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Stability criteria for linear Hamiltonian dynamic systems on time scales

Xiaofei He[1,2], Xianhua Tang[1*] and Qi-Ming Zhang[1]

* Correspondence: tangxh@mail.csu.edu.cn
[1]School of Mathematical Sciences and Computing Technology, Central South University, Changsha 410083, Hunan, P.R. China
Full list of author information is available at the end of the article

Abstract

In this article, we establish some stability criteria for the polar linear Hamiltonian dynamic system on time scales

$$x^{\Delta}(t) = \alpha(t)x(\sigma(t)) + \beta(t)y(t), \quad y^{\Delta}(t) = -\gamma(t)x(\sigma(t)) - \alpha(t)y(t), \quad t \in \mathbb{T}$$

by using Floquet theory and Lyapunov-type inequalities.

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1 Introduction

A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume that $\mathbb{T}$ is a time scale. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) : \inf\{s \in \mathbb{T} : s > t\}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) : \sup\{s \in \mathbb{T} : s < t\}$, and the graininess function $\mu : \mathbb{T} \to [0, \infty\}$ is defined by $\mu(t) = \sigma(t) - t$. For other related basic concepts of time scales, we refer the reader to the original studies by Hilger [1-3], and for further details, we refer the reader to the books of Bohner and Peterson [4,5] and Kaymakcalan et al. [6].

Definition 1.1. *If there exists a positive number $\omega \in \mathbb{R}$ such that $t + n\omega \in \mathbb{T}$ for all $t \in \mathbb{T}$ and $n \in \mathbb{Z}$, then we call $\mathbb{T}$ a periodic time scale with period ω.*

Suppose $\mathbb{T}$ is a ω-periodic time scale and $0 \in \mathbb{T}$. Consider the polar linear Hamiltonian dynamic system on time scale $\mathbb{T}$

$$x^{\Delta}(t) = \alpha(t)x(\sigma(t)) + \beta(t)y(t), \quad y^{\Delta}(t) = -\gamma(t)x(\sigma(t)) - \alpha(t)y(t), \quad t \in \mathbb{T}, \quad (1.1)$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are real-valued rd-continuous functions defined on $\mathbb{T}$. Throughout this article, we always assume that

$$1 - \mu(t)\alpha(t) > 0, \quad \forall\, t \in \mathbb{T} \quad (1.2)$$

and

$$\beta(t) \geq 0, \quad \forall\, t \in \mathbb{T}. \quad (1.3)$$

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For the second-order linear dynamic equation

$$[p(t)x^{\Delta}(t)]^{\Delta} + q(t)x(\sigma(t)) = 0, \quad t \in \mathbb{T}, \tag{1.4}$$

if let $y(t) = p(t)x^{\Delta}(t)$, then we can rewrite (1.4) as an equivalent polar linear Hamiltonian dynamic system of type (1.1):

$$x^{\Delta}(t) = \frac{1}{p(t)}y(t), \quad y^{\Delta}(t) = -q(t)x(\sigma(t)), \quad t \in \mathbb{T}, \tag{1.5}$$

where $p(t)$ and $q(t)$ are real-valued rd-continuous functions defined on $\mathbb{T}$ with $p(t) > 0$, and

$$\alpha(t) = 0, \quad \beta(t) = \frac{1}{p(t)}, \quad \gamma(t) = q(t).$$

Recently, Agarwal et al. [7], Jiang and Zhou [8], Wong et al. [9] and He et al. [10] established some Lyapunov-type inequalities for dynamic equations on time scales, which generalize the corresponding results on differential and difference equations. Lyapunov-type inequalities are very useful in oscillation theory, stability, disconjugacy, eigenvalue problems and numerous other applications in the theory of differential and difference equations. In particular, the stability criteria for the polar continuous and discrete Hamiltonian systems can be obtained by Lyapunov-type inequalities and Floquet theory, see [11-16]. In 2000, Atici et al. [17] established the following stablity criterion for the second-order linear dynamic equation (1.4):

Theorem 1.2 [17]. *Assume $p(t) > 0$ for $t \in \mathbb{T}$, and that*

$$p(t+\omega) = p(t), \quad q(t+\omega) = q(t), \quad \forall\, t \in \mathbb{T}. \tag{1.6}$$

