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# Three extensions of Ćirić quasicontraction on partial metric spaces

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## Abstract

In this paper we define and study three extensions of the notion of Ćirić quasicontraction to the context of partial metric spaces. For such mappings, we prove fixed point theorems. Among other things, we generalize a recent result of Altun, Sola and Simsek, of Ilić *et al.*, of Matthews, and the main result of Ćirić is also recovered. The theory is illustrated by some examples.

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**Keywords:** fixed point; partial metric space; quasicontraction

## 1 Introduction and preliminaries

Let  $(X, \rho)$  be a complete metric space. A map  $T : X \mapsto X$ , such that for some constant  $\lambda \in (0, 1)$  and for every  $x, y \in X$ , there is the inequality

$$\rho(Tx, Ty) \leq \lambda \cdot \max \{ \rho(x, y), \rho(x, Tx), \rho(y, Ty), \rho(x, Ty), \rho(y, Tx) \}, \quad (1.1)$$

is called a *quasicontraction*. Let us remark that Ćirić [1] (see also [2, 3]) introduced and studied quasicontractions as one of the most general types of contractive maps. The well-known Ćirić result is that every quasicontraction  $T$  possesses a unique fixed point.

There exist many generalizations of the concept of metric spaces in the literature. In particular, Matthews [4] introduced the notion of partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. After that, fixed point results in partial metric spaces have been studied by many other authors.

References [5–31] are some works in this line of research. The existence of several connections between partial metrics and topological aspects of domain theory were pointed out in, *e.g.*, [11, 24, 32–34].

In this paper we study fixed point results about certain extensions of the notion of Ćirić quasicontraction to the setting of partial metric spaces, and we give some generalized versions of the fixed point theorem of Matthews as well as the main result of Ilić *et al.* [35]. The theory is illustrated by some examples.

Throughout this paper, the letters  $\mathbb{R}, \mathbb{R}^+, \mathbb{Q}, \mathbb{N}$  denote the sets of real numbers, nonnegative real numbers, rational numbers and positive integers, respectively.

Let us recall [4] that a mapping  $p : X \times X \rightarrow \mathbb{R}^+$ , where  $X$  is a nonempty set, is said to be a *partial metric on  $X$*  if for any  $x, y, z \in X$ , the following four conditions hold true:

(P1)  $x = y$  if and only if  $p(x, x) = p(y, y) = p(x, y)$ ,

(P2)  $p(x, x) \leq p(x, y)$ ,

(P3)  $p(x, y) = p(y, x)$ ,

(P4)  $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ .

The pair  $(X, p)$  is then called a *partial metric space*. A sequence  $\{x_m\}_{m=0}^\infty$  of elements of  $X$  is called *p-Cauchy* if the limit  $\lim_{m,n} p(x_n, x_m)$  exists and is finite. The partial metric space  $(X, p)$  is called *complete* if for each *p-Cauchy* sequence  $\{x_m\}_{m=0}^\infty$ , there is some  $z \in X$  such that

$$p(z, z) = \lim_n p(z, x_n) = \lim_{n,m} p(x_n, x_m). \tag{1.2}$$

If  $(X, p)$  is a partial metric space, then  $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ ,  $x, y \in X$ , is a metric on  $X$ ,  $\{x_n\}_{n \geq 1}$  converges to  $z \in X$  with respect to  $p^s$  if and only if (1.2) holds, and  $(X, p)$  is a complete partial metric space if and only if  $(X, p^s)$  is a complete metric space (see [4, 21]).

A sequence  $x_n$  in a partial metric space  $(X, p)$  is called 0-Cauchy [27] if  $\lim_{m,n} p(x_n, x_m) = 0$ . We say that  $(X, p)$  is 0-complete if every 0-Cauchy sequence in  $X$  converges, with respect to  $p$ , to a point  $x \in X$  such that  $p(x, x) = 0$ . Note that every 0-Cauchy sequence in  $(X, p)$  is Cauchy in  $(X, p^s)$ , and that every complete partial metric space is 0-complete. A paradigm for partial metric spaces is the pair  $(X, p)$  where  $X = \mathbb{Q} \cap [0, +\infty)$  and  $p(x, y) = \max\{x, y\}$  for  $x, y \geq 0$ , which provides an example of a 0-complete partial metric space which is not complete.

## 2 Auxiliary results

In this section we define three extensions of the notion of Ćirić quasicontraction to the context partial metric spaces and establish a few auxiliary results that will be used in the next, main section.

**Definition 2.1** Let  $(X, p)$  be a partial metric space and  $T : X \rightarrow X$  be a mapping. If for some  $\alpha \in [0, 1)$  and all  $x, y \in X$ , there holds

$$p(Tx, Ty) \leq \max\{\alpha M(x, y), p(x, x), p(y, y)\}, \tag{2.1}$$

where  $M(x, y) := \max\{p(x, Tx), p(x, Ty), p(x, y), p(Tx, y), p(y, Ty)\}$ , then we shall say that  $T$  is a  $p_1$ -quasicontraction. If for all  $x, y \in X$ , the stronger condition

$$p(Tx, Ty) \leq \max\left\{\alpha M(x, y), \frac{p(x, x) + p(y, y)}{2}\right\} \tag{2.2}$$

is satisfied, we shall call  $T$  a  $p_2$ -quasicontraction. If the even stronger condition

$$p(Tx, Ty) \leq \alpha M(x, y) \tag{2.3}$$

holds for all  $x, y \in X$ , then we shall say that  $T$  is a  $p$ -quasicontraction.

**Lemma 2.1** Let  $(X, p)$  be a partial metric space,  $T : X \rightarrow X$  be a  $p_1$ -quasicontraction,  $x \in X$  and  $n, m$  and  $k$  be integers such that  $m \geq n \geq k \geq 1$ .

