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Approximate analytic solutions of the modified Kawahara equation with homotopy analysis method

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Abstract

In this paper, we applied the homotopy analysis method (HAM) to solve the modified Kawahara equation. Numerical results demonstrate that the methods provide efficient approaches to solving the modified Kawahara equation. It is shown that the method, with the help of symbolic computation, is very effective and powerful for discrete nonlinear evolution equations in mathematical physics.

Keywords: the modified Kawahara equation; homotopy analysis method

1 Introduction

In the past several decades, the investigation of traveling-wave solutions for nonlinear equations has played an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction, and convection are very important in nonlinear wave equations. In recent years, many authors paid attention to study solitonic solutions of nonlinear equations by using a variety of powerful methods such as the variational iteration method (VIM) [1, 2] and homotopy perturbation method (HPM) [3, 4]. Exp-function method [5–8], sine-cosine method [9], and homogeneous balance method [10–12] have been proposed for obtaining exact and approximate analytic solutions.

The aim of this paper is to directly apply the optimal HAM [13, 14] to reconsider the traveling-wave solution of the Kawahara equation. The method used here contains three convergence-control parameters to accelerate the convergence of homotopy series solution. The optimal convergence-control parameters can be determined by minimizing the averaged residual error. The results obtained in this paper show that the solutions given by the optimal HAM give much better approximations and converge much faster than those given by the usual HAM. The homotopy analysis method (HAM) [15–18] is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-integral equations, differential-difference equation, and coupled equations of them.

2 The homotopy analysis method

In this paper, we use the homotopy analysis method to solve the problem.

This method was proposed by the Chinese mathematician Liao [15]. We apply Liao's basic ideas to the nonlinear partial differential equations. Let us consider the nonlinear partial differential equation

$$ND\big(u(x,t)\big) = 0. \tag{2.1}$$

Based on the constructed zero-order deformation equation, we give the following zero-order deformation equation in the similar way:

$$(1-q)L\big(U(x,t;q) - u_0(x,t)\big) = qhND\big(U(x,t;q)\big), \quad q \in [0,1], h \neq 0. \tag{2.2}$$

L is an auxiliary linear integer-order operator and it possesses the property $L(C) = 0$. U is an unknown function. Expanding U in Taylor series with respect to q, one has

$$U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{2.3}$$

where

$$u_m(x,t) = \frac{\partial^m U(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{2.4}$$

As $h = -1$, Eq. (2.2) becomes

$$(1-q)L\big(U(x,t;q) - u_0(x,t)\big) + qNDU(x,t;q) = 0, \quad q \in [0,1], \tag{2.5}$$

which is used mostly in the homotopy perturbation method (HPM) [19–22]. Thus, HPM is a special case of HAM.

Differentiating the equation m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the mth-order deformation equation

$$L\big[u_m(x,t) - \chi_m u_{m-1}(x,t)\big] = hR_m\big[\vec{u}_{m-1}(x,t)\big], \tag{2.6}$$

where

$$R_m\big[\vec{u}_{m-1}(x,t)\big] = \frac{1}{(m-1)!}\frac{\partial^{m-1}ND(U(x,t;q))}{\partial q^{m-1}}\bigg|_{q=0} \tag{2.7}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{2.8}$$

These equations can be easily solved with software such as Maple, Matlab and so on.

The mth-order deformation Eq. (2.6) is linear, and thus can be easily solved, especially by means of a symbolic computation software such as Maple, Matlab and so on.

3 Test problem

We first consider the modified Kawahara equation [23]

$$u_t + u^2 u_x + p u_{xxx} + q u_{xxxxx} = 0, \tag{3.1}$$

where p, q are nonzero real constants. We solve the nonlinear partial differential equation with the HAM method. We consider Eq. (3.1) with initial condition

$$u(x,0) = \frac{3p}{\sqrt{-10q}} \sec h^2(Kx), \tag{3.2}$$

where $K = \frac{1}{2}\sqrt{\frac{-p}{5q}}$ is constant. The exact solution is given for modified Kawahara equation by [23]

$$u(x,t) = \frac{3p}{\sqrt{-10q}} \sec h^2\bigl(K(x-ct)\bigr) \tag{3.3}$$

with $c = \frac{25q-4p^2}{25q}$.

Furthermore, Eq. (3.1) suggests defining the nonlinear fractional partial differential operator

$$\begin{aligned} ND\bigl(u(x,t;q)\bigr) &= u_t(x,t;q) + \bigl(u(x,t;q)\bigr)^2 u_x(x,t;q) \\ &\quad + p u_{xxx}(x,t;q) + q u_{xxxxx}(x,t;q). \end{aligned} \tag{3.4}$$

Applying the above definition, we construct the zeroth-order deformation equation

$$(1-q)L\bigl(u(x,t;q) - u_0(x,t)\bigr) = hqNDu(x,t;q). \tag{3.5}$$

For $q = 0$ and $q = 1$ respectively, we can write

$$u(x,t;0) = u_0(x,t) = u(x,0), \qquad v(x,t;1) = u(x,t). \tag{3.6}$$

According to Eqs. (2.6)-(2.7), we gain the mth-order deformation equation

$$L\bigl(u_m(x,t) - x_m u_{m-1}(x,t)\bigr) = hNR\bigl(\vec{u}_{m-1}(x,t)\bigr), \tag{3.7}$$

where

$$\begin{aligned} NR\bigl(u_m(x,t)\bigr) &= D_t u_{m-1}(x,t) + \sum_{i=0}^{j} u_i u_{j-i} \sum_{j=0}^{m-1} (u_{m-1-j})_x(x,t) \\ &\quad + p(u_{m-1})_{xxx}(x,t) + q(u_{m-1})_{xxxxx}(x,t). \end{aligned} \tag{3.8}$$

Now, the solution of Eq. (3.7) for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + hL^{-1}NR\bigl[\vec{u}_{m-1}(x,t)\bigr]. \tag{3.9}$$

Table 1 The numerical results for the approximate solutions obtained by HPM [4] and HAM in comparison with the exact solutions of (3.1)

X	Time	HPM	HAM	Exact solution
–0.5	0.02	9.474889415e–4	9.474889415e–4	9.474889415e–4
–2.5	0.04	9.483868961e–4	9.483868961e–4	9.483773375e–4
0.0	0.06	9.486832980e–4	9.486832980e–4	9.486831272e–4
2.5	0.08	9.483868961e–4	9.483868961e–4	9.484055589e–4
5.0	1.0	9.474984315e–4	9.474984315e–4	9.475453144e–4

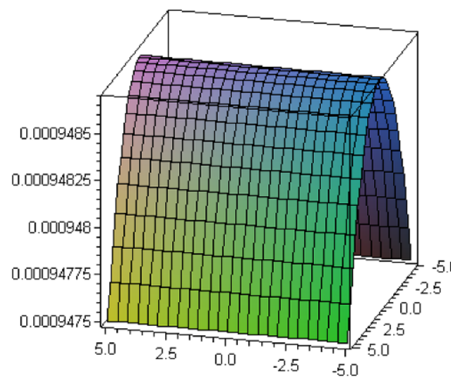

Figure 1 The surface of the exact solutions to Eq. (3.1) obtained in this work.

From Eqs. (3.1), (3.6), and (3.9), for $h = -1$, we now successively get

$$\begin{aligned} u(x,t) &= \frac{27}{5}\frac{p^3 \sec h(Kx)^6 \tanh(kx)kt}{\sqrt{-10q}q} + p\biggl(-\frac{24p \sec h(Kx)^2 \tanh(kx)^3 k^3}{\sqrt{-10q}} \\ &\quad + \frac{48p \sec h(Kx)^2 \tanh(kx)k^3(1-\tanh(kx)^2)}{\sqrt{-10q}}\biggr)t \\ &\quad + \biggl(\frac{96p \sec h(Kx)^2 \tanh(kx)^5 k^5}{\sqrt{-10q}} + \frac{1{,}248p \sec h(Kx)^2 \tanh(kx)^3 k^5(1-\tanh(kx)^2)}{\sqrt{-10q}} \\ &\quad - \frac{816p \sec h(Kx)^2 \tanh(kx)k^5(1-\tanh(kx)^2)^2}{\sqrt{-10q}}\biggr)qt. \end{aligned}$$

As shown in Table 1, we note through the results of the preceding table that the solutions we have obtained are very precise and that we have compared our solution (HAM) to HPM and exact solution. HAM is easily more than the other method. It is obvious that two components only were sufficient to determine the exact solution of Eq. (3.1). Figures 1 and 2 show the evolution results. From Figures 1 and 2, it is easy to conclude that the solution continuously depends on the derivative. Where, Figures 1 and 2 are approximation and exact solution respectly. The exact solution of this test problem is as follows [23]:

$$u(x,t) = \frac{3p}{\sqrt{-10q}} \sec h^2\bigl(K(x-ct)\bigr).$$

4 Conclusions

In this paper, we applied the homotopy analysis method to the Kawahara equation. The homotopy analysis method was successfully used to obtain the exact solutions of Kawa-

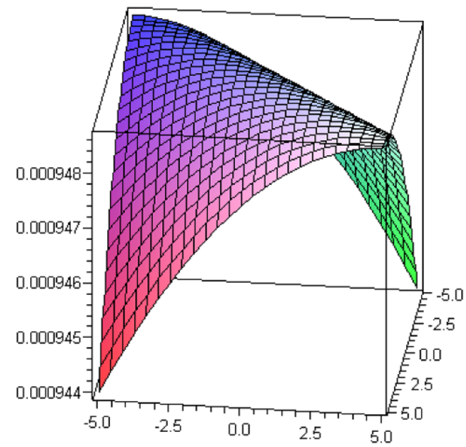

Figure 2 The approximation solutions to Eq. (3.1) obtained in this work.

hara equation. As a result, some new generalized solitary solutions with parameters are obtained. It may be important to explain some physical phenomena by setting the parameters as special values. Finally, the method is straightforward, concise, and is a powerful mathematical method for solving nonlinear problems.

Competing interests
The author declares that they have no competing interests.

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