If

$$\int_0^{\omega} q(t)\Delta t \geq 0, \quad q(t) \not\equiv 0 \tag{1.7}$$

and

$$\left[p_0 + \int_0^{\omega} \frac{1}{p(t)}\,\Delta t\right] \int_0^{\omega} q^{+}(t)\Delta t \leq 4, \tag{1.8}$$

then equation (1.4) is stable, where

$$p_0 = \max_{t \in [0, \rho(\omega)]} \frac{\sigma(t) - t}{p(t)}, \quad q^{+}(t) = \max\{q(t), 0\}, \tag{1.9}$$

where and in the sequel, system (1.1) or Equation (1.4) is said to be unstable if all nontrivial solutions are unbounded on $\mathbb{T}$; conditionally stable if there exist a nontrivial solution which is bounded on $\mathbb{T}$; and stable if all solutions are bounded on $\mathbb{T}$.

In this article, we will use the Floquet theory in [18,19] and the Lyapunov-type inequalities in [10] to establish two stability criteria for system (1.1) and equation (1.4). Our main results are the following two theorems.

Theorem 1.3. *Suppose (1.2) and (1.3) hold and*

$$\alpha(t+\omega) = \alpha(t), \quad \beta(t+\omega) = \beta(t), \quad \gamma(t+\omega) = \gamma(t), \quad \forall\, t \in \mathbb{T}. \tag{1.10}$$

Assume that there exists a non-negative rd-continuous function $\theta(t)$ defined on $\mathbb{T}$ such that

$$|\alpha(t)| \leq \theta(t)\beta(t), \quad \forall\, t \in \mathbb{T}[0,\omega] = [0,\omega] \cap \mathbb{T}, \tag{1.11}$$

$$\int_0^{\omega} [\gamma(t) - \theta^2(t)\beta(t)]\, \Delta t > 0, \tag{1.12}$$

and

$$\int_0^{\omega} |\alpha(t)|\Delta t + \left[\int_0^{\omega} \beta(t)\Delta t \int_0^{\omega} \gamma^+(t)\Delta t\right]^{1/2} < 2. \tag{1.13}$$

Then system (1.1) is stable.

Theorem 1.4. *Assume that (1.6) and (1.7) hold, and that*

$$\int_0^{\omega} \frac{1}{p(t)} \Delta t \int_0^{\omega} q^+(t)\, \Delta t \leq 4. \tag{1.14}$$

Then equation (1.4) is stable.

Remark 1.5. *Clearly, condition (1.14) improves (1.8) by removing term p_0.*

We dwell on the three special cases as follows:

1. If $\mathbb{T} = \mathbb{R}$, system (1.1) takes the form:

$$x'(t) = \alpha(t)x(t) + \beta(t)y(t), \quad y'(t) = -\gamma(t)x(t) - \alpha(t)y(t), \quad t \in \mathbb{R}. \tag{1.15}$$

In this case, the conditions (1.12) and (1.13) of Theorem 1.3 can be transformed into

$$\int_0^{\omega} \left[\gamma(t) - \theta^2(t)\beta(t)\right] dt > 0, \tag{1.16}$$

and

$$\int_0^{\omega} |\alpha(t)|dt + \left[\int_0^{\omega} \beta(t)dt \int_0^{\omega} \gamma^+(t)dt\right]^{1/2} < 2. \tag{1.17}$$

Condition (1.17) is the same as (3.10) in [12], but (1.11) and (1.16) are better than (3.9) in [12] by taking $\theta(t) = |\alpha(t)|/\beta(t)$. A better condition than (1.17) can be found in [14,15].

2. If $\mathbb{T} = \mathbb{Z}$, system (1.1) takes the form:

$$\Delta x(n) = \alpha(n)x(n+1) + \beta(n)y(n), \quad \Delta y(n) = -\gamma(n)x(n+1) - \alpha(n)y(n), \quad n \in \mathbb{Z}. \tag{1.18}$$

In this case, the conditions (1.11), (1.12), and (1.13) of Theorem 1.3 can be transformed into

$$|\alpha(n)| \leq \theta(n)\beta(n), \quad \forall\, n \in \{0, 1, \ldots, \omega - 1\}, \tag{1.19}$$

$$\sum_{n=0}^{\omega-1} \left[\gamma(n) - \theta^2(n)\beta(n)\right] > 0, \tag{1.20}$$

and

$$\sum_{n=0}^{\omega-1}|\alpha(n)| + \left[\sum_{n=0}^{\omega-1}\beta(n)\sum_{n=0}^{\omega-1}\gamma^{+}(n)\right]^{1/2} < 2. \tag{1.21}$$

Conditions (1.19), (1.20), and (1.21) are the same as (1.17), (1.18) and (1.19) in [16], i. e., Theorem 1.3 coincides with Theorem 3.4 in [16]. However, when $p(n)$ and $q(n)$ are ω-periodic functions defined on $\mathbb{Z}$, the stability conditions

$$0 \le \sum_{n=0}^{\omega-1}q(n) \le \sum_{n=0}^{\omega-1}q^{+}(n) \le \frac{4}{\sum_{n=0}^{\omega-1}\frac{1}{p(n)}}, \quad q(n) \not\equiv 0, \quad \forall n \in \{0, 1, \dots, \omega-1\} \tag{1.22}$$

in Theorem 1.4 are better than the one

$$0 < \sum_{n=0}^{\omega-1}q(n) \le \sum_{n=0}^{\omega-1}q^{+}(n) < \frac{4}{\sum_{n=0}^{\omega-1}\frac{1}{p(n)}} \tag{1.23}$$

in [16, Corollary 3.4]. More related results on stability for discrete linear Hamiltonian systems can be found in [20-24].

3. Let $\delta > 0$ and $N \in \{2, 3, 4, ...\}$. Set $\omega = \delta + N$, define the time scale $\mathbb{T}$ as follows:

$$\mathbb{T} = \bigcup_{k\in\mathbb{Z}}[k\omega, k\omega+\delta] \cup \{k\omega+\delta+n : n = 1, 2, \dots, N-1\}. \tag{1.24}$$

Then system (1.1) takes the form:

$$x'(t) = \alpha(t)x(t) + \beta(t)y(t), \quad y'(t) = -\gamma(t)x(t) - \alpha(t)y(t), \quad t \in \bigcup_{k\in\mathbb{Z}}[k\omega, k\omega+\delta), \tag{1.25}$$

and

$$\begin{gathered}\Delta x(t) = \alpha(t)x(t+1) + \beta(t)y(t), \quad \Delta y(t) = -\gamma(t)x(t+1) - \alpha(t)y(t), \\ t \in \bigcup_{k\in\mathbb{Z}}\{k\omega+\delta+n : n = 0, 1, \dots, N-2\}.\end{gathered} \tag{1.26}$$

In this case, the conditions (1.11), (1.12), and (1.13) of Theorem 1.3 can be transformed into

$$|\alpha(t)| \le \theta(t)\beta(t), \quad \forall t \in [0, \delta] \cup \{\delta+1, \delta+2, \dots, \delta+N-1\}, \tag{1.27}$$

$$\int_0^{\delta}\left[\gamma(t) - \theta^2(t)\beta(t)\right]dt + \sum_{n=0}^{N-1}\left[\gamma(\delta+n) - \theta^2(\delta+n)\beta(\delta+n)\right] > 0, \tag{1.28}$$

and

$$\begin{aligned}&\left(\int_0^{\delta}|\alpha(t)|dt + \sum_{n=0}^{N-1}|\alpha(\delta+n)|\right) \\ &+\left[\left(\int_0^{\delta}\beta(t)dt + \sum_{n=0}^{N-1}|\beta(\delta+n)|\right)\left(\int_0^{\delta}\gamma^{+}(t)dt + \sum_{n=0}^{N-1}|\gamma^{+}(\delta+n)|\right)\right]^{1/2} < 2.\end{aligned} \tag{1.29}$$

2 Proofs of theorems

Let $u(t) = (x(t), y(t))^\top$, $u^\sigma(t) = (x(\sigma(t)), y(t))^\top$ and

$$A(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ -\gamma(t) & -\alpha(t) \end{pmatrix}.$$

Then, we can rewrite (1.1) as a standard linear Hamiltonian dynamic system

$$u^\Delta(t) = A(t)u^\sigma(t), \quad t \in \mathbb{T}. \tag{2.1}$$

Let $u_1(t) = (x_{10}(t), y_{10}(t))^\top$ and $u_2(t) = (x_{20}(t), y_{20}(t))^\top$ be two solutions of system (1.1) with $(u_1(0), u_2(0)) = I_2$. Denote by $\Phi(t) = (u_1(t), u_2(t))$. Then $\Phi(t)$ is a fundamental matrix solution for (1.1) and satisfies $\Phi(0) = I_2$. Suppose that $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are ω-periodic functions defined on $\mathbb{T}$ (i.e. (1.10) holds), then $\Phi(t + \omega)$ is also a fundamental matrix solution for (1.1) (see [18]). Therefore, it follows from the uniqueness of solutions of system (1.1) with initial condition (see [9,18,19]) that

$$\Phi(t+\omega) = \Phi(t)\Phi(\omega), \quad \forall\, t \in \mathbb{T}. \tag{2.2}$$

From (1.1), we have

$$(\det \Phi(t))^\Delta = \begin{vmatrix} x_{10}^\Delta(t) & x_{20}^\Delta(t) \\ y_{10}(t) & y_{20}(t) \end{vmatrix} + \begin{vmatrix} x_{10}(\sigma(t)) & x_{20}(\sigma(t)) \\ y_{10}^\Delta(t) & y_{20}^\Delta(t) \end{vmatrix} = 0, \quad \forall t \in \mathbb{T}. \tag{2.3}$$

It follows that $\det \Phi(t) = \det \Phi(0) = 1$ for all $t \in \mathbb{T}$. Let λ_1 and λ_2 be the roots (real or complex) of the characteristic equation of $\Phi(\omega)$

$$\det(\lambda I_2 - \Phi(\omega)) = 0,$$

which is equivalent to

$$\lambda^2 - H\lambda + 1 = 0, \tag{2.4}$$

where

$$H = x_{10}(\omega) + y_{20}(\omega).$$

Hence

$$\lambda_1 + \lambda_2 = H, \quad \lambda_1\lambda_2 = 1. \tag{2.5}$$

Let $v_1 = (c_{11}, c_{21})^\top$ and $v_2 = (c_{12}, c_{22})^\top$ be the characteristic vectors associated with the characteristic roots λ_1 and λ_2 of $\Phi(\omega)$, respectively, i.e.

$$\Phi(\omega)v_j = \lambda_j v_j, \quad j = 1, 2. \tag{2.6}$$

Let $v_j(t) = \Phi(t)v_j$, $j = 1, 2$. Then it follows from (2.2) and (2.6) that

$$v_j(t+\omega) = \lambda_j v_j(t), \quad \forall t \in \mathbb{T}, \quad j = 1, 2. \tag{2.7}$$

On the other hand, it follows from (2.1) that

$$\begin{aligned} v_j^\Delta(t) &= \Phi^\Delta(t)v_j \\ &= \left(u_1^\Delta(t), u_2^\Delta(t)\right) v_j \\ &= A(t)\left(u_1^\sigma(t), u_2^\sigma(t)\right) v_j \\ &= A(t)v_j^\sigma(t), \quad j = 1, 2. \end{aligned} \tag{2.8}$$

This shows that $v_1(t)$ and $v_2(t)$ are two solutions of system (1.1) which satisfy (2.7). Hence, we obtain the following lemma.

Lemma 2.1. *Let* $\Phi(t)$ *be a fundamental matrix solution for (1.1) with* $\Phi(0) = I_2$, *and let* λ_1 *and* λ_2 *be the roots (real or complex) of the characteristic equation (2.4) of* $\Phi(\omega)$. *Then system (1.1) has two solutions* $v_1(t)$ *and* $v_2(t)$ *which satisfy (2.7).*

Similar to the continuous case, we have the following lemma.

Lemma 2.2. *System (1.1) is unstable if* $|H| > 2$, *and stable if* $|H| < 2$.

Instead of the usual zero, we adopt the following concept of generalized zero on time scales.

Definition 2.3. *A function* $f : \mathbb{T} \to \mathbb{R}$ *is said to have a generalized zero at* $t_0 \in \mathbb{T}$ *provided either* $f(t_0) = 0$ *or* $f(t_0)f(\sigma(t_0)) < 0$.

Lemma 2.4. [4]*Assume* $f, g : \mathbb{T} \to \mathbb{R}$ *are differential at* $t \in \mathbb{T}^k$. *If* $f^\Delta(t)$ *exists, then* $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$.

Lemma 2.5. [4] (Cauchy-Schwarz inequality). *Let* $a, b \in \mathbb{T}$. *For* $f,g \in C_{rd}$ *we have*

$$\int_a^b |f(t)g(t)|\,\Delta t \le \left[\int_a^b f^2(t)\Delta t \cdot \int_a^b g^2(t)\Delta t\right]^{\frac{1}{2}}.$$

The above inequality can be equality only if there exists a constant c such that $f(t) = cg(t)$ for $t \in \mathbb{T}[a, b]$.

Lemma 2.6. *Let* $v_1(t) = (x_1(t), y_1(t))^\top$ *and* $v_2(t) = (x_2(t), y_2(t))^\top$ *be two solutions of system (1.1) which satisfy (2.7). Assume that (1.2), (1.3) and (1.10) hold, and that exists a non-negative function* $\theta(t)$ *such that (1.11) and (1.12) hold. If* $H^2 \ge 4$, *then both* $x_1(t)$ *and* $x_2(t)$ *have generalized zeros in* $\mathbb{T}[0, \omega]$.

Proof. Since $|H| \ge 2$, then λ_1 and λ_2 are real numbers, and $v_1(t)$ and $v_2(t)$ are also real functions. We only prove that $x_1(t)$ must have at least one generalized zero in $\mathbb{T}[0, \omega]$. Otherwise, we assume that $x_1(t) > 0$ for $t \in \mathbb{T}[0, \omega]$ and so (2.7) implies that $x_1(t) > 0$ for $t \in \mathbb{T}$. Define $z(t) := y_1(t)/x_1(t)$. Due to (2.7), one sees that $z(t)$ is ω-periodic, i.e. $z(t + \omega) = z(t)$, $\forall t \in \mathbb{T}$. From (1.1), we have

$$\begin{aligned} z^\Delta(t) &= \frac{x_1(t)y_1^\Delta(t) - x_1^\Delta(t)y_1(t)}{x_1(t)x_1(\sigma(t))} \\ &= \frac{-\gamma(t)x_1(t)x_1(\sigma(t)) - \alpha(t)[x_1(t) + x_1(\sigma(t))]y_1(t) - \beta(t)y_1^2(t)}{x_1(t)x_1(\sigma(t))} \\ &= -\gamma(t) - \alpha(t)\left[\frac{y_1(t)}{x_1(t)} + \frac{y_1(t)}{x_1(\sigma(t))}\right] - \beta(t)\left[\frac{y_1(t)}{x_1(t)}\,\frac{y_1(t)}{x_1(\sigma(t))}\right] \\ &= -\gamma(t) - \alpha(t)\left[z(t) + \frac{y_1(t)}{x_1(\sigma(t))}\right] - \beta(t)z(t)\left[\frac{y_1(t)}{x_1(\sigma(t))}\right]. \end{aligned} \tag{2.9}$$

From the first equation of (1.1), and using Lemma 2.4, we have

$$[1 - \mu(t)\alpha(t)]x_1(\sigma(t)) = x_1(t) + \mu(t)\beta(t)y_1(t), \quad t \in \mathbb{T}. \tag{2.10}$$

Since $x_1(t) > 0$ for all $t \in \mathbb{T}$, it follows from (1.2) and (2.10) that

$$1 + \mu(t)\beta(t)z(t) = 1 + \mu(t)\beta(t)\frac{y_1(t)}{x_1(t)} = [1 - \mu(t)\alpha(t)]\frac{x_1(\sigma(t))}{x_1(t)} > 0, \tag{2.11}$$

which yields

$$\frac{y_1(t)}{x_1(\sigma(t))} = \frac{[1 - \mu(t)\alpha(t)]z(t)}{1 + \mu(t)\beta(t)z(t)}. \tag{2.12}$$

Substituting (2.12) into (2.9), we obtain

$$z^{\Delta}(t) = -\gamma(t) + \frac{[-2\alpha(t) + \mu(t)\alpha^2(t)]z(t) - \beta(t)z^2(t)}{1 + \mu(t)\beta(t)z(t)}. \tag{2.13}$$

If $\beta(t) > 0$, together with (1.11), it is easy to verify that

$$\frac{[-2\alpha(t) + \mu(t)\alpha^2(t)]z(t) - \beta(t)z^2(t)}{1 + \mu(t)\beta(t)z(t)} \leq \frac{\alpha^2(t)}{\beta(t)} \leq \theta^2(t)\beta(t); \tag{2.14}$$

If $\beta(t) = 0$, it follows from (1.11) that $\alpha(t) = 0$, hence

$$\frac{[-2\alpha(t) + \mu(t)\alpha^2(t)]z(t) - \beta(t)z^2(t)}{1 + \mu(t)\beta(t)z(t)} = 0 = \theta^2(t)\beta(t). \tag{2.15}$$

Combining (2.14) with (2.15), we have

$$\frac{[-2\alpha(t) + \mu(t)\alpha^2(t)]z(t) - \beta(t)z^2(t)}{1 + \mu(t)\beta(t)z(t)} \leq \theta^2(t)\beta(t). \tag{2.16}$$

Substituting (2.16) into (2.13), we obtain

$$z^{\Delta}(t) \leq -\gamma(t) + \theta^2(t)\beta(t). \tag{2.17}$$

Integrating equation (2.17) from 0 to ω, and noticing that $z(t)$ is ω-periodic, we obtain

$$0 \leq -\int_0^{\omega} \left[\gamma(t) - \theta^2(t)\beta(t)\right] \Delta t,$$

which contradicts condition (1.12). □

Lemma 2.7. *Let $v_1(t) = (x_1(t), y_1(t))^{\top}$ and $v_2(t) = (x_2(t), y_2(t))^{\top}$ be two solutions of system (1.1) which satisfy (2.7). Assume that*

$$\alpha(t) = 0, \quad \beta(t) > 0, \quad \gamma(t) \not\equiv 0, \quad \forall t \in \mathbb{T}, \tag{2.18}$$

$$\beta(t + \omega) = \beta(t), \quad \gamma(t + \omega) = \gamma(t), \quad \forall t \in \mathbb{T}, \tag{2.19}$$

and

$$\int_0^{\omega} \gamma(t)\Delta t \geq 0. \tag{2.20}$$

If $H^2 \geq 4$, then both $x_1(t)$ and $x_2(t)$ have generalized zeros in $\mathbb{T}[0, \omega]$.

Proof. Except (1.12), (2.18), and (2.19) imply all assumptions in Lemma 2.6 hold. In view of the proof of Lemma 2.6, it is sufficient to derive an inequality which contradicts (2.20) instead of (1.12). From (2.11), (2.13), and (2.18), we have

$$1 + \mu(t)\beta(t)z(t) = 1 + \mu(t)\beta(t)\frac{y_1(t)}{x_1(t)} = \frac{x_1(\sigma(t))}{x_1(t)} > 0 \tag{2.21}$$

and

$$z^{\Delta}(t) = -\gamma(t) - \frac{\beta(t)z^2(t)}{1 + \mu(t)\beta(t)z(t)}. \tag{2.22}$$

Since $z(t)$ is ω-periodic and $\gamma(t)$ ⊠ 0,, it follows from (2.22) that $z^2(t)$ ⊠ 0 on $\mathbb{T}[0, \omega]$. Integrating equation (2.22) from 0 to ω, we obtain

$$0 = -\int_0^{\omega} \left[\gamma(t) + \frac{\beta(t)z^2(t)}{1 + \mu(t)\beta(t)z(t)}\right] \Delta t < -\int_0^{\omega} \gamma(t)\Delta t,$$

which contradicts condition (2.20). □

Lemma 2.8. [10]*Suppose that (1.2) and (1.3) hold and let* $a, b \in \mathbb{T}^k$ *with* $\sigma(a) \le b$. *Assume (1.1) has a real solution* $(x(t), y(t))$ *such that* $x(t)$ *has a generalized zero at end-point a and* $(x(b), y(b)) = (\kappa_1 x(a), \kappa_2 y(a))$ *with* $0 < \kappa_1^2 \le \kappa_1\kappa_2 \le 1$*and* $x(t)$ ⊠ 0 *on* $\mathbb{T}[a, b]$. *Then one has the following inequality*

$$\int_a^b |\alpha(t)|\,\Delta t + \left[\int_a^b \beta(t)\Delta t \int_a^b \gamma^{+}(t)\Delta t\right]^{1/2} \ge 2. \tag{2.23}$$

Lemma 2.9. *Suppose that (2.18) holds and let* $a, b \in \mathbb{T}^k$ *with* $\sigma(a) \le b$. *Assume (1.1) has a real solution* $(x(t), y(t))$ *such that* $x(t)$ *has a generalized zero at end-point a and* $(x(b), y(b)) = (\kappa x(a), \kappa y(a))$ *with* $0 < \kappa^2 \le 1$ *and* $x(t)$ *is not identically zero on* $\mathbb{T}[a, b]$. *Then one has the following inequality*

$$\int_a^b \beta(t)\Delta t \int_a^b \gamma^{+}(t)\Delta t > 4. \tag{2.24}$$

Proof. In view of the proof of [10, Theorem 3.5] (see (3.8), (3.29)-(3.34) in [10]), we have

$$x(a) = -\xi\mu(a)\beta(a)y(a), \tag{2.25}$$

$$x(\tau) = (1 - \xi)\mu(a)\beta(a)y(a) + \int_{\sigma(a)}^{\tau} \beta(t)y(t)\Delta t, \quad \sigma(a) \le \tau \le b, \tag{2.26}$$

$$\vartheta_1\mu(a)\beta(a)y^2(a) + \int_{\sigma(a)}^b \beta(t)y^2(t)\Delta t = \int_a^b \gamma(t)x^2(\sigma(t))\Delta t, \tag{2.27}$$

and

$$2|x(\tau)| \le \vartheta_2\mu(a)\beta(a)|y(a)| + \int_{\sigma(a)}^b \beta(t)|y(t)|\Delta t, \quad \sigma(a) \le \tau \le b, \tag{2.28}$$

where $\xi \in [0, 1)$, and

$$\vartheta_1 = 1 - \xi + \kappa^2\xi, \quad \vartheta_2 = 1 - \xi + |\kappa|\xi. \tag{2.29}$$

Let $|x(\tau^*)| = \max_{\sigma(a)\le\tau\le b} |x(\tau)|$. There are three possible cases:

(1) $y(t) \equiv y(a) \ne 0$, $\forall\, t \in \mathbb{T}[a, b]$;

(2) $y(t)$ ⊠ $y(a)$, $|y(t)| \equiv |y(a)|$, $\forall\, t \in \mathbb{T}[a, b]$;

(3) $|y(t)| \boxtimes |y(a)|, \ \forall\, t \in \mathbb{T}[a, b]$.

Case (1). In this case, $\kappa = 1$. It follows from (2.25) and (2.26) that

$$\begin{aligned} x(b) &= (1-\xi)\mu(a)\beta(a)y(a) + \int_{\sigma(a)}^{b} \beta(t)y(t)\Delta t \\ &= y(a)\left[(1-\xi)\mu(a)\beta(a) + \int_{\sigma(a)}^{b} \beta(t)\Delta t\right] \\ &= x(a) + y(a)\int_{a}^{b} \beta(t)\Delta t \\ &\neq x(a), \end{aligned}$$

which contradicts the assumption that $x(b) = \kappa x(a) = x(a)$.

Case (2). In this case, we have

$$2|x(\tau)| < \vartheta_2\mu(a)\beta(a)|y(a)| + \int_{\sigma(a)}^{b} \beta(t)|y(t)|\Delta t, \quad \sigma(a) \le \tau \le b \tag{2.30}$$

instead of (2.28). Applying Lemma 2.5 and using (2.27) and (2.30), we have

$$\begin{aligned} &2|x(\tau^*)| \\ <\ & \vartheta_2\mu(a)\beta(a)|y(a)| + \int_{\sigma(a)}^{b} \beta(t)|y(t)|\Delta t \\ \le\ & \left\{\left[\frac{\vartheta_2^2}{\vartheta_1}\mu(a)\beta(a) + \int_{\sigma(a)}^{b} \beta(t)\Delta t\right]\left[\vartheta_1\mu(a)\beta(a)y^2(a) + \int_{\sigma(a)}^{b} \beta(t)y^2(t)\Delta t\right]\right\}^{1/2} \\ =\ & \left\{\left[\frac{\vartheta_2^2}{\vartheta_1}\mu(a)\beta(a) + \int_{\sigma(a)}^{b} \beta(t)\Delta t\right]\int_{a}^{b} \gamma(t)x^2(\sigma(t))\Delta t\right\}^{1/2} \\ \le\ & |x(\tau^*)|\left[\left(\frac{\vartheta_2^2}{\vartheta_1}\mu(a)\beta(a) + \int_{\sigma(a)}^{b} \beta(t)\Delta t\right)\int_{a}^{b} \gamma^+(t)\Delta t\right]^{1/2}. \end{aligned} \tag{2.31}$$

Dividing the latter inequality of (2.31) by $|x(\tau^*)|$, we obtain

$$\left[\left(\frac{\vartheta_2^2}{\vartheta_1}\mu(a)\beta(a) + \int_{\sigma(a)}^{b} \beta(t)\Delta t\right)\int_{a}^{b} \gamma^+(t)\Delta t\right]^{1/2} > 2. \tag{2.32}$$

Case (3). In this case, applying Lemma 2.5 and using (2.27) and (2.28), we have

$$\begin{aligned} &2|x(\tau^*)| \\ \le\ & \vartheta_2\mu(a)\beta(a)|y(a)| + \int_{\sigma(a)}^{b} \beta(t)|y(t)|\Delta t \\ <\ & \left\{\left[\frac{\vartheta_2^2}{\vartheta_1}\mu(a)\beta(a) + \int_{\sigma(a)}^{b} \beta(t)\Delta t\right]\left[\vartheta_1\mu(a)\beta(a)y^2(a) + \int_{\sigma(a)}^{b} \beta(t)y^2(t)\Delta t\right]\right\}^{1/2} \\ =\ & \left\{\left[\frac{\vartheta_2^2}{\vartheta_1}\mu(a)\beta(a) + \int_{\sigma(a)}^{b} \beta(t)\Delta t\right]\int_{a}^{b} \gamma(t)x^2(\sigma(t))\Delta t\right\}^{1/2} \\ \le\ & |x(\tau^*)|\left[\left(\frac{\vartheta_2^2}{\vartheta_1}\mu(a)\beta(a) + \int_{\sigma(a)}^{b} \beta(t)\Delta t\right)\int_{a}^{b} \gamma^+(t)\Delta t\right]^{1/2}. \end{aligned} \tag{2.33}$$

Dividing the latter inequality of (2.33) by $|x(\tau^*)|$, we also obtain (2.32). It is easy to verify that

$$\frac{\vartheta_2^2}{\vartheta_1} = \frac{[1 - \xi + |\kappa|\xi]^2}{1 - \xi + \kappa^2\xi} \leq 1.$$

Substituting this into (2.32), we obtain (2.24). □

Proof of Theorem 1.3. If $|H| \geq 2$, then λ_1 and λ_2 are real numbers and $\lambda_1\lambda_2 = 1$, it follows that $0 < \min\{\lambda_1^2, \lambda_2^2\} \leq 1$. Suppose $\lambda_1^2 \leq 1$. By Lemma 2.6, system (1.1) has a non-zero solution $\nu_1(t) = (x_1(t), y_1(t))^\top$ such that (2.7) holds and $x_1(t)$ has a generalized zero in $\mathbb{T}[0, \omega]$, say t_1. It follows from (2.7) that $(x_1(t_1 + \omega), y_1(t_1 + \omega)) = \lambda_1(x_1(t_1), y_1(t_1))$. Applying Lemma 2.8 to the solution $(x_1(t), y_1(t))$ with $a = t_1$, $b = t_1 + \omega$ and $\kappa_1 = \kappa_2 = \lambda_1$, we get

$$\int_{t_1}^{t_1+\omega} |\alpha(t)|\Delta t + \left[\int_{t_1}^{t_1+\omega} \beta(t)\Delta t \int_{t_1}^{t_1+\omega} \gamma^+(t)\Delta t\right]^{1/2} \geq 2. \tag{2.34}$$

Next, noticing that for any ω-periodic function $f(t)$ on $\mathbb{T}$, the equality

$$\int_{t_0}^{t_0+\omega} f(t)\Delta t = \int_0^\omega f(t)\Delta t$$

holds for all $t_0 \in \mathbb{T}$. It follows from (3.1) that

$$\int_0^\omega |\alpha(t)|\Delta t + \left[\int_0^\omega \beta(t)\Delta t \int_0^\omega \gamma^+(t)\Delta t\right]^{1/2} \geq 2. \tag{2.35}$$

which contradicts condition (1.13). Thus $|H| < 2$ and hence system (1.1) is stable. □

Proof of Theorem 1.4. By using Lemmas 2.7 and 2.9 instead of Lemmas 2.6 and 2.8, respectively, we can prove Theorem 1.4 in a similar fashion as the proof of Theorem 1.3. So, we omit the proof here. □

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Author details
[1]School of Mathematical Sciences and Computing Technology, Central South University, Changsha 410083, Hunan, P. R. China [2]College of Mathematics and Computer Science, Jishou University, Jishou 416000, Hunan, P.R.China

Authors' contributions
XH carried out the theoretical proof and drafted the manuscript. Both XT and QZ participated in the design and coordination. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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