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A generalized weak contraction principle with applications to coupled coincidence point problems

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Abstract

In this paper we establish some coincidence point results for generalized weak contractions with discontinuous control functions. The theorems are proved in metric spaces with a partial order. Our theorems extend several existing results in the current literature. We also discuss several corollaries and give illustrative examples. We apply our result to obtain some coupled coincidence point results which effectively generalize a number of established results.

MSC: 54H10; 54H25

Keywords: partially ordered set; control function; coincidence point; coupled coincidence point

1 Introduction

In this paper we prove certain coincidence point results in partially ordered metric spaces for functions which satisfy a certain inequality involving three control functions. Two of the control functions are discontinuous. Fixed point theory in partially ordered metric spaces is of relatively recent origin. An early result in this direction is due to Turinici [1], in which fixed point problems were studied in partially ordered uniform spaces. Later, this branch of fixed point theory has developed through a number of works, some of which are in [2–6].

Weak contraction was studied in partially ordered metric spaces by Harjani *et al.* [3]. In a recent result by Choudhury *et al.* [2], a generalization of the above result to a coincidence point theorem has been done using three control functions. Here we prove coincidence point results by assuming a weak contraction inequality with three control functions, two of which are not continuous. The results are obtained under two sets of additional conditions. A fixed point theorem is also established. There are several corollaries and two examples. One of the examples shows that the corollaries are properly contained in their respective theorem. The corollaries are generalizations of several existing works.

We apply our result to obtain some coupled coincidence point results. Coupled fixed theorems and coupled coincidence point theorems have appeared prominently in recent literature. Although the concept of coupled fixed points was introduced by Guo *et al.* [7], starting with the work of Gnana Bhaskar and Lakshmikantham [8], where they established a coupled contraction principle, this line of research has developed rapidly in partially ordered metric spaces. References [9–20] are some examples of these works. There is a

viewpoint from which coupled fixed and coincidence point theorems can be considered as problems in product spaces [21]. We adopt this approach here. Specifically, we apply our theorem to a product of two metric spaces on which a metric is defined from the metric of the original spaces. We establish a generalization of several results. We also discuss an example which shows that our result is an actual improvement over the results it generalizes.

2 Mathematical preliminaries

Let (X, \preceq) be a partially ordered set and $T: X \rightarrow X$. The mapping T is said to be nondecreasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $Tx_1 \preceq Tx_2$ and nonincreasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $Tx_1 \succeq Tx_2$.

Definition 2.1 ([22]) Let (X, \preceq) be a partially ordered set and $T: X \rightarrow X$ and $G: X \rightarrow X$. The mapping T is said to be *G-nondecreasing* if for all $x, y \in X$, $Gx \preceq Gy$ implies $Tx \preceq Ty$ and *G-nonincreasing* if for all $x, y \in X$, $Gx \preceq Gy$ implies $Tx \succeq Ty$.

Definition 2.2 Two self-mappings G and T of a nonempty set X are said to be *commutative* if $GTx = TGx$ for all $x \in X$.

Definition 2.3 ([23]) Two self-mappings G and T of a metric space (X, d) are said to be *compatible* if the following relation holds:

$$\lim_{n \rightarrow \infty} d(GTx_n, TGx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Gx_n = \lim_{n \rightarrow \infty} Tx_n = x$ for some $x \in X$ is satisfied.

Definition 2.4 ([24]) Two self-mappings G and T of a nonempty set X are said to be *weakly compatible* if they commute at their coincidence points; that is, if $Gx = Tx$ for some $x \in X$, then $GTx = TGx$.

Definition 2.5 ([8]) Let (X, \preceq) be a partially ordered set and $F: X \times X \rightarrow X$. The mapping F is said to have the *mixed monotone property* if F is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument; that is, if

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \implies F(x_1, y) \preceq F(x_2, y) \quad \text{for all } y \in X;$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \implies F(x, y_1) \succeq F(x, y_2) \quad \text{for all } x \in X.$$

Definition 2.6 ([17]) Let (X, \preceq) be a partially ordered set, $F: X \times X \rightarrow X$ and $g: X \rightarrow X$. We say that F has the *mixed g-monotone property* if

$$x_1, x_2 \in X, \quad gx_1 \preceq gx_2 \implies F(x_1, y) \preceq F(x_2, y) \quad \text{for all } y \in X;$$

and

$$y_1, y_2 \in X, \quad gy_1 \preceq gy_2 \implies F(x, y_1) \succeq F(x, y_2) \quad \text{for all } x \in X.$$

Definition 2.7 ([8]) An element $(x, y) \in X \times X$ is called a *coupled fixed point* of the mapping $F: X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.8 ([17]) An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the mappings $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.9 ([17]) Let X be a nonempty set. The mappings g and F , where $g: X \rightarrow X$ and $F: X \times X \rightarrow X$, are said to be *commutative* if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.

Definition 2.10 ([12]) Let (X, d) be a metric space. The mappings g and F , where $g: X \rightarrow X$ and $F: X \times X \rightarrow X$, are said to be *compatible* if the following relations hold:

$$\lim_{n \rightarrow \infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0, \quad \lim_{n \rightarrow \infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} F(x_n, y_n) = x$ and $\lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} F(y_n, x_n) = y$ for some $x, y \in X$ are satisfied.

Definition 2.11 Let X be a nonempty set. The mappings g and F , where $g: X \rightarrow X$ and $F: X \times X \rightarrow X$, are said to be *weakly compatible* if they commute at their coupled coincidence points, that is, if $F(x, y) = gx$ and $F(y, x) = gy$ for some $(x, y) \in X \times X$, then $gF(x, y) = F(gx, gy)$ and $gF(y, x) = F(gy, gx)$.

Definition 2.12 ([25]) A function $\psi: [0, \infty) \rightarrow [0, \infty)$ is called an *altering distance function* if the following properties are satisfied:

- (i) ψ is monotone increasing and continuous;
- (ii) $\psi(t) = 0$ if and only if $t = 0$.

In our results in the following sections, we use the following classes of functions.

We denote by Ψ the set of all functions $\psi: [0, \infty) \rightarrow [0, \infty)$ satisfying

- (i $_{\psi}$) ψ is continuous and monotone non-decreasing,
- (ii $_{\psi}$) $\psi(t) = 0$ if and only if $t = 0$;

and by Θ we denote the set of all functions $\alpha: [0, \infty) \rightarrow [0, \infty)$ such that

- (i $_{\alpha}$) α is bounded on any bounded interval in $[0, \infty)$,
- (ii $_{\alpha}$) α is continuous at 0 and $\alpha(0) = 0$.

3 Main results

Let (X, \leq, d) be an ordered metric space. X is called *regular* if it has the following properties:

- (i) if a nondecreasing sequence $\{x_n\} \rightarrow x$ in (X, d) , then $x_n \leq x$ for all $n \geq 0$;
- (ii) if a nonincreasing sequence $\{y_n\} \rightarrow y$ in (X, d) , then $y \leq y_n$ for all $n \geq 0$.

Theorem 3.1 Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T, G: X \rightarrow X$ be two mappings such that G is continuous and nondecreasing, $T(X) \subseteq G(X)$, T is G -nondecreasing with respect to \leq and the pair (G, T) is compatible. Suppose that there exist $\psi \in \Psi$ and $\varphi, \theta \in \Theta$ such that

$$\psi(x) \leq \varphi(y) \implies x \leq y, \tag{3.1}$$

for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \rightarrow t > 0$,

$$\psi(t) - \overline{\lim} \varphi(x_n) + \underline{\lim} \theta(x_n) > 0, \tag{3.2}$$

and for all $x, y \in X$ with $Gx \succeq Gy$

$$\psi(d(Tx, Ty)) \leq \varphi(d(Gx, Gy)) - \theta(d(Gx, Gy)). \tag{3.3}$$

Also, suppose that

- (a) T is continuous, or
- (b) X is regular.

If there exists $x_0 \in X$ such that $Gx_0 \preceq Tx_0$, then G and T have a coincidence point in X .

Proof Let $x_0 \in X$ be such that $Gx_0 \preceq Tx_0$. Since $T(X) \subseteq G(X)$, we can choose $x_1 \in X$ such that $Gx_1 = Tx_0$. Again, we can choose $x_2 \in X$ such that $Gx_2 = Tx_1$. Continuing this process, we construct a sequence $\{x_n\}$ in X such that

$$Gx_{n+1} = Tx_n \quad \text{for all } n \geq 0. \tag{3.4}$$

Since $Gx_0 \preceq Tx_0$ and $Gx_1 = Tx_0$, we have $Gx_0 \preceq Gx_1$, which implies that $Tx_0 \preceq Tx_1$. Now, $Tx_0 \preceq Tx_1$, that is, $Gx_1 \preceq Gx_2$ implies that $Tx_1 \preceq Tx_2$. Again, $Tx_1 \preceq Tx_2$, that is, $Gx_2 \preceq Gx_3$ implies that $Tx_2 \preceq Tx_3$. Continuing this process, we have

$$Gx_0 \preceq Gx_1 \preceq Gx_2 \preceq Gx_3 \preceq \dots \preceq Gx_n \preceq Gx_{n+1} \preceq \dots, \tag{3.5}$$

and

$$Tx_0 \preceq Tx_1 \preceq Tx_2 \preceq Tx_3 \preceq \dots \preceq Tx_n \preceq Tx_{n+1} \preceq \dots. \tag{3.6}$$

Let $R_n = d(Gx_{n+1}, Gx_n)$ for all $n \geq 0$.

Since $Gx_{n+1} \succeq Gx_n$, from (3.3) and (3.4), we have

$$\psi(d(Gx_{n+2}, Gx_{n+1})) = \psi(d(Tx_{n+1}, Tx_n)) \leq \varphi(d(Gx_{n+1}, Gx_n)) - \theta(d(Gx_{n+1}, Gx_n)),$$

that is,

$$\psi(R_{n+1}) \leq \varphi(R_n) - \theta(R_n), \tag{3.7}$$

which, in view of the fact that $\theta \geq 0$, yields $\psi(R_{n+1}) \leq \varphi(R_n)$, which by (3.1) implies that $R_{n+1} \leq R_n$ for all positive integer n , that is, $\{R_n\}$ is a monotone decreasing sequence. Hence there exists an $r \geq 0$ such that

$$R_n = d(Gx_{n+1}, Gx_n) \rightarrow r \quad \text{as } n \rightarrow \infty. \tag{3.8}$$

Taking limit supremum on both sides of (3.7), using (3.8), the property (i_α) of φ and θ , and the continuity of ψ , we obtain

$$\psi(r) \leq \overline{\lim} \varphi(R_n) + \overline{\lim} (-\theta(R_n)).$$

Since $\overline{\lim}(-\theta(R_n)) = -\underline{\lim}\theta(R_n)$, it follows that

$$\psi(r) \leq \overline{\lim}\varphi(R_n) - \underline{\lim}\theta(R_n),$$

that is,

$$\psi(r) - \overline{\lim}\varphi(R_n) + \underline{\lim}\theta(R_n) \leq 0,$$

which by (3.2) is a contradiction unless $r = 0$. Therefore,

$$R_n = d(Gx_{n+1}, Gx_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.9}$$

Next we show that $\{Gx_n\}$ is a Cauchy sequence.

Suppose that $\{Gx_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k , $n(k) > m(k) > k$ and $d(Gx_{m(k)}, Gx_{n(k)}) \geq \epsilon$. Assuming that $n(k)$ is the smallest such positive integer, we get

$$\begin{aligned} n(k) &> m(k) > k, \\ d(Gx_{m(k)}, Gx_{n(k)}) &\geq \epsilon \quad \text{and} \\ d(Gx_{m(k)}, Gx_{n(k)-1}) &< \epsilon. \end{aligned}$$

Now,

$$\epsilon \leq d(Gx_{m(k)}, Gx_{n(k)}) \leq d(Gx_{m(k)}, Gx_{n(k)-1}) + d(Gx_{n(k)-1}, Gx_{n(k)}),$$

that is,

$$\epsilon \leq d(Gx_{m(k)}, Gx_{n(k)}) \leq \epsilon + d(Gx_{n(k)-1}, Gx_{n(k)}).$$

Letting $k \longrightarrow \infty$ in the above inequality and using (3.9), we have

$$\lim_{k \rightarrow \infty} d(Gx_{m(k)}, Gx_{n(k)}) = \epsilon. \tag{3.10}$$

Again,

$$d(Gx_{m(k)+1}, Gx_{n(k)+1}) \leq d(Gx_{m(k)+1}, Gx_{m(k)}) + d(Gx_{m(k)}, Gx_{n(k)}) + d(Gx_{n(k)+1}, Gx_{n(k)})$$

and

$$d(Gx_{m(k)}, Gx_{n(k)}) \leq d(Gx_{m(k)+1}, Gx_{m(k)}) + d(Gx_{m(k)+1}, Gx_{n(k)+1}) + d(Gx_{n(k)+1}, Gx_{n(k)}).$$

Letting $k \longrightarrow \infty$ in the above inequalities, using (3.9) and (3.10), we have

$$\lim_{k \rightarrow \infty} d(Gx_{m(k)+1}, Gx_{n(k)+1}) = \epsilon. \tag{3.11}$$

As $n(k) > m(k)$, $Gx_{n(k)} \geq Gx_{m(k)}$, from (3.3) and (3.4), we have

$$\begin{aligned} \psi(d(Gx_{n(k)+1}, Gx_{m(k)+1})) &= \psi(d(Tx_{n(k)}, Tx_{m(k)})) \\ &\leq \varphi(d(Gx_{n(k)}, Gx_{m(k)})) - \theta(d(Gx_{n(k)}, Gx_{m(k)})). \end{aligned}$$

Taking limit supremum on both sides of the above inequality, using (3.10), (3.11), the property (i_α) of φ and θ , and the continuity of ψ , we obtain

$$\psi(\epsilon) \leq \overline{\lim} \varphi(d(Gx_{n(k)}, Gx_{m(k)})) + \overline{\lim}(-\theta(d(Gx_{n(k)}, Gx_{m(k)}))).$$

Since $\overline{\lim}(-\theta(d(Gx_{n(k)}, Gx_{m(k)}))) = -\underline{\lim} \theta(d(Gx_{n(k)}, Gx_{m(k)}))$, it follows that

$$\psi(\epsilon) \leq \overline{\lim} \varphi(d(Gx_{n(k)}, Gx_{m(k)})) - \underline{\lim} \theta(d(Gx_{n(k)}, Gx_{m(k)})),$$

that is,

$$\psi(\epsilon) - \overline{\lim} \varphi(d(Gx_{n(k)}, Gx_{m(k)})) + \underline{\lim} \theta(d(Gx_{n(k)}, Gx_{m(k)})) \leq 0,$$

which is a contradiction by (3.2). Therefore, $\{Gx_n\}$ is a Cauchy sequence in X . From the completeness of X , there exists $x \in X$ such that

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Gx_n = x. \tag{3.12}$$

Since the pair (G, T) is compatible, from (3.12), we have

$$\lim_{n \rightarrow \infty} d(GTx_n, TGx_n) = 0. \tag{3.13}$$

Let the condition (a) hold.

By the triangular inequality, we have

$$d(Gx, TGx_n) \leq d(Gx, GTx_n) + d(GTx_n, TGx_n).$$

Taking $n \rightarrow \infty$ in the above inequality, using (3.12), (3.13) and the continuities of T and G , we have $d(Gx, Tx) = 0$, that is, $Gx = Tx$, that is, x is a coincidence point of the mappings G and T .

Next we suppose that the condition (b) holds.

By (3.5) and (3.12), we have $Gx_n \leq x$ for all $n \geq 0$. Using the monotone property of G , we obtain

$$GGx_n \leq Gx. \tag{3.14}$$

As G is continuous and the pair (G, T) is compatible, by (3.12) and (3.13), we have

$$\lim_{n \rightarrow \infty} GGx_n = Gx = \lim_{n \rightarrow \infty} GTx_n = \lim_{n \rightarrow \infty} TGx_n. \tag{3.15}$$

Then

$$d(Tx, Gx) = \lim_{n \rightarrow \infty} d(Tx, TGx_n).$$

Since ψ is continuous, from the above inequality, we obtain

$$\psi(d(Tx, Gx)) = \psi\left(\lim_{n \rightarrow \infty} d(Tx, TGx_n)\right) = \lim_{n \rightarrow \infty} \psi(d(Tx, TGx_n)),$$

which, by (3.3) and (3.14), implies that

$$\psi(d(Tx, Gx)) \leq \lim_{n \rightarrow \infty} [\varphi(d(Gx, GGx_n)) - \theta(d(Gx, GGx_n))].$$

Using (3.15) and the property (ii_α) of φ and θ , we have

$$\psi(d(Tx, Gx)) = 0,$$

which, by the property of ψ , implies that $d(Tx, Gx) = 0$, that is, $Gx = Tx$, that is, x is a coincidence point of the mappings G and T . □

Next we discuss some corollaries of Theorem 3.1. By an example, we show that Theorem 3.1 properly contains all its corollaries.

Every commuting pair (G, T) is also a compatible pair. Then considering (G, T) to be the commuting pair in Theorem 3.1, we have the following corollary.

Corollary 3.1 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T, G: X \rightarrow X$ be two mappings such that G is continuous and nondecreasing, $T(X) \subseteq G(X)$, T is G -nondecreasing with respect to \leq and the pair (G, T) is commutative. Suppose that there exist $\psi \in \Psi$ and $\varphi, \theta \in \Theta$ such that (3.1), (3.2) and (3.3) are satisfied. Also, suppose that*

- (a) T is continuous, or
- (b) X is regular.

If there exists $x_0 \in X$ such that $Gx_0 \leq Tx_0$, then G and T have a coincidence point in X .

Considering ψ to be the identity mapping and $\theta(t) = 0$ for all $t \in [0, \infty)$ in Theorem 3.1, we have the following corollary.

Corollary 3.2 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T, G: X \rightarrow X$ be two mappings such that G is continuous and nondecreasing, $T(X) \subseteq G(X)$, T is G -nondecreasing with respect to \leq and the pair (G, T) is compatible. Suppose that there exists $\varphi \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \rightarrow t > 0$,*

$$\overline{\lim} \varphi(x_n) < t, \tag{3.16}$$

and for all $x, y \in X$ with $Gx \geq Gy$,

$$d(Tx, Ty) \leq \varphi(d(Gx, Gy)). \tag{3.17}$$

Also, suppose that

- (a) T is continuous, or
- (b) X is regular.

If there exists $x_0 \in X$ such that $Gx_0 \leq Tx_0$, then G and T have a coincidence point in X .

Considering $\theta(t) = 0$ for all $t \in [0, \infty)$ and $\varphi(t) = k\psi(t)$ with $k \in [0, 1)$ in Theorem 3.1, we have the following corollary.

Corollary 3.3 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T, G: X \rightarrow X$ be two mappings such that G is continuous and nondecreasing, $T(X) \subseteq G(X)$, T is G -nondecreasing with respect to \leq and the pair (G, T) is compatible. Suppose that there exists $\psi \in \Psi$ and $k \in [0, 1)$ such that for all $x, y \in X$ with $Gx \geq Gy$,*

$$\psi(d(Tx, Ty)) \leq k\psi(d(Gx, Gy)). \tag{3.18}$$

Also, suppose that

- (a) T is continuous, or
- (b) X is regular.

If there exists $x_0 \in X$ such that $Gx_0 \leq Tx_0$, then G and T have a coincidence point in X .

Considering φ to be the function ψ in Theorem 3.1, we have the following corollary.

Corollary 3.4 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T, G: X \rightarrow X$ be two mappings such that G is continuous and nondecreasing, $T(X) \subseteq G(X)$, T is G -nondecreasing with respect to \leq and the pair (G, T) is compatible. Suppose that there exist $\psi \in \Psi$ and $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \rightarrow t > 0$,*

$$\liminf \theta(x_n) > 0, \tag{3.19}$$

and for all $x, y \in X$ with $Gx \geq Gy$,

$$\psi(d(Tx, Ty)) \leq \psi(d(Gx, Gy)) - \theta(d(Gx, Gy)). \tag{3.20}$$

Also, suppose that

- (a) T is continuous or
- (b) X is regular.

If there exists $x_0 \in X$ such that $Gx_0 \leq Tx_0$, then G and T have a coincidence point in X .

If ψ and φ are the identity mappings in Theorem 3.1, we have the following corollary.

Corollary 3.5 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T, G: X \rightarrow X$ be two mappings such that G is continuous and nondecreasing, $T(X) \subseteq G(X)$, T is G -nondecreasing with respect*

to \leq and the pair (G, T) is compatible. Suppose that there exists $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \rightarrow t > 0$, $\liminf \theta(x_n) > 0$ and for all $x, y \in X$ with $Gx \succeq Gy$,

$$d(Tx, Ty) \leq d(Gx, Gy) - \theta(d(Gx, Gy)). \tag{3.21}$$

Also, suppose that

- (a) T is continuous, or
- (b) X is regular.

If there exists $x_0 \in X$ such that $Gx_0 \leq Tx_0$, then G and T have a coincidence point in X .

Considering ψ and φ to be the identity mappings and $\theta(t) = (1 - k)t$, where $0 \leq k < 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.6 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T, G: X \rightarrow X$ be two mappings such that G is continuous and nondecreasing, $T(X) \subseteq G(X)$, T is G -nondecreasing with respect to \leq and the pair (G, T) is compatible. Assume that there exists $k \in [0, 1)$ such that for all $x, y \in X$ with $Gx \succeq Gy$,*

$$d(Tx, Ty) \leq kd(Gx, Gy). \tag{3.22}$$

Also, suppose that

- (a) T is continuous, or
- (b) X is regular.

If there exists $x_0 \in X$ such that $Gx_0 \leq Tx_0$, then G and T have a coincidence point in X .

The condition (i), the continuity and the monotone property of the function G , and (ii), the compatibility condition of the pairs (G, T) , which were considered in Theorem 3.1, are relaxed in our next theorem by taking $G(X)$ to be closed in (X, d) .

Theorem 3.2 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T, G: X \rightarrow X$ be two mappings such that $T(X) \subseteq G(X)$ and T is G -nondecreasing with respect to \leq and $G(X)$ is closed in X . Suppose that there exist $\psi \in \Psi$ and $\varphi, \theta \in \Theta$ such that (3.1), (3.2) and (3.3) are satisfied. Also, suppose that X is regular.*

If there exists $x_0 \in X$ such that $Gx_0 \leq Tx_0$, then G and T have a coincidence point in X .

Proof We take the same sequence $\{x_n\}$ as in the proof of Theorem 3.1. Then we have (3.12), that is,

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Gx_n = x.$$

Since $\{Gx_n\}$ is a sequence in $G(X)$ and $G(X)$ is closed in X , $x \in G(X)$. As $x \in G(X)$, there exists $z \in X$ such that $x = Gz$. Then

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Gx_n = Gz. \tag{3.23}$$

Now, $\{Gx_n\}$ is nondecreasing and converges to Gz . So, by the order condition of the metric space X , we have

$$Gx_n \preceq Gz. \tag{3.24}$$

Putting $x = z$ and $y = x_n$ in (3.3), by the virtue of (3.24), we get

$$\psi(d(Tz, Tx_n)) \leq \varphi(d(Gz, Gx_n)) - \theta(d(Gz, Gx_n)).$$

Taking $n \rightarrow \infty$ in the above inequality, using (3.23), the property (ii_α) of φ and θ and the continuity of ψ , we have

$$\psi(d(Tz, Gz)) = 0,$$

which, by the property of ψ , implies that $d(Tz, Gz) = 0$, that is, $Tz = Gz$, that is, z is a coincidence point of the mappings G and T . \square

In the following, our aim is to prove the existence and uniqueness of the common fixed point in Theorems 3.1 and 3.2.

Theorem 3.3 *In addition to the hypotheses of Theorems 3.1 and 3.2, in both of the theorems, suppose that for every $x, y \in X$ there exists $u \in X$ such that Tu is comparable to Tx and Ty , and also the pair (G, T) is weakly compatible. Then G and T have a unique common fixed point.*

Proof From Theorem 3.1 or Theorem 3.2, the set of coincidence points of G and T is non-empty. Suppose x and y are coincidence points of G and T , that is, $Gx = Tx$ and $Gy = Ty$. Now, we show

$$Gx = Gy. \tag{3.25}$$

By the assumption, there exists $u \in X$ such that Tu is comparable with Tx and Ty .

Put $u_0 = u$ and choose $u_1 \in X$ so that $Gu_1 = Tu_0$. Then, similarly to the proof of Theorem 3.1, we can inductively define sequences $\{Gu_n\}$ where $Gu_{n+1} = Tu_n$ for all $n \geq 0$. Hence $Tx = Gx$ and $Tu = Tu_0 = Gu_1$ are comparable.

Suppose that $Gu_1 \preceq Gx$ (the proof is similar to that in the other case).

We claim that $Gu_n \preceq Gx$ for each $n \in N$.

In fact, we will use mathematical induction. Since $Gu_1 \preceq Gx$, our claim is true for $n = 1$. We presume that $Gu_n \preceq Gx$ holds for some $n > 1$. Since T is G -nondecreasing with respect to \preceq , we get

$$Gu_{n+1} = Tu_n \preceq Tx = Gx;$$

and this proves our claim.

Let $R_n = d(Gx, Gu_n)$. Since $Gu_n \preceq Gx$, using the contractive condition (3.3), for all $n \geq 1$, we have

$$\psi(d(Gx, Gu_{n+1})) = \psi(d(Tx, Tu_n)) \leq \varphi(d(Gx, Gu_n)) - \theta(d(Gx, Gu_n)),$$

that is, $\psi(R_{n+1}) \leq \varphi(R_n) - \theta(R_n)$, which, in view of the fact that $\theta \geq 0$, yields $\psi(R_{n+1}) \leq \varphi(R_n)$, which by (3.1) implies that $R_{n+1} \leq R_n$ for all positive integer n , that is, $\{R_n\}$ is a monotone decreasing sequence.

Then, as in the proof of Theorem 3.1, we have

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} d(Gx, Gu_n) = 0. \tag{3.26}$$

Similarly, we show that

$$\lim_{n \rightarrow \infty} d(Gy, Gu_n) = 0. \tag{3.27}$$

By the triangle inequality, using (3.26) and (3.27), we have

$$d(Gx, Gy) \leq [d(Gx, Gu_n) + d(Gu_n, Gy)] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence $Gx = Gy$. Thus (3.25) is proved.

Since $Gx = Tx$, by weak compatibility of G and T , we have

$$GGx = GTx = TGx. \tag{3.28}$$

Denote

$$Gx = z. \tag{3.29}$$

Then from (3.28) we have

$$Gz = Tz.$$

Thus z is a coincidence point of G and T . Then from (3.25) with $y = z$ it follows that

$$Gx = Gz.$$

By (3.29) it follows that

$$z = Gz. \tag{3.30}$$

From (3.29) and (3.30), we get $z = Gz = Tz$.

Therefore, z is a common fixed point of G and T .

To prove the uniqueness, assume that r is another common fixed point of G and T . Then by (3.25) we have $r = Gr = Gz = z$. Hence the common fixed point of G and T is unique. \square

Example 3.1 Let $X = [0, \infty)$. Then (X, \leq) is a partially ordered set with the natural ordering of real numbers. Let $d(x, y) = |x - y|$ for $x, y \in X$. Then (X, d) is a complete metric space.

Let $T, G: X \rightarrow X$ be given respectively by the formulas $Tx = \frac{1}{3}x^2$ and $Gx = x^2$ for all $x \in X$. Then T and G satisfy all the properties mentioned in Theorem 3.1.

Let $\psi, \varphi, \theta: [0, \infty) \rightarrow [0, \infty)$ be given respectively by the formulas

$$\psi(t) = t^2, \quad \varphi(t) = \begin{cases} \frac{1}{2}[t]^2 & \text{if } 3 < t < 4, \\ \frac{4}{9}t^2 & \text{otherwise,} \end{cases} \quad \theta(t) = \begin{cases} \frac{1}{9}[t]^2 & \text{if } 3 < t < 4, \\ 0 & \text{otherwise.} \end{cases}$$

Then ψ, φ and θ have the properties mentioned in Theorem 3.1.

It can be verified that (3.3) is satisfied for all $x, y \in X$ with $Gx \geq Gy$. Hence the required conditions of Theorem 3.1 are satisfied and it is seen that 0 is a coincidence point of G and T . Also, the conditions of Theorem 3.3 are satisfied and it is seen that 0 is the unique common fixed point of G and T .

Remark 3.1 In the above example, the pair (G, T) is compatible but not commuting so that Corollary 3.1 is not applicable to this example and hence Theorem 3.1 properly contains its Corollary 3.1.

Remark 3.2 In the above example, ψ is not the identity mapping and $\theta(t) \neq 0$ for all t in $[0, \infty)$. Let us consider the sequence $\{x_n\}$ in $[0, \infty)$, where $x_n = 3$ for all n . Then $\varphi(x_n) = 4$ for all n . Now $x_n \rightarrow t = 3 > 0$, but $\overline{\lim} \varphi(x_n) = 4 \not\leq t = 3$. Therefore, Corollary 3.2 is not applicable to this example, and hence Theorem 3.1 properly contains its Corollary 3.2.

Remark 3.3 The above example $\theta(t) \neq 0$ for all $t \in [0, \infty)$, and hence Corollary 3.3 is not applicable to the example, and so Theorem 3.1 properly contains its Corollary 3.3.

Remark 3.4 In the above example, φ is not identical to the function ψ , and also $\lim \theta(x_n) \not\leq 0$ for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \rightarrow t > 0$. Therefore, Corollaries 3.4 and 3.5 are not applicable to this example, and hence Theorem 3.1 properly contains its Corollaries 3.4 and 3.5.

Remark 3.5 In the above example, ψ and φ are not the identity functions and $\theta(t) \neq (1 - k)t$ with $0 \leq k < 1$. Therefore, Corollary 3.6 is not applicable to the above example. Hence Theorem 3.1 properly contains its Corollary 3.6.

Remark 3.6 Theorem 3.1 generalizes the results in [2–4, 6, 25–29].

Example 3.2 Let $X = [0, 1]$. Then (X, \leq) is a partially ordered set with the natural ordering of real numbers. Let $d(x, y) = |x - y|$ for $x, y \in X$. Then (X, d) is a metric space with the required properties of Theorem 3.2.

Let $T, G: X \rightarrow X$ be given respectively by the formulas

$$Tx = 1 \quad \text{for all } x \in X, \quad Gx = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ \frac{1}{2} & \text{if } x \text{ is irrational.} \end{cases}$$

Then T and G have the properties mentioned in Theorem 3.2.

Let $\psi, \varphi, \theta: [0, \infty) \rightarrow [0, \infty)$ be given respectively by the formulas

$$\psi(t) = t^2, \quad \varphi(t) = \begin{cases} \frac{1}{2}[t]^2 & \text{if } 3 < t < 4, \\ \frac{4}{9}t^2 & \text{otherwise,} \end{cases} \quad \theta(t) = \begin{cases} \frac{1}{9}[t]^2 & \text{if } 3 < t < 4, \\ 0 & \text{otherwise.} \end{cases}$$

Then ψ, φ and θ have the properties mentioned in Theorem 3.2.

All the required conditions of Theorem 3.2 are satisfied. It is seen that every rational number $x \in X$ is a coincidence point of G and T . Also, the conditions of Theorem 3.3 are satisfied and it is seen that 1 is the unique common fixed point of G and T .

Remark 3.7 In the above example, the function g is not continuous. Therefore, Theorem 3.1 is not applicable to the above example.

4 Applications to coupled coincidence point results

In this section, we use the results of the previous section to establish new coupled coincidence point results in partially ordered metric spaces. Our results are extensions of some existing results.

Let (X, \preceq) be a partially ordered set. Now, we endow the product space $X \times X$ with the following partial order:

$$\text{for } (x, y), (u, v) \in X \times X, \quad (u, v) \preceq (x, y) \iff x \succeq u, y \preceq v.$$

Let (X, d) be a metric space. Then d_1 given by the law

$$d_1((x, y), (u, v)) = d(x, u) + d(y, v) \quad \text{for } (x, y), (u, v) \in X \times X$$

is a metric on $X \times X$.

Let $g: X \rightarrow X$ and $F: X \times X \rightarrow X$ be two mappings. Then we define two functions $G: X \times X \rightarrow X \times X$ and $T: X \times X \rightarrow X \times X$ respectively as follows:

$$G(x, y) = (gx, gy) \quad \text{for } x, y \in X, \quad T(x, y) = (F(x, y), F(y, x)) \quad \text{for } x, y \in X.$$

Lemma 4.1 *Let (X, \preceq) be a partially ordered set, $F: X \times X \rightarrow X$ and $g: X \rightarrow X$. If F has the mixed g -monotone property, then T is G -nondecreasing.*

Proof Let $(x_1, y_1), (x_2, y_2) \in X \times X$ such that $G(x_1, y_1) \preceq G(x_2, y_2)$. Then, by the definition of G , it follows that $(gx_1, gy_1) \preceq (gx_2, gy_2)$, that is, $gx_1 \preceq gx_2$ and $gy_1 \succeq gy_2$. Since F has the mixed g -monotone property, we have

$$F(x_1, y_1) \preceq F(x_2, y_1) \preceq F(x_2, y_2),$$

and

$$F(y_1, x_1) \succeq F(y_2, x_1) \succeq F(y_2, x_2).$$

It follows that $(F(x_1, y_1), F(y_1, x_1)) \preceq (F(x_2, y_2), F(y_2, x_2))$, that is, $T(x_1, y_1) \preceq T(x_2, y_2)$. Therefore, T is G -nondecreasing. □

Lemma 4.2 *Let X be a nonempty set, $g: X \rightarrow X$ and $F: X \times X \rightarrow X$. If g and F are commutative, then the mappings G and T are also commutative.*

Proof Let $(x, y) \in X \times X$. Since g and F are commutative, by the definition of G and T , we have

$$\begin{aligned} GT(x, y) &= G(F(x, y), F(y, x)) = (gF(x, y), gF(y, x)) = (F(gx, gy), F(gy, gx)) \\ &= T(gx, gy) = TG(x, y), \end{aligned}$$

which shows that G and T are commutative. □

Lemma 4.3 *Let (X, d) be metric space and $g: X \rightarrow X, F: X \times X \rightarrow X$. If g and F are compatible, then the mappings G and T are also compatible.*

Proof Let $\{(x_n, y_n)\}$ be a sequence in $X \times X$ such that $\lim_{n \rightarrow \infty} G(x_n, y_n) = \lim_{n \rightarrow \infty} T(x_n, y_n) = (x, y)$ for some $(x, y) \in X \times X$. By the definition of G and T , we have $\lim_{n \rightarrow \infty} (gx_n, gy_n) = \lim_{n \rightarrow \infty} (F(x_n, y_n), F(y_n, y_n)) = (x, y)$, which implies that

$$\lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} gx_n = x, \quad \lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} gy_n = y.$$

Now

$$\begin{aligned} d_1(GT(x_n, y_n), TG(x_n, y_n)) &= d_1(G(F(x_n, y_n), F(y_n, x_n)), T(gx_n, gy_n)) \\ &= d_1((gF(x_n, y_n), gF(y_n, x_n)), (F(gx_n, gy_n), F(gy_n, gx_n))) \\ &= d(gF(x_n, y_n), F(gx_n, gy_n)) + d(gF(y_n, x_n), F(gy_n, gx_n)). \end{aligned}$$

Since g and F are compatible, we have

$$\lim_{n \rightarrow \infty} d_1(GT(x_n, y_n), TG(x_n, y_n)) = 0.$$

It follows that G and T are compatible. □

Lemma 4.4 *Let X be a nonempty set, $g: X \rightarrow X$ and $F: X \times X \rightarrow X$. If g and F are weak compatible, then the mappings G and T are also weak compatible.*

Proof Let $(x, y) \in X \times X$ be a coincidence point G and T . Then $G(x, y) = T(x, y)$, that is, $(gx, gy) = (F(x, y), F(y, x))$, that is, $gx = F(x, y)$ and $gy = F(y, x)$. Since g and F are weak compatible, by the definition of G and T , we have

$$\begin{aligned} GT(x, y) &= G(F(x, y), F(y, x)) = (gF(x, y), gF(y, x)) \\ &= (F(gx, gy), F(gy, gx)) = T(gx, gy) = TG(x, y), \end{aligned}$$

which shows that G and T commute at their coincidence point, that is, G and T are weak compatible. □

Theorem 4.1 *Let (X, \preceq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings such that g is continuous and nondecreasing, $F(X \times X) \subseteq g(X)$, F has the mixed g -monotone property on X and the pair (g, F) is compatible. Suppose that there exist $\psi \in \Psi$*

and $\varphi, \theta \in \Theta$ such that (3.1) and (3.2) are satisfied and for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$,

$$\begin{aligned} &\psi(d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))) \\ &\leq \varphi(d(gx, gu) + d(gy, gv)) - \theta(d(gx, gu) + d(gy, gv)). \end{aligned} \tag{4.1}$$

Also, suppose that

- (a) F is continuous, or
- (b) X is regular.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$; that is, g and F have a coupled coincidence point in X .

Proof We consider the product space $(X \times X, \preceq)$, the metric d_1 on $X \times X$ and the functions $G: X \times X \rightarrow X \times X$ and $T: X \times X \rightarrow X \times X$ as mentioned above. Denote $S = X \times X$. Then (S, d_1) is a complete metric space. By the definition of G and T , we have that

- (i) G is continuous and nondecreasing; and T is continuous,
- (ii) $T(S) \subseteq G(S)$,
- (iii) T is G -nondecreasing with respect to \preceq ,
- (iv) the pair (G, T) is compatible.

Let $p = (x, y), q = (u, v) \in S = X \times X$ such that $gx \succeq gu$ and $gy \preceq gv$, that is, $(gx, gy) \succeq (gu, gv)$, that is, $Gp \succeq Gq$. Then (4.1) reduces to

$$\psi(d_1(Tp, Tq)) \leq \varphi(d_1(Gp, Gq)) - \theta(d_1(Gp, Gq)).$$

Now, the existence of $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$ implies the existence of a point $p_0 = (x_0, y_0) \in S$ such that $(gx_0, gy_0) \preceq (F(x_0, y_0), F(y_0, x_0))$, that is, $Gp_0 \preceq Tp_0$. Therefore, the theorem reduces to Theorem 3.1, and hence there exists $w = (x, y) \in S = X \times X$ such that $Gw = Tw$, that is, $G(x, y) = T(x, y)$, that is, $(gx, gy) = (F(x, y), F(y, x))$, that is, $gx = F(x, y)$ and $gy = F(y, x)$, that is, (x, y) is a coupled coincidence point of g and F . □

The following corollary is a consequence of Corollary 3.1.

Corollary 4.1 *Let (X, \preceq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings such that g is continuous and nondecreasing, $F(X \times X) \subseteq g(X)$, F has the mixed g -monotone property on X and the pair (g, F) is commutative. Suppose that there exist $\psi \in \Psi, \varphi, \theta \in \Theta$ such that (3.1), (3.2) and (4.1) are satisfied. Also, suppose that*

- (a) F is continuous, or
- (b) X is regular.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$; that is, g and F have a coupled coincidence point in X .

The following corollary is a consequence of Corollary 3.2.

Corollary 4.2 *Let (X, \preceq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings such that g is continuous and nondecreasing, $F(X \times X) \subseteq g(X)$, F has the mixed g -monotone property on X and the pair (g, F) is compatible. Suppose that there exists $\varphi \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \rightarrow t > 0$,*

$$\overline{\lim} \varphi(x_n) < t,$$

and for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$,

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq \varphi(d(gx, gu) + d(gy, gv)).$$

Also, suppose that

- (a) *F is continuous, or*
- (b) *X is regular.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$; that is, g and F have a coupled coincidence point in X .

The following corollary is a consequence of Corollary 3.3.

Corollary 4.3 *Let (X, \preceq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings such that g is continuous and nondecreasing, $F(X \times X) \subseteq g(X)$, F has the mixed g -monotone property on X and the pair (g, F) is compatible. Suppose that there exist $\psi \in \Psi$ and $k \in [0, 1)$ such that for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$,*

$$\psi(d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))) \leq k\psi(d(gx, gu) + d(gy, gv)).$$

Also, suppose that

- (a) *F is continuous, or*
- (b) *X is regular.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$; that is, g and F have a coupled coincidence point in X .

The following corollary is a consequence of Corollary 3.4.

Corollary 4.4 *Let (X, \preceq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings such that g is continuous and nondecreasing, $F(X \times X) \subseteq g(X)$, F has the mixed g -monotone property on X and the pair (g, F) is compatible. Suppose that there exist $\psi \in \Psi$ and $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \rightarrow t > 0$, $\underline{\lim} \theta(x_n) > 0$ and for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$,*

$$\begin{aligned} &\psi(d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))) \\ &\leq \psi(d(gx, gu) + d(gy, gv)) - \theta(d(gx, gu) + d(gy, gv)). \end{aligned} \tag{4.2}$$

Also, suppose that

- (a) F is continuous, or
- (b) X is regular.

If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$; that is, g and F have a coupled coincidence point in X .

Remark 4.1 The above result is also true if the arguments of ψ and θ in (4.2) are replaced by their half values, that is, when (4.2) is replaced by

$$\begin{aligned} & \psi \left(\frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2} \right) \\ & \leq \psi \left(\frac{d(gx, gu) + d(gy, gv)}{2} \right) - \theta \left(\frac{d(gx, gu) + d(gy, gv)}{2} \right). \end{aligned}$$

In this case, we can write $\psi_1(t) = \psi(\frac{t}{2})$ and $\theta_1(t) = \theta(\frac{t}{2})$ and proceed with the same proof by replacing ψ, θ by ψ_1, θ_1 respectively. Then we obtain a generalization of Theorem 2 of Berinde in [10].

The following corollary is a consequence of Corollary 3.5.

Corollary 4.5 Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings such that g is continuous and nondecreasing, $F(X \times X) \subseteq g(X)$, F has the mixed g -monotone property on X and the pair (g, F) is compatible. Suppose that there exists $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \rightarrow t > 0$, $\liminf \theta(x_n) > 0$ and for all $x, y, u, v \in X$ with $gx \geq gu$ and $gy \leq gv$,

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq d(gx, gu) + d(gy, gv) - \theta(d(gx, gu) + d(gy, gv)).$$

Also, suppose that

- (a) F is continuous, or
- (b) X is regular.

If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$; that is, g and F have a coupled coincidence point in X .

The following corollary is a consequence of Corollary 3.6.

Corollary 4.6 Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings such that g is continuous and nondecreasing, $F(X \times X) \subseteq g(X)$, F has the mixed g -monotone property on X and the pair (g, F) is compatible. Assume that there exists $k \in [0, 1)$ such that for all $x, y, u, v \in X$ with $gx \geq gu, gy \leq gv$,

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k[d(gx, gu) + