamma_1})^{\frac{1}{4}}$ and $B = (\frac{\alpha - \beta}{\gamma})^{\frac{1}{4}}$.

Theorem 2.1 *Let* (x_n, y_n) *be a positive solution of the system* (1.1), *then for every* $m \geq 0$, *the following results hold*:

$$0 \leq x_n \leq \left(\frac{\alpha}{\beta}\right)^{m+1} x_{-3}, \quad \textit{if } n = 4m + 1,$$

$$0 \leq x_n \leq \left(\frac{\alpha}{\beta}\right)^{m+1} x_{-2}, \quad \textit{if } n = 4m + 2,$$

$$0 \leq x_n \leq \left(\frac{\alpha}{\beta}\right)^{m+1} x_{-1}, \quad \textit{if } n = 4m + 3,$$

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{m+1} x_0, \quad \text{if } n = 4m+4,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_{-3}, \quad \text{if } n = 4m+1,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_{-2}, \quad \text{if } n = 4m+2,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_{-1}, \quad \text{if } n = 4m+3,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_0, \quad \text{if } n = 4m+4.$$

Proof The results are obviously true for $m = 0$. Suppose that results are true for $m = k \ge 1$, *i.e.*,

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{k+1} x_{-3}, \quad \text{if } n = 4k+1,$$

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{k+1} x_{-2}, \quad \text{if } n = 4k+2,$$

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{k+1} x_{-1}, \quad \text{if } n = 4k+3,$$

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{k+1} x_0, \quad \text{if } n = 4k+4,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{k+1} y_{-3}, \quad \text{if } n = 4k+1,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{k+1} y_{-2}, \quad \text{if } n = 4k+2,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{k+1} y_{-1}, \quad \text{if } n = 4k+3,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{k+1} y_0, \quad \text{if } n = 4k+4.$$

Now, for $m = k+1$ using (1.1), one has

$$0 \le x_{4k+5} = \frac{\alpha x_{4k+1}}{\beta + \gamma y_{4k+4} y_{4k+3} y_{4k+2} y_{4k+1}} \le \frac{\alpha x_{4k+1}}{\beta} \le \left(\frac{\alpha}{\beta}\right)^{k+2} x_{-3},$$

$$0 \le x_{4k+6} = \frac{\alpha x_{4k+2}}{\beta + \gamma y_{4k+5} y_{4k+4} y_{4k+3} y_{4k+2}} \le \frac{\alpha x_{4k+2}}{\beta} \le \left(\frac{\alpha}{\beta}\right)^{k+2} x_{-2},$$

$$0 \le x_{4k+7} = \frac{\alpha x_{4k+3}}{\beta + \gamma y_{4k+6} y_{4k+5} y_{4k+4} y_{4k+3}}$$

$$\le \frac{\alpha x_{4k+3}}{\beta} \le \left(\frac{\alpha}{\beta}\right)^{k+2} x_{-1},$$

$$0 \le x_{4k+8} = \frac{\alpha x_{4k+4}}{\beta + \gamma y_{4k+7} y_{4k+6} y_{4k+5} y_{4k+4}}$$

$$\le \frac{\alpha x_{4k+4}}{\beta} \le \left(\frac{\alpha}{\beta}\right)^{k+2} x_{0},$$

$$0 \le y_{4k+5} = \frac{\alpha_1 y_{4k+1}}{\beta_1 + \gamma_1 x_{4k+4} x_{4k+3} x_{4k+2} x_{4k+1}}$$

$$\le \frac{\alpha_1 y_{4k+1}}{\beta_1} \le \left(\frac{\alpha_1}{\beta_1}\right)^{k+2} y_{-3},$$

$$0 \le y_{4k+6} = \frac{\alpha_1 y_{4k+2}}{\beta_1 + \gamma_1 x_{4k+5} x_{4k+4} x_{4k+3} x_{4k+2}}$$

$$\le \frac{\alpha_1 y_{4k+2}}{\beta_1} \le \left(\frac{\alpha_1}{\beta_1}\right)^{k+2} y_{-2},$$

$$0 \le y_{4k+7} = \frac{\alpha_1 y_{4k+3}}{\beta_1 + \gamma_1 x_{4k+6} x_{4k+5} x_{4k+4} x_{4k+3}}$$

$$\le \frac{\alpha_1 y_{4k+3}}{\beta_1} \le \left(\frac{\alpha_1}{\beta_1}\right)^{k+2} y_{-1},$$

$$0 \le y_{4k+8} = \frac{\alpha_1 y_{4k+4}}{\beta_1 + \gamma_1 x_{4k+7} x_{4k+6} x_{4k+5} x_{4k+4}}$$

$$\le \frac{\alpha_1 y_{4k+4}}{\beta_1} \le \left(\frac{\alpha_1}{\beta_1}\right)^{k+2} y_{0}.$$

□

Theorem 2.2 *For the equilibrium point* $P_0 = (0,0)$ *of Equation* (1.1), *the following results hold*:

(i) *Let* $\alpha < \beta$ *and* $\alpha_1 < \beta_1$, *then the equilibrium point* $P_0 = (0,0)$ *of the system* (1.1) *is locally asymptotically stable.*

(ii) *If* $\alpha > \beta$ *or* $\alpha_1 > \beta_1$, *then the equilibrium point* $P_0 = (0,0)$ *of the system* (1.1) *is unstable.*

Proof (i) The linearized system of (1.1) about the equilibrium point $(0,0)$ is given by

$$X_{n+1} = F_J(0,0)X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{pmatrix}$$

and

$$F_J(0,0) = \begin{pmatrix} 0 & 0 & 0 & \frac{\alpha}{\beta} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha_1}{\beta_1} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $F_J(0,0)$ is given by

$$P(\lambda) = \lambda^8 - \left(\frac{\alpha}{\beta} + \frac{\alpha_1}{\beta_1}\right)\lambda^4 - \frac{\alpha\alpha_1}{\beta\beta_1}. \tag{2.1}$$

The roots of $P(\lambda)$ are $\lambda = \pm\frac{\alpha}{\beta}$, $\lambda = \pm\frac{\alpha_1}{\beta_1}$, $\lambda = \pm\iota\frac{\alpha}{\beta}$, $\lambda = \pm\iota\frac{\alpha_1}{\beta_1}$. Since all eigenvalues of the Jacobian matrix $F_J(0,0)$ about $(0,0)$ lie in an open unit dick $|\lambda| < 1$, the equilibrium point $(0,0)$ is locally asymptotically stable.

(ii) It is easy to see that if $\alpha > \beta$ or $\alpha_1 > \beta_1$, then there exists at least one root λ of Equation (2.1) such that $|\lambda| > 1$. Hence, by Theorem 1.3 if $\alpha > \beta$ or $\alpha_1 > \beta_1$, then $(0,0)$ is unstable. □

Theorem 2.3 *If $\alpha > \beta$ and $\alpha_1 > \beta_1$, then a positive equilibrium point $P_1 = ((\frac{\alpha_1-\beta_1}{\gamma_1})^{\frac{1}{4}}, (\frac{\alpha-\beta}{\gamma})^{\frac{1}{4}})$ of Equation* (1.1) *is unstable.*

Proof The linearized system of (1.1) about the equilibrium point P_1 is given by

$$X_{n+1} = F_J(P_1)X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{pmatrix}$$

and

$$F_J(P_1) = \begin{pmatrix} 0 & 0 & 0 & 1 & L & L & L & L \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ M & M & M & M & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where

$$L = -\left(\frac{\gamma_1}{\gamma}\right)^{\frac{1}{4}} \frac{(\alpha_1 - \beta_1)^{\frac{1}{4}(\alpha-\beta)^{\frac{3}{4}}}}{\alpha}$$

and

$$M = -\left(\frac{\gamma}{\gamma_1}\right)^{\frac{1}{4}} \frac{(\alpha - \beta)^{\frac{1}{4}(\alpha_1-\beta_1)^{\frac{3}{4}}}}{\alpha_1}.$$

The characteristic polynomial of $F_J(P_1)$ is given by

$$P(\lambda) = \lambda^8 - LM\lambda^6 - 2LM\lambda^5 - 3LM\lambda^4 - 4LM\lambda^3 - 3LM\lambda^2 - 2LM\lambda - LM + 1. \tag{2.2}$$

The roots of the characteristic polynomial $P(\lambda)$ given in Equation (2.2) are given by

$$-1, \qquad \pm\iota, \qquad 1 \pm \sqrt{L}\sqrt{M}.$$

It is sufficient to prove that any one of these roots has absolute value greater than one. For this, consider

$$\begin{aligned} |1 - \sqrt{L}\sqrt{M}| &= \left|1 + \sqrt{\left(\frac{\alpha-\beta}{\alpha}\right)\left(\frac{\alpha_1-\beta_1}{\alpha_1}\right)}\right| \\ &= \left|1 + \sqrt{\left(1 - \frac{\beta}{\alpha}\right)\left(1 - \frac{\beta_1}{\alpha_1}\right)}\right| > 1. \end{aligned}$$

Hence, by Theorem 1.3 if $\alpha > \beta$ and $\alpha_1 > \beta_1$, then P_1 is unstable. □

Theorem 2.4 *If $\alpha > \beta$ and $\alpha_1 > \beta_1$, then the equilibrium points P_2, P_3, P_4 of Equation* (1.1) *are unstable.*

Proof The proof is similar to Theorem 2.3, so it is omitted. □

The following theorem is similar to Theorem 3.4 of [9].

Theorem 2.5 *Let $\alpha > \beta$ and $\alpha_1 > \beta_1$, and let (x_n, y_n) be a solution of the system* (1.1). *Then, for $k = -3, -2, -1, 0$, the following statements are true*:

(i) *If* $(x_k, y_k) \in (0, (\frac{\alpha_1-\beta_1}{\gamma_1})^{\frac{1}{4}}) \times ((\frac{\alpha-\beta}{\gamma})^{\frac{1}{4}}, \infty)$, *then* $(x_n, y_n) \in (0, (\frac{\alpha_1-\beta_1}{\gamma_1})^{\frac{1}{4}}) \times ((\frac{\alpha-\beta}{\gamma})^{\frac{1}{4}}, \infty)$.
(ii) *If* $(x_k, y_k) \in ((\frac{\alpha_1-\beta_1}{\gamma_1})^{\frac{1}{4}}, \infty) \times (0, (\frac{\alpha-\beta}{\gamma})^{\frac{1}{4}})$, *then* $(x_n, y_n) \in ((\frac{\alpha_1-\beta_1}{\gamma_1})^{\frac{1}{4}}, \infty) \times (0, (\frac{\alpha-\beta}{\gamma})^{\frac{1}{4}})$.

Theorem 2.6 *The system* (1.1) *has no prime period-two solutions.*

Proof Assume that $(p_1, q_1), (p_2, q_2), (p_1, q_1), \ldots$ is a prime period-two solution of Equation (1.1) such that $p_i, q_i \neq 0$ and $p_i \neq q_i$ for $i = 1, 2$. Then, from the system (1.1), one has

$$p_1 = \frac{\alpha p_1}{\beta + \gamma (q_1 q_2)^2}, \qquad p_2 = \frac{\alpha p_2}{\beta + \gamma (q_1 q_2)^2}, \tag{2.3}$$

and

$$q_1 = \frac{\alpha_1 q_1}{\beta_1 + \gamma_1 (p_1 p_2)^2}, \qquad q_2 = \frac{\alpha_1 q_2}{\beta_1 + \gamma_1 (p_1 p_2)^2}. \tag{2.4}$$

From (2.3) and (2.4), one has $p_i, q_i = 0$ for $i = 1, 2$. Which is a contradiction. Hence, the system (1.1) has no prime period-two solutions. □

Theorem 2.7 *Let* $\alpha < \beta$ *and* $\alpha_1 < \beta_1$, *then the equilibrium point* $P_0 = (0, 0)$ *of Equation* (1.1) *is globally asymptotically stable.*

Proof For $\alpha < \beta$ and $\alpha_1 < \beta_1$, from Theorem 2.2, $(0, 0)$ is locally asymptotically stable. From Theorem 2.1, it is easy to see that every positive solution (x_n, y_n) is bounded, *i.e.*, $0 \leq x_n \leq \mu$ and $0 \leq y_n \leq \nu$ for all $n = 0, 1, 2, \ldots$, where $\mu = \max\{x_{-3}, x_{-2}, x_{-1}, x_0\}$ and $\nu = \max\{y_{-3}, y_{-2}, y_{-1}, y_0\}$. Now, it is sufficient to prove that (x_n, y_n) is decreasing. From the system (1.1), one has

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}} \\ &\leq \frac{\alpha x_{n-3}}{\beta} < x_{n-3}. \end{aligned}$$

This implies that $x_{4n+1} < x_{4n-3}$ and $x_{4n+5} < x_{4n+1}$. Hence, the subsequences $\{x_{4n+1}\}$, $\{x_{4n+2}\}$, $\{x_{4n+3}\}$, $\{x_{4n+4}\}$ are decreasing, *i.e.*, the sequence $\{x_n\}$ is decreasing. Similarly, one has

$$\begin{aligned} y_{n+1} &= \frac{\alpha_1 x_{n-3}}{\beta_1 + \gamma_1 y_n y_{n-1} y_{n-2} y_{n-3}} \\ &\leq \frac{\alpha_1 y_{n-3}}{\beta_1} < y_{n-3}. \end{aligned}$$

This implies that $y_{4n+1} < y_{4n-3}$ and $y_{4n+5} < y_{4n+1}$. Hence, the subsequences $\{y_{4n+1}\}$, $\{y_{4n+2}\}$, $\{y_{4n+3}\}$, $\{y_{4n+4}\}$ are decreasing, *i.e.*, the sequence $\{y_n\}$ is decreasing. Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. □

Theorem 2.8 *Let* $\alpha > \beta$ *and* $\alpha_1 > \beta_1$. *Then, for a solution* (x_n, y_n) *of the system* (1.1), *the following statements are true*:
(i) *If* $x_n \to 0$, *then* $y_n \to \infty$.
(ii) *If* $y_n \to 0$, *then* $x_n \to \infty$.

3 Examples

In order to verify our theoretical results and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the system of nonlinear difference equations (1.1). All plots in this section are drawn with mathematica.

Example Consider the system (1.1) with initial conditions $x_{-3} = 1.1$, $x_{-2} = 2.6$, $x_{-1} = 1.6$, $x_0 = 1.7$, $y_{-3} = 1.5$, $y_{-2} = 1.3$, $y_{-1} = 2.5$, $y_0 = 0.5$. Moreover, choosing the parameters $\alpha = 0.01$, $\beta = 0.011$, $\gamma = 50$, $\alpha_1 = 0.03$, $\beta_1 = 0.031$, $\gamma_1 = 70$, the system (1.1) can be written as follows:

$$x_{n+1} = \frac{0.01x_{n-3}}{0.011 + 50y_n y_{n-1} y_{n-2} y_{n-3}}, \qquad y_{n+1} = \frac{0.03y_{n-3}}{0.031 + 70x_n x_{n-1} x_{n-2} x_{n-3}}, \tag{3.1}$$

$n = 0, 1, \dots$, and with initial conditions $x_{-3} = 1.1$, $x_{-2} = 2.6$, $x_{-1} = 1.6$, $x_0 = 1.7$, $y_{-3} = 1.5$, $y_{-2} = 1.3$, $y_{-1} = 2.5$, $y_0 = 0.5$. The plot of the system (3.1) is shown in Figure 1 and its global attractor is shown in Figure 2.

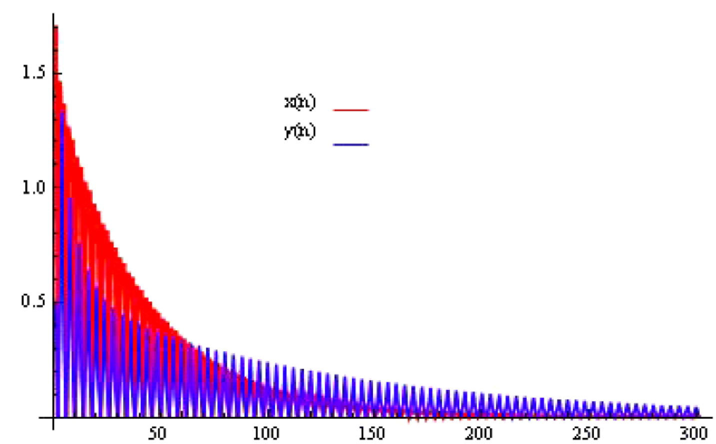

Figure 1 Plot of the system (3.1).

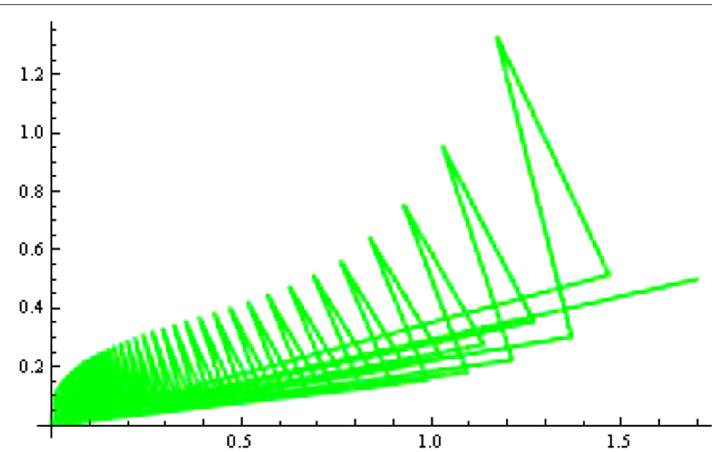

Figure 2 An attractor of the system (3.1).

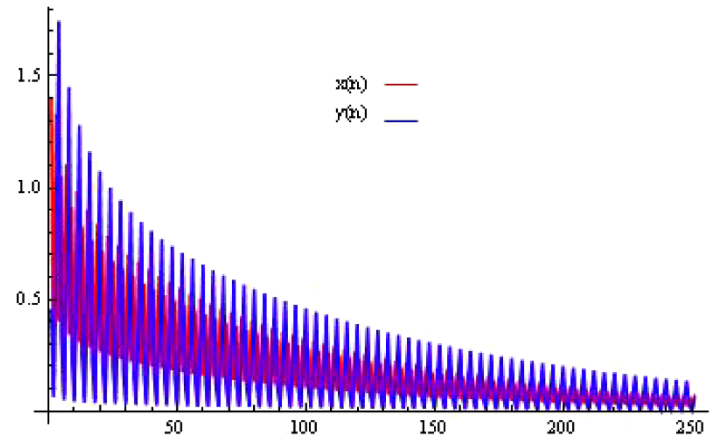

Figure 3 Plot of the system (3.1).

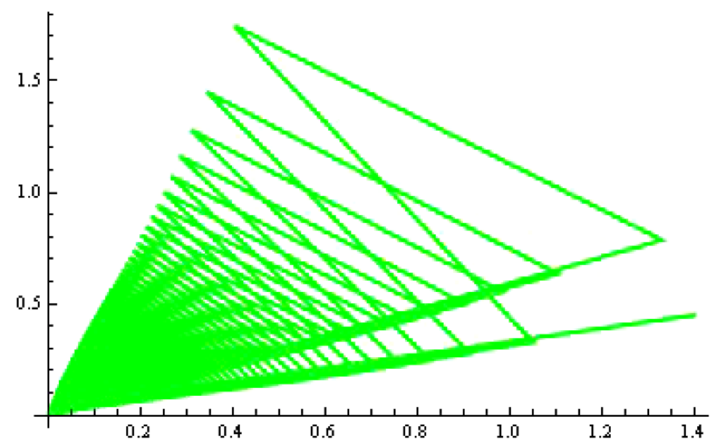

Figure 4 An attractor of the system (3.1).

Example Consider the system (1.1) with initial conditions $x_{-3} = 2.1$, $x_{-2} = 2.6$, $x_{-1} = 0.6$, $x_0 = 1.4$, $y_{-3} = 1.5$, $y_{-2} = 1.6$, $y_{-1} = 2.7$, $y_0 = 0.45$. Moreover, choosing the parameters $\alpha = 12$, $\beta = 12.5$, $\gamma = 90$, $\alpha_1 = 15$, $\beta_1 = 15.5$, $\gamma_1 = 75$, the system (1.1) can be written as follows:

$$x_{n+1} = \frac{12x_{n-3}}{12.5 + 90y_n y_{n-1} y_{n-2} y_{n-3}}, \qquad y_{n+1} = \frac{15y_{n-3}}{15.5 + 75x_n x_{n-1} x_{n-2} x_{n-3}}, \tag{3.2}$$

$n = 0, 1, \ldots$, and with initial conditions $x_{-3} = 2.1$, $x_{-2} = 2.6$, $x_{-1} = 0.6$, $x_0 = 1.4$, $y_{-3} = 1.5$, $y_{-2} = 1.6$, $y_{-1} = 2.7$, $y_0 = 0.45$. The plot of the system (3.2) is shown in Figure 3 and its global attractor is shown in Figure 4.

Example Consider the system (1.1) with initial conditions $x_{-3} = 9.2$, $x_{-2} = 1.8$, $x_{-1} = 0.76$, $x_0 = 1.1$, $y_{-3} = 1.1$, $y_{-2} = 1.2$, $y_{-1} = 8.1$, $y_0 = 0.52$. Moreover, choosing the parameters $\alpha = 200$, $\beta = 225$, $\gamma = 1000$, $\alpha_1 = 150$, $\beta_1 = 160$, $\gamma_1 = 700$, the system (1.1) can be written as follows:

$$x_{n+1} = \frac{200x_{n-3}}{225 + 1000y_n y_{n-1} y_{n-2} y_{n-3}}, \qquad y_{n+1} = \frac{150y_{n-3}}{160 + 700x_n x_{n-1} x_{n-2} x_{n-3}}, \tag{3.3}$$

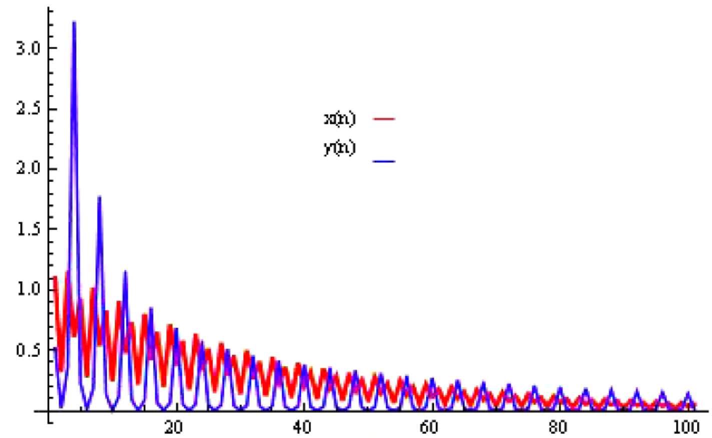

Figure 5 Plot of the system (3.3).

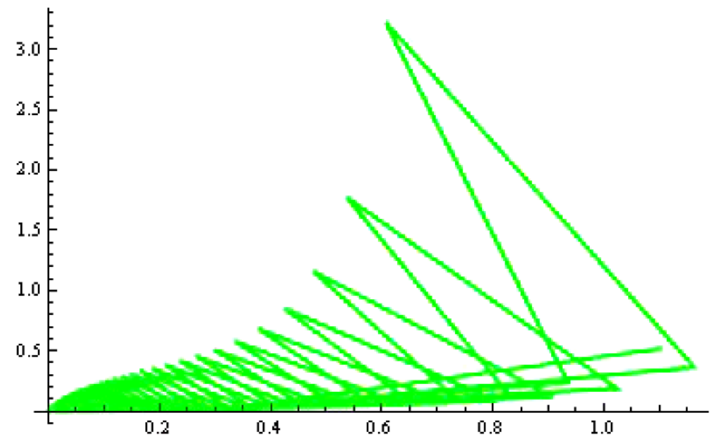

Figure 6 An attractor of the system (3.3).

$n = 0, 1, \ldots$, and with initial conditions $x_{-3} = 9.2$, $x_{-2} = 1.8$, $x_{-1} = 0.76$, $x_0 = 1.1$, $y_{-3} = 1.1$, $y_{-2} = 1.2$, $y_{-1} = 8.1$, $y_0 = 0.52$. The plot of the system (3.3) is shown in Figure 5 and its global attractor is shown in Figure 6.

Example Consider the system (1.1) with initial conditions $x_{-3} = 1.3$, $x_{-2} = 1.8$, $x_{-1} = 2.6$, $x_0 = 2.1$, $y_{-3} = 0.01$, $y_{-2} = 1.2$, $y_{-1} = 2.8$, $y_0 = 1.5$. Moreover, choosing the parameters $\alpha = 12$, $\beta = 10.5$, $\gamma = 15$, $\alpha_1 = 14$, $\beta_1 = 13$, $\gamma_1 = 0.2$, the system (1.1) can be written as follows:

$$x_{n+1} = \frac{12x_{n-3}}{10.5 + 15y_n y_{n-1} y_{n-2} y_{n-3}}, \qquad y_{n+1} = \frac{14y_{n-3}}{13 + 0.2x_n x_{n-1} x_{n-2} x_{n-3}}, \tag{3.4}$$

$n = 0, 1, \ldots$, with initial conditions $x_{-3} = 1.3$, $x_{-2} = 1.8$, $x_{-1} = 2.6$, $x_0 = 2.1$, $y_{-3} = 0.01$, $y_{-2} = 1.2$, $y_{-1} = 2.8$, $y_0 = 1.5$. The plot of the system (3.4) is shown in Figure 7.

4 Conclusion

This work is a natural extension of [9, 10]. In the paper, we investigated some dynamics of an eight-dimensional discrete system. The system has five equilibrium points all of

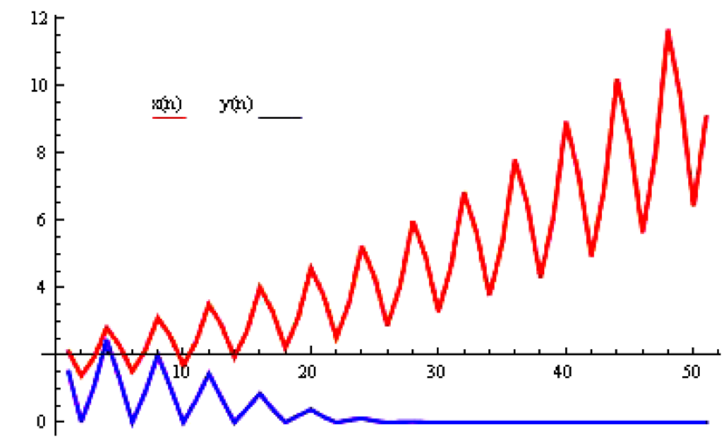

Figure 7 Plot of the system (3.4).

which except $(0,0)$ are unstable. The linearization method is used to show that the equilibrium point $(0,0)$ is locally asymptotically stable. We prove that the system has no prime period-two solutions. The main objective of dynamical systems theory is to predict the global behavior of a system based on the knowledge of its present state. An approach to this problem consists of determining the possible global behaviors of the system and determining which initial conditions lead to these long-term behaviors. In case of higher-order dynamical systems, it is crucial to discuss global behavior of the system. Some powerful tools such as semiconjugacy and weak contraction cannot be used to analyze global behavior of the system (1.1). In the paper, we prove the global asymptotic stability of the equilibrium point $(0,0)$ by using simple techniques. Some numerical examples are provided to support our theoretical results. These examples are experimental verifications of theoretical discussions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QD and MNQ carried out the theoretical proof and drafted the manuscript. AQK participated in the design and coordination. All authors read and approved the final manuscript.

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