If  $p(T^n x, T^m x) > p(T^i x, T^j x)$  for all  $i \in \{n - k, \dots, n - 1\} \cup \{m - k, \dots, m - 1\}$ , then for some  $i_0, j_0 \in \{n - k, \dots, m\}$ , there must hold

$$p(T^n x, T^m x) \leq \alpha^k p(T^{i_0} x, T^{j_0} x).$$

*Proof* This is an easy induction on  $k$ . For  $k = 1$  this follows from (2.1) and the assumptions made. Now suppose that the assertion holds for some  $k \geq 1$  and all  $m \geq n \geq k$ , and let  $m, n \geq k + 1$  be such that we have  $p(T^n x, T^m x) > p(T^i x, T^j x)$  for all  $i \in \{n - (k + 1), \dots, n - 1\} \cup \{m - (k + 1), \dots, m - 1\}$ .

From  $p(T^n x, T^m x) > \max\{p(T^{n-1} x, T^{n-1} x), p(T^{m-1} x, T^{m-1} x)\}$  and (2.1) we see that there must hold  $p(T^n x, T^m x) \leq \alpha p(T^{n_1} x, T^{m_1} x)$  for some  $(n_1, m_1) \in \{(n - 1, n), (m - 1, m), (n - 1, m - 1), (n - 1, m), (n, m - 1)\}$  if  $n < m$ , or for some  $(n_1, m_1) \in \{(n - 1, n), (n - 1, n - 1)\}$  if  $n = m$ ; in either case, we have  $m_1 \geq n_1 \geq n - 1 \geq k$ .

There cannot be any  $i \in \{n_1 - k, \dots, n_1 - 1\} \cup \{m_1 - k, \dots, m_1 - 1\} \subseteq \{n - (k + 1), \dots, n - 1\} \cup \{m - (k + 1), \dots, m - 1\}$  such that  $p(T^{n_1} x, T^{m_1} x) \leq p(T^i x, T^j x)$  because this would imply  $p(T^n x, T^m x) \leq p(T^i x, T^j x)$ . Thus, by the induction hypothesis, there are  $i_0, j_0 \in \{n_1 - k, \dots, m_1\} \subseteq \{n - (k + 1), \dots, m\}$  such that  $p(T^{n_1} x, T^{m_1} x) \leq \alpha^k p(T^{i_0} x, T^{j_0} x)$ , whence  $p(T^n x, T^m x) \leq \alpha^{k+1} p(T^{i_0} x, T^{j_0} x)$ .  $\square$

**Lemma 2.2** Let  $(X, p)$  be a partial metric space,  $T : X \rightarrow X$  be a  $p_1$ -quasicontraction,  $x \in X$  and  $N_x := \frac{1}{(1-\alpha)}p(x, Tx) + p(x, x)$ . Then

(1) for all  $i, j \geq 0$ , there holds

$$p(T^i x, T^j x) \leq N_x, \tag{2.4}$$

(2) if we put  $S_x := \sup_{i \geq 0} p(T^i x, T^i x)$ , then for some  $i \geq 0$  we have

$$p(T^i x, T^i x) = S_x. \tag{2.5}$$

*Proof* (1) Fix  $m \geq 1$  and put  $d := \max\{p(T^i x, T^j x) : 0 \leq i, j \leq m\}$ .

It is easy to see that there must be some  $i \in \{0, 1, \dots, m\}$  with  $d = p(x, T^i x)$ . Indeed, take any  $i, j \in \{0, 1, \dots, m\}$  such that  $d = p(T^i x, T^j x)$ . If  $0 \in \{i, j\}$ , we are done. Otherwise  $i, j \geq 1$ , so  $p(T^i x, T^j x) \leq \max\{\alpha p(T^{i_1} x, T^{j_1} x), p(T^{i-1} x, T^{i-1} x), p(T^{j-1} x, T^{j-1} x)\}$  for some  $i_1, j_1 \in \{0, 1, \dots, m\}$ . Therefore either  $d \leq \alpha p(T^{i_1} x, T^{j_1} x) \leq \alpha d$ , i.e.,  $d = 0 = p(x, T^k x)$  for any  $0 \leq k \leq m$ , or  $d \leq \max\{p(T^{i-1} x, T^{i-1} x), p(T^{j-1} x, T^{j-1} x)\} \leq d$ , thus  $d \in \{p(x, T^{i-1} x), p(x, T^{j-1} x)\}$ .

Now take any  $i_0 \in \{0, 1, \dots, m\}$  such that  $d = p(x, T^{i_0} x)$ . If  $i_0 = 0$ , then  $d \leq N_x$  follows trivially. If  $i_0 \geq 1$ , then for some  $i_1, i_2 \in \{0, 1, \dots, m\}$ ,

$$\begin{aligned} d &= p(x, T^{i_0} x) \leq p(x, Tx) + p(Tx, T^{i_0} x) \\ &\leq p(x, Tx) + \max\{\alpha p(T^{i_1} x, T^{i_2} x), p(x, x), p(T^{i_0-1} x, T^{i_0-1} x)\}, \end{aligned}$$

so  $d \leq p(x, Tx) + \max\{\alpha d, p(T^l x, T^l x)\}$ , where  $l$  is the least integer such that  $0 \leq l \leq m$  and  $p(T^l x, T^l x) = \max\{p(T^i x, T^i x) : 0 \leq i \leq m\}$ . So it must either be  $d \leq \frac{d(x, Tx)}{1-\alpha} \leq N_x$  or  $d \leq p(x, Tx) + p(T^l x, T^l x)$ . The latter possibility in the case  $l = 0$  gives  $d \leq N_x$  directly, and if  $l \geq 1$ , then, since the minimality of  $l$  implies  $p(T^l x, T^l x) > p(T^{l-1} x, T^{l-1} x)$ , we must have for some  $j_1, j_2 \in \{l - 1, l\}$  that  $d \leq p(x, Tx) + \alpha p(T^{j_1} x, T^{j_2} x) \leq p(x, Tx) + \alpha d$  and again  $d \leq N_x$ .

(2) By part (1) we have  $S_x < \infty$ . If  $S_x = 0$ , then there is nothing to prove. Thus let  $S_x > 0$ . Suppose to the contrary that for all  $i \geq 0$ ,

$$p(T^i x, T^i x) < S_x. \tag{2.6}$$

Take  $k \in \mathbb{N}$  such that  $\alpha^k N_x < \frac{S_x}{2}$  and  $i_0 \in \mathbb{N}$  such that  $p(T^{i_0} x, T^{i_0} x) > \frac{S_x}{2}$ . By (2.6) we may assume that  $i_0 \geq k$ . Also by (2.6) there must be some  $n \in \mathbb{N}$  such that  $n > i_0$  and  $p(T^n x, T^n x) > p(T^{i_0} x, T^{i_0} x)$  for all  $0 \leq i \leq i_0$ . Denote by  $n_0$  the least such integer.

Since  $p(T^{n_0} x, T^{n_0} x) > p(T^{i_0} x, T^{i_0} x)$  for all  $0 \leq i < n_0$  and since  $n_0 \geq k$ , Lemma 2.1 implies that for some  $j_1, j_2 \geq n_0 - k$ , we have  $p(T^{n_0} x, T^{n_0} x) \leq \alpha^k p(T^{j_1} x, T^{j_2} x) \leq \alpha^k N_x < \frac{S_x}{2}$ . But  $\frac{S_x}{2} < p(T^{i_0} x, T^{i_0} x) < p(T^{n_0} x, T^{n_0} x)$ , a contradiction.  $\square$

**Lemma 2.3** *Let  $(X, p)$  be a partial metric space,  $T : X \rightarrow X$  be a  $p_1$ -quasicontraction,  $x \in X$  and  $r_x := \limsup_i p(T^i x, T^i x)$ . Then*

$$\lim_{n,m} p(T^n x, T^m x) = r_x. \tag{2.7}$$

*Proof* Notice first that  $r_x < \infty$  by Lemma 2.2. The lemma easily follows from the following two claims:

(1) for each  $\epsilon > 0$ , there is some  $k \in \mathbb{N}$  such that for all  $n, m \geq k$ , we have

$$p(T^n x, T^m x) < r_x + \epsilon;$$

and

(2) if  $r_x > 0$ , then

$$p(T^i x, T^i x) \geq p(T^{i+1} x, T^{i+1} x) \text{ holds for all but finitely many } i \geq 0. \tag{2.8}$$

To prove (1), fix any  $\epsilon > 0$ . From  $r_x = \limsup_i p(T^i x, T^i x) < r_x + \epsilon$ , we see that there is some  $k_0 \in \mathbb{N}$  such that  $\alpha^{k_0} N_x < r_x + \epsilon$  and such that for all  $i \geq k_0$  we have  $p(T^i x, T^i x) < r_x + \epsilon$ .

Let  $m \geq n \geq 2k_0$ . If  $p(T^n x, T^m x) \leq p(T^i x, T^i x)$  for some  $i \geq k_0$ , then  $p(T^n x, T^m x) < r_x + \epsilon$ . Otherwise, as  $n - k_0 \geq k_0$ , by Lemma 2.1 there are  $i_0, j_0 \geq n - k_0$  such that  $p(T^n x, T^m x) \leq \alpha^{k_0} p(T^{i_0} x, T^{j_0} x) \leq \alpha^{k_0} N_x < r_x + \epsilon$ .

To prove (2) suppose  $r_x > 0$ . Take any positive  $\epsilon < \frac{r_x(1-\alpha)}{\alpha}$  and let  $k \in \mathbb{N}$  be as claimed to exist in (1). We will show that for any  $n \geq k$ , it must be that  $p(T^n x, T^n x) = \max\{p(T^i x, T^i x) \mid i \geq n\}$ . Indeed, suppose that for some  $n \geq k$  this were not true. By (2) of Lemma 2.2, there is an integer  $l$  such that  $l \geq n$  and  $p(T^l x, T^l x) = \max\{p(T^i x, T^i x) \mid i \geq n\}$ . Let  $m$  be the least such integer. Then by our assumption  $m - 1 \geq n$ , so, by the choice of  $m$ , we must have  $p(T^m x, T^m x) > p(T^{m-1} x, T^{m-1} x)$ . Hence  $p(T^m x, T^m x) \leq \alpha p(T^{m-1} x, T^{m-1} x) \leq \alpha(r_x + \epsilon)$ . Since by definition of  $r_x$  and  $m$  it must be  $r_x \leq p(T^m x, T^m x)$ , the last inequality now yields  $r_x \leq \alpha(r_x + \epsilon)$ , i.e.,  $\frac{r_x(1-\alpha)}{\alpha} \leq \epsilon$ , a contradiction.  $\square$

**Lemma 2.4** *Let  $(X, p)$  be a partial metric space,  $T : X \rightarrow X$  be a  $p_1$ -quasicontraction and  $x, y \in X$  be such that*

$$\lim_{n,m} p(T^n x, T^m x) = \lim_n p(y, T^n x) = p(y, y).$$

*Then  $p(y, y) = p(y, Ty)$ .*

*Proof* Denote  $\delta_n := p(y, T^n x) - p(T^n x, T^n x)$  and  $\theta_n := p(y, T^{n-1} x) - p(y, y)$  and notice that

$$\begin{aligned} p(y, Ty) &\leq \delta_n + p(T^n x, Ty) \\ &\leq \delta_n + \max\{\alpha p(T^{n-1} x, Ty), \alpha p(T^n x, y), \alpha p(T^{n-1} x, y), \\ &\quad \alpha p(T^{n-1} x, T^n x), \alpha p(y, Ty), p(T^{n-1} x, T^{n-1} x), p(y, y)\} \end{aligned}$$

implies, using  $p(T^{n-1} x, Ty) \leq \theta_n + p(y, Ty)$ , that

$$\begin{aligned} p(y, Ty) &\leq \frac{\delta_n}{(1-\alpha)} + \max\left\{\frac{\alpha}{(1-\alpha)}\theta_n, \alpha p(T^n x, y), \alpha p(T^{n-1} x, y), \alpha p(T^{n-1} x, T^n x), \right. \\ &\quad \left. p(T^{n-1} x, T^{n-1} x), p(y, y)\right\}. \end{aligned}$$

Now simply take the limit as  $n \rightarrow \infty$  in the above inequality. □

### 3 Main results

In this paper we study three extensions of Ćirić quasicontraction to a partial metric space. For such mappings, we prove fixed point theorems. Among other things, we generalize a recent result of Altun, Sola and Simsek, and we give some generalized versions of the fixed point theorem of Matthews, and the main result of Ćirić is also recovered. The theory is illustrated by some examples.

**Theorem 3.1** *Let  $(X, p)$  be a complete partial metric space and  $T : X \rightarrow X$  be a  $p_1$ -quasicontraction. Then*

- (1) *for each  $x \in X$ , the sequence  $\{T^n x\}_{n \geq 1}$  converges with respect to  $p^s$  to some point  $\dot{x}$  such that  $p(\dot{x}, \dot{x}) = p(\dot{x}, T\dot{x})$ ;*
- (2) *there is some  $u \in X$  such that  $u = Tu$  and  $p(u, u) = \inf_{x \in X} p(\dot{x}, \dot{x})$ .*

*Proof* By Lemmas 2.3 and 2.4 and completeness of  $(X, p)$ , for each  $x \in X$ , there must be some  $\dot{x} \in X$  with  $p(\dot{x}, \dot{x}) = p(\dot{x}, T\dot{x})$  such that

$$p(\dot{x}, \dot{x}) = \lim_n p(\dot{x}, T^n x) = \lim_{n,m} p(T^n x, T^m x) = r_x.$$

Also notice that we must have

$$p(T^n x, T^n x) \geq r_x \quad \text{for all but finitely many } n. \tag{3.1}$$

Denote  $I := \inf_{x \in X} r_x = \inf_{x \in X} p(\dot{x}, \dot{x})$ . Using the construction described bellow, we will find a specific  $u \in X$  such that  $p(u, u) = p(u, Tu) = I$  and later on we prove that we must actually have  $Tu = u$  for that particular point  $u$ . This will complete the proof.

Use (2.7) and (3.1) to find for each  $n \in \mathbb{N}$  some  $x_n \in X$  and  $i_n \in \mathbb{N}$  such that

$$\begin{aligned} \{p(T^{i_n-1} x_n, T^{i_n-1} x_n), p(T^{i_n-1} x_n, T^{i_n} x_n), \\ p(T^{i_n} x_n, T^{i_n} x_n), p(T^{i_n} x_n, T^{i_n+1} x_n)\} \subseteq \left[ I, I + \frac{1}{n} \right). \end{aligned} \tag{3.2}$$

Set  $\delta'_{n,m} := [p(T^{in}x_n, T^{in+1}x_n) - p(T^{in+1}x_n, T^{in+1}x_n)] + [p(T^{im}x_m, T^{im+1}x_m) - p(T^{im+1}x_m, T^{im+1}x_m)]$  and  $\theta'_n := p(T^{in}x_n, T^{in+1}x_n) - p(T^{in}x_n, T^{in}x_n)$ .

Now

$$\begin{aligned} & p(T^{in}x_n, T^{im}x_m) \\ & \leq \delta'_{n,m} + p(T^{in+1}x_n, T^{im+1}x_m) \\ & \leq \delta'_{n,m} + \max\{\alpha p(T^{in}x_n, T^{im+1}x_m), \alpha p(T^{in+1}x_n, T^{im}x_m), \alpha p(T^{in}x_n, T^{im}x_m), \\ & \quad \alpha p(T^{in}x_n, T^{in+1}x_n), \alpha p(T^{im}x_m, T^{im+1}x_m), p(T^{in}x_n, T^{in}x_n), p(T^{im}x_m, T^{im}x_m)\} \end{aligned}$$

gives, using  $p(T^{in}x_n, T^{im+1}x_m) \leq p(T^{in}x_n, T^{im}x_m) + \theta'_m$  and similarly  $p(T^{im}x_m, T^{in+1}x_n) \leq p(T^{im}x_m, T^{in}x_n + \theta'_n)$ ,

$$\begin{aligned} I & \leq p(T^{in}x_n, T^{in}x_n) \leq p(T^{in}x_n, T^{im}x_m) \\ & \leq \frac{\delta'_{n,m}}{1-\alpha} + \max\left\{\frac{\alpha\theta'_m}{1-\alpha}, \frac{\alpha\theta'_n}{1-\alpha}, I + \frac{1}{\min\{n, m\}}\right\}. \end{aligned}$$

So  $\lim_{n,m} p(T^{in}x_n, T^{im}x_m) = I$ , and thus there is some  $u \in X$  such that

$$p(u, u) = \lim_{n,m} p(T^{in}x_n, T^{im}x_m) = \lim_n p(u, T^{in}x_n) = I = \inf_{x \in X} r_x. \tag{3.3}$$

Let us show that

$$p(u, u) = p(u, Tu). \tag{3.4}$$

Set  $\lambda_n := p(u, T^{in}x_n) - p(T^{in}x_n, T^{in}x_n)$  and  $\mu_n := p(u, T^{in-1}x_n) - p(u, u)$ . From

$$\begin{aligned} p(u, Tu) & \leq \lambda_n + p(T^{in}x_n, Tu) \\ & \leq \lambda_n + \max\{\alpha p(T^{in-1}x_n, Tu), \alpha p(T^{in}x_n, u), \alpha p(T^{in-1}x_n, u), \\ & \quad \alpha p(T^{in-1}x_n, T^{in}x_n), \alpha p(u, Tu), p(T^{in-1}x_n, T^{in-1}x_n), p(u, u)\} \end{aligned}$$

it follows, using  $p(T^{in-1}x_n, Tu) \leq p(u, Tu) + \mu_n$ , that

$$\begin{aligned} p(u, Tu) & \leq \frac{\lambda_n}{1-\alpha} + \max\left\{\frac{\alpha\mu_n}{1-\alpha}, \alpha p(T^{in}x_n, u), \alpha p(T^{in-1}x_n, u), \right. \\ & \quad \left. \alpha p(T^{in-1}x_n, T^{in}x_n), p(T^{in-1}x_n, T^{in-1}x_n), p(u, u)\right\}, \end{aligned}$$

so we must have  $p(u, Tu) \leq p(u, u)$ .

We now prove by induction on  $n \in \mathbb{N} \cup \{0\}$  that

$$\lim_k p(T^{ik}x_k, T^nu) = p(u, u) = I. \tag{3.5}$$

For  $n = 0$ , this is just (3.3).

Now suppose that (3.5) holds for some  $n \geq 0$ . We have

$$\begin{aligned} I &\leq p(T^{im}x_m, T^{n+1}u) \\ &\leq \max\{\alpha p(T^{im-1}x_m, T^{n+1}u), \alpha p(T^{im}x_m, T^n u), \alpha p(T^{im-1}x_m, T^n u), \\ &\quad \alpha p(T^{im-1}x_m, T^{im}x_m), \alpha p(T^n u, T^{n+1}u), p(T^{im-1}x_m, T^{im-1}x_m), p(T^n u, T^n u)\}. \end{aligned}$$

From this, using

$$\begin{aligned} p(T^{im-1}x_m, T^{n+1}u) &\leq p(T^{im}x_m, T^{n+1}u) + [p(T^{im}x_m, T^{im-1}x_m) - p(T^{im}x_m, T^{im}x_m)] \\ &\leq p(T^{im}x_m, T^{n+1}u) + \frac{1}{m} \quad (\text{here we recall (3.2)}) \end{aligned}$$

and

$$\begin{aligned} p(T^{im-1}x_m, T^n u) &\leq p(T^{im}x_m, T^n u) + [p(T^{im}x_m, T^{im-1}x_m) - p(T^{im}x_m, T^{im}x_m)] \\ &\leq p(T^{im}x_m, T^n u) + \frac{1}{m} \end{aligned}$$

and

$$p(T^n u, T^{n+1}u) \leq p(T^{im}x_m, T^{n+1}u) + [p(T^n u, T^{im}x_m) - p(T^{im}x_m, T^{im}x_m)]$$

and

$$\begin{aligned} p(T^n u, T^n u) &\leq p(u, u), \quad \text{which follows after taking the limit as } k \rightarrow \infty \text{ in} \\ p(T^n u, T^n u) &\leq 2p(T^n u, T^{ik}x_k) - p(T^{ik}x_k, T^{ik}x_k), \end{aligned}$$

we deduce that

$$\begin{aligned} I &\leq p(T^{im}x_m, T^{n+1}u) \\ &\leq \max\left\{\frac{\alpha}{(1-\alpha)m}, \alpha\left[p(T^{im}x_m, T^n u) + \frac{1}{m}\right], \right. \\ &\quad \left. \frac{\alpha}{(1-\alpha)}[p(T^{im}x_m, T^n u) - p(T^{im}x_m, T^{im}x_m)], I + \frac{1}{m}, p(u, u)\right\}. \end{aligned}$$

The last inequality, in view of the induction hypothesis and (3.3), immediately gives

$$\lim_m p(T^{im}x_m, T^{n+1}u) = p(u, u) = I.$$

We finally prove  $Tu = u$ . By (3.4) we only need to show  $p(Tu, Tu) = p(u, u)$ . For a proof by contradiction, assume this is not the case. Then  $p(Tu, Tu) < p(u, u) = I$ .

Set  $y := Tu$ . By (3.1) and  $r_y \geq I$ , there must be some  $j \geq 0$  such that  $p(T^j y, T^j y) \geq I$ . Let  $m$  be the least such  $j$ . By our assumption, we must have  $m \geq 1$ . Then  $p(T^j y, T^j y) < p(T^m y, T^m y)$  for all  $0 \leq j < m$ . Thus by Lemma 2.1 there must be some  $i, j \in \{0, 1, \dots, m\}$  such that  $p(T^m y, T^m y) \leq \alpha^m p(T^i y, T^j y)$ . So

$$I \leq p(T^m y, T^m y) \leq \alpha^m p(T^{i+1}u, T^{j+1}u).$$

We have  $p(T^{i+1}u, T^{j+1}u) \leq I$ , which follows by (3.3) and (3.5) after taking the limit as  $n \rightarrow \infty$  in  $p(T^{i+1}u, T^{j+1}u) \leq p(T^{i+1}u, T^{i_n}x_n) + p(T^{i_n}x_n, T^{j+1}u) - p(T^{i_n}x_n, T^{i_n}x_n)$ . But  $m \geq 1$  now yields  $I \leq \alpha I$ , i.e.,  $0 = I > p(Tu, Tu)$ , a contradiction.  $\square$

**Theorem 3.2** *Let  $(X, p)$  be a complete partial metric space and  $T : X \rightarrow X$  be a  $p_2$ -quasicontraction. Then there is a unique fixed point  $u \in X$  of  $T$ . For each  $x \in X$ , the sequence  $\{T^n x\}_{n \geq 1}$  converges with respect to  $p^s$  to some point  $\hat{x}$  such that  $p(\hat{x}, \hat{x}) = p(\hat{x}, T\hat{x})$ , and there is the equality  $p(u, u) = \inf_{x \in X} p(\hat{x}, \hat{x})$ .*

*Proof* By Theorem 3.1 it remains to verify the uniqueness of the fixed point, so let  $u, v \in X$  be such that  $Tu = u$  and  $Tv = v$ . By (2.2) we have

$$p(u, v) = p(Tu, Tv) \leq \max \left\{ \alpha p(u, v), \frac{p(u, u) + p(v, v)}{2} \right\}.$$

Hence either  $p(u, v) = 0$  or  $0 \geq 2p(u, v) - p(u, u) - p(v, v) = p^s(u, v)$  and the assertion follows.  $\square$

**Theorem 3.3** *Let  $(X, p)$  be a 0-complete partial metric space and  $T : X \rightarrow X$  be a  $p$ -quasicontraction. Then there is a unique fixed point  $u \in X$  of  $T$ . Furthermore, we have  $p(u, u) = 0$  and for each  $x \in X$  the sequence  $\{T^n x\}_{n \geq 1}$  converges to  $u$  with respect to  $p^s$ .*

*Proof* By Lemma 2.3 we have  $\lim_{n,m} p(T^n x, T^m x) = r_x$ . Also  $r_x < \infty$  by Lemma 2.2.

If we had  $r_x > 0$ , then there would be some positive  $\epsilon < \frac{r_x(1-\alpha)}{1+\alpha}$  and  $n \in \mathbb{N}$  such that  $p(T^i x, T^j x) \in (r_x - \epsilon, r_x + \epsilon)$  for all  $i, j \in \{n-1, n\}$ . Thus by (2.3) there would be some  $i, j \in \{n-1, n\}$  such that

$$r_x - \epsilon < p(T^n x, T^n x) \leq \alpha p(T^i x, T^j x) \leq \alpha(r_x + \epsilon),$$

i.e.,  $\frac{r_x(1-\alpha)}{1+\alpha} < \epsilon$ , a contradiction.

So  $r_x = 0$  and thus by 0-completeness of  $(X, p)$  there is some  $u \in X$  such that  $\lim_{n,m} p(T^n x, T^m x) = \lim_n p(u, T^n x) = p(u, u) = 0$ . But by Lemma 2.4 we have  $p(u, Tu) = p(u, u) = 0$  so  $Tu = u$ . The argument for uniqueness of the fixed point is standard.  $\square$

**Remark 3.1** Recently a very interesting paper by Haghi, Rezapour and Shahzad [36] showed up in which the authors associated to each partial metric space  $(X, p)$  a metric space  $(X, d)$  by setting  $d(x, x) = 0$  and  $d(x, y) = p(x, y)$  if  $x \neq y$  and proved that  $(X, p)$  is 0-complete if and only if  $(X, d)$  is complete. They then proceeded to demonstrate how using the associated metric  $d$  some of the fixed point results in partial metric spaces can easily be deduced from the corresponding known results in metric spaces.

Let us point out that these considerations can apply neither to  $p_1$ -quasicontractions nor to  $p_2$ -quasicontractions, since the terms  $p(x, x)$  and  $p(y, y)$  on the right-hand side of (2.1) and  $\frac{1}{2}(p(x, x) + p(y, y))$  on the right-hand side of (2.2) do not get multiplied by  $\alpha$ . Thus Theorems 3.1 and 3.2 cannot follow from the result of Ćirić they generalize.

On the other hand, using the approach of Haghi, Rezapour and Shahzad, we now show how Theorem 3.3 can be directly deduced from Ćirić's result [1].



Given the assumptions of Theorem 3.3, let  $d$  be defined as said above. So  $(X, d)$  is a complete metric space (see Proposition 2.1 of [36]). Observe that we have  $d(x, y) \leq p(x, y)$  for all  $x, y \in X$ . For  $x, y \in X$ , set

$$S(x, y) = \{(x, y), (x, Tx), (x, Ty), (y, Ty), (y, Tx)\}$$

and also  $M_p(x, y) = \max\{p(a, b) \mid (a, b) \in S(x, y)\}$  and  $M_d(x, y) = \max\{d(a, b) \mid (a, b) \in S(x, y)\}$ . We thus have that

$$p(Tx, Ty) \leq \alpha M_p(x, y)$$

for all  $x, y \in X$ . We check that for all  $x, y \in X$  it holds  $d(Tx, Ty) \leq \alpha M_d(x, y)$ , so that the main result from [1] can immediately be applied. Since in the case  $x = y$  the inequality trivially holds, suppose  $x \neq y$ . So  $p(x, y) = d(x, y)$ .

Since  $d(Tx, Ty) \leq p(Tx, Ty) \leq \alpha M_p(x, y)$ , it suffices to show that  $M_p(x, y) \leq M_d(x, y)$ . Let  $(c, e) \in S(x, y)$  be such that  $M_p(x, y) = p(c, e)$ . If  $c \neq e$ , then  $M_p(x, y) = p(c, e) = d(c, e) \leq M_d(x, y)$ . If  $c = e$ , then, since  $c \in \{x, y\}$ , it follows that  $M_p(x, y) = p(c, c) \leq \max\{p(x, x), p(y, y)\} \leq p(x, y) = d(x, y) \leq M_d(x, y)$ .

**Remark 3.2** Even though the results of Haghi *et al.* can deduce the same fixed point as the corresponding partial metric fixed point result, using the partial metric version computers evaluate faster since many nonsense terms are omitted. This is very important in computer science due to its cost and explains the vast body of partial metric fixed point results found in literature.

Now we give corollaries of the above theorems.

**Corollary 3.1** ([35]) *Let  $(X, p)$  be a complete partial metric space,  $\alpha \in [0, 1)$  and  $T : X \rightarrow X$  be a given mapping. Suppose that for each  $x, y \in X$ , the following condition holds:*

$$p(Tx, Ty) \leq \max\{\alpha p(x, y), p(x, x), p(y, y)\}. \tag{3.6}$$

Then

- (1) the set  $X_p := \{x \in X \mid p(x, x) = \inf_{y \in X} p(y, y)\}$  is nonempty;
- (2) there is a unique  $u \in X_p$  such that  $Tu = u$ ;
- (3) for each  $x \in X_p$ , the sequence  $\{T^n x\}_{n \geq 1}$  converges with respect to the metric  $p^s$  to  $u$ .

*Proof* Put  $\rho_p := \inf_{x \in X} p(x, x)$  and  $I := \inf_{x \in X} p(\dot{x}, \dot{x})$ . Note that (3.6) implies  $p(Tx, Tx) \leq p(x, x)$  for all  $x \in X$ . So, from  $p(\dot{x}, \dot{x}) = \lim_n p(T^n x, T^n x)$  and  $p(T^{n+1} x, T^{n+1} x) \leq p(T^n x, T^n x)$ ,  $n \geq 0$ , we see that  $p(\dot{x}, \dot{x}) \leq p(T^0 x, T^0 x) = p(x, x)$ . Thus  $\rho_p = I$ . By Theorem 3.1, there is some  $u \in X$  with  $Tu = u$  and  $p(u, u) = \rho_p$ . This means that  $u \in X_p \neq \emptyset$ . Now let  $x \in X_p$  be arbitrary. Since  $\rho_p \leq p(\dot{x}, \dot{x}) \leq p(T^n x, T^n x) \leq p(x, x) = \rho_p$ , for all  $n \in \mathbb{N}$ , we have  $p(\dot{x}, \dot{x}) = p(T^n x, T^n x) = \rho_p = p(u, u)$ , and so  $\lim_{n, m} p(T^n x, T^m x) = p(u, u)$ . Also, from  $p(u, T^{n+1} x) \leq \{\alpha p(u, T^n x), \rho_p\}$  using  $p(u, T^n x) \leq p(u, T^{n+1} x) + p(T^{n+1} x, T^n x) - p(T^{n+1} x, T^{n+1} x)$ , it follows

$$\rho_p \leq p(u, T^{n+1} x) \leq \left\{ \frac{\alpha}{1 - \alpha} [p(T^{n+1} x, T^n x) - p(T^{n+1} x, T^{n+1} x)], \rho_p \right\}.$$

Hence  $\lim_n p(u, T^n x) = \rho_p = p(u, u)$ .

Finally, if  $v \in X_p$  is a fixed point, then by the preceding discussion  $p(u, u) = \lim_n p(T^n v, T^n v) = \lim_n p(u, T^n v)$ , i.e.,  $p(u, u) = p(v, v) = p(u, v)$  so  $u = v$ .  $\square$

**Corollary 3.2** *Let  $(X, p)$  be a complete partial metric space,  $\alpha \in [0, 1)$  and  $T : X \rightarrow X$  be a given mapping. Suppose that for each  $x, y \in X$ , the following condition holds:*

$$p(Tx, Ty) \leq \max \left\{ \alpha p(x, y), \frac{p(x, x) + p(y, y)}{2} \right\}. \tag{3.7}$$

*Then there is a unique  $z \in X$  such that  $Tz = z$ . Furthermore,  $z \in X_p$  and for each  $x \in X_p$ , the sequence  $\{T^n x\}_{n \geq 1}$  converges with respect to the metric  $p^s$  to  $z$ .*

As a corollary we obtain the already mentioned result of Matthews (see also Corollary 2 of [8] and [21]). Let us remark that the result of Matthews is for a complete partial metric space, but it is true for a 0-complete partial metric space.

**Corollary 3.3** (Matthews [4]) *Let  $(X, p)$  be a 0-complete partial metric space,  $\alpha \in [0, 1)$  and  $T : X \rightarrow X$  be a given mapping. Suppose that for each  $x, y \in X$ , the following condition holds:*

$$p(Tx, Ty) \leq \alpha p(x, y). \tag{3.8}$$

*Then there is a unique  $z \in X$  such that  $Tz = z$ . Also  $p(z, z) = 0$  and for each  $x \in X$  the sequence  $\{T^n x\}_{n \geq 1}$  converges with respect to the metric  $p^s$  to  $z$ .*

**Remark 3.3** In the case  $p = \rho$  is a metric, by Theorem 3.3, the main result of Ćirić [1] is recovered. Theorem 3.3 also implies Corollaries 1-4 of [8], and the next Hardy and Rogers type [37] fixed point result. This result, under some extra conditions, was proved as one of the main results, Theorem 2 of [8].

**Corollary 3.4** *Let  $(X, p)$  be a 0-complete partial metric space,  $a, b, c, d, e \geq 0$ ,  $0 \leq a + b + c + d + e < 1$ , and  $T : X \rightarrow X$  be a given mapping. Suppose that for each  $x, y \in X$ , the following condition holds:*

$$p(Tx, Ty) \leq ap(x, y) + bp(x, Tx) + cp(x, Ty) + dp(Tx, y) + ep(y, Ty). \tag{3.9}$$

*Then there is a unique  $z \in X$  such that  $Tz = z$ . Also  $p(z, z) = 0$  and for each  $x \in X$  the sequence  $\{T^n x\}_{n \geq 1}$  converges with respect to the metric  $p^s$  to  $z$ .*

**Example 3.1** Let  $X := [0, 1] \cup [2, 3]$  and define  $p : X^2 \rightarrow \mathbb{R}$  by

$$p(x, y) = \begin{cases} \max\{x, y\}, & \{x, y\} \cap [2, 3] \neq \emptyset, \\ |x - y|, & \{x, y\} \subseteq [0, 1]. \end{cases}$$

Then  $(X, p)$  is a complete partial metric space. Define  $T : X \rightarrow X$  by

$$Tx = \begin{cases} \frac{x+1}{2}, & 0 \leq x \leq 1, \\ 1, & x = 2, \\ \frac{2+x}{2}, & 2 < x \leq 3. \end{cases}$$

If  $2 < x \leq y \leq 3$ , then it is easy to see that  $p(Tx, Ty) = \frac{2+y}{2}$  and  $M(x, y) = y$ . Given any  $\lambda \in [0, 1)$ , we have that  $p(Tx, Ty) > \lambda M(x, y)$  holds for all  $2 < x \leq y \leq 3$  if  $\lambda \leq \frac{1}{2}$ , i.e., for all  $2 < x \leq y < \frac{2}{2\lambda-1}$  if  $\lambda > \frac{1}{2}$ . On the other hand, we have that

$$p(Tx, Ty) \leq \frac{1}{2}p(x, y), \quad \{x, y\} \subseteq [0, 1]$$

and

$$p(Tx, Ty) \leq \frac{p(x, x) + p(y, y)}{2}, \quad \{x, y\} \cap [2, 3] \neq \emptyset.$$

Thus  $T$  is a  $p_2$ -quasicontraction on  $X$  which is not a  $p$ -quasicontraction. By Theorem 3.2, there is a unique fixed point  $z = 1$ . Also we have  $p(1, 1) = 0 = \min\{p(x, x) \mid x \in X\}$ .

**Example 3.2** Denote by  $X_\infty$  the set of all sequences  $x : \mathbb{N} \rightarrow \mathbb{N}$  and for  $n \in \mathbb{N}$  by  $X_n$  the set of all  $n$ -tuples  $x : \{1, \dots, n\} \rightarrow \mathbb{N}$  of positive integers. Put  $X := X_\infty \cup \bigcup_{n \in \mathbb{N}} X_n$ . For  $x, y \in X$ , set

$$I(x, y) = \{i \in \mathbb{N} \cup \{0\} \mid [j \in \text{dom}(x) \cap \text{dom}(y) \wedge j \leq i] \Rightarrow x(j) = y(j)\}$$

and define  $p(x, y) := \inf\{\frac{1}{2^i} \mid i \in I(x, y)\}$  (thus if  $x(1) \neq y(1)$ , then  $I(x, y) = \{0\}$  and  $p(x, y) = 1$ ). Here ‘dom( $x$ )’ stands for the domain of the function  $x$ . Then  $(X, p)$  is a partial metric space (see [4]) and a complete one as can easily be verified.

Define  $T : X \rightarrow X$  by  $Tx = y$ , where:

- $x \in X_\infty \Rightarrow y \in X_\infty$  and  $x \in X_n \Rightarrow y \in X_{n+1}$ ;
- $y(1) = 1, i \in \text{dom}(x) \setminus \{1\} \Rightarrow y(i) = \min\{x(i), x(i-1)\}$  (this condition is vacuous if  $x \in X_1$ ) and if in addition  $x \in X_n$ , then  $y(n+1) = 1$ .

Note that taking, e.g.,  $x = (3, 3, 1)$  and  $y = (3, 3, 2)$ , we have  $Tx = (1, 3, 1, 1)$  and  $Ty = (1, 3, 2, 1)$  so  $p(Tx, Ty) = \frac{1}{4} = p(x, y) > \frac{1}{8} = p(x, x) = p(y, y)$ . Thus the contractive condition of Corollary 3.1 is not satisfied. Nevertheless, there is a unique fixed point of  $T$  - the sequence  $s_\infty : \mathbb{N} \rightarrow \mathbb{N}$  defined by  $s_\infty(i) = 1$  for all  $i \in \mathbb{N}$ . We will show that  $T$  is a  $p$ -quasicontraction. Consider arbitrary  $x, y \in X$ .

*Case 1.* There is a nonnegative integer  $i$  with  $i+1 \in \text{dom}(x) \cap \text{dom}(y)$  such that  $x(i+1) \neq 1 \vee y(i+1) \neq 1$ . Denote by  $k$  the least such nonnegative integer. Thus  $k = 0$  simply means that  $x(1) \neq 1 \vee y(1) \neq 1$  and if  $k \geq 1$ , then for all  $i \in \mathbb{N}$  with  $i \leq k$ , we must have  $x(i) = y(i) = 1$ .

If  $x(k+1) \neq y(k+1)$ , then  $p(x, y) = \frac{1}{2^k}$  so  $p(Tx, Ty) \leq \frac{1}{2^{k+1}} = \frac{1}{2}p(x, y)$ .

If  $x(k+1) = y(k+1)$ , then  $p(x, Tx) = \frac{1}{2^k}$  (because in this case we must have  $x(k+1) = y(k+1) \neq 1$  but  $(Tx)(k+1) = 1$ ). Hence  $p(Tx, Ty) \leq \frac{1}{2^{k+1}} = \frac{1}{2}p(x, Tx)$ .

*Case 2.*  $x(i) = 1$  for all  $i \in \text{dom}(x)$  and  $x \subseteq y$  (meaning  $\text{dom}(x) \subseteq \text{dom}(y)$  and  $x$  is the restriction of  $y$  to the set  $\text{dom}(x)$ ).

If  $x \in X_n$ , then  $p(Tx, Ty) = \frac{1}{2^{n+1}} = \frac{1}{2}p(x, y)$ .

If  $x \in X_\infty$ , then  $x = y = s_\infty = Tx$ , and so  $p(Tx, Ty) = 0$ .

*Case 3.*  $y(i) = 1$  for all  $i \in \text{dom}(y)$  and  $y \subseteq x$ . This reduces to the previous case.

To illustrate the role condition (2.3) plays in ensuring the uniqueness of the fixed point, we modify a bit the definition of the operator  $T$  to obtain the operator  $T_1 : X \rightarrow X$  determined by  $T_1x = y$  if and only if

- $\text{dom}(y) = \text{dom}(x)$ ;
- $y(1) = 1$  and  $i \in \text{dom}(x) \setminus \{1\} \Rightarrow y(i) = \min\{x(i), x(i-1)\}$ .

$T_1$  has infinitely many fixed points - these are exactly the sequence  $s_\infty$  and the restrictions  $s_n = (1, \dots, n)$  of  $s_\infty$  to the sets  $\{1, \dots, n\}$ . Due to the existence of infinitely many fixed points,  $T_1$  cannot be a  $p_2$ -quasicontraction. We verify that  $T_1$  is a  $p_1$ -quasicontraction.

Given arbitrary  $x, y \in X$ , we distinguish three cases exactly as we did with the operator  $T$ .

**Case 1.** There is a nonnegative integer  $i$  with  $i + 1 \in \text{dom}(x) \cap \text{dom}(y)$  such that  $x(i + 1) \neq 1 \vee y(i + 1) \neq 1$ . This is handled exactly as in the corresponding case with the operator  $T$ : if  $k$  is the least such nonnegative integer, then  $p(Tx, Ty) \leq \frac{1}{2}p(x, y)$  or  $p(Tx, Ty) \leq \frac{1}{2}p(x, Tx)$  according to whether  $x(k + 1) \neq y(k + 1)$  or  $x(k + 1) = y(k + 1)$ , respectively.

**Case 2.**  $x(i) = 1$  for all  $i \in \text{dom}(x)$  and  $x \subseteq y$ . If  $x \in X_n$ , then  $T_1x = x$ , and so  $p(T_1x, T_1y) = \frac{1}{2^n}p(x, x)$ . If  $x \in X_\infty$ , then  $x = y = s_\infty = T_1x$ , so  $p(T_1x, T_1y) = 0$ .

**Case 3.**  $y(i) = 1$  for all  $i \in \text{dom}(y)$  and  $y \subseteq x$ . This reduces to the previous case.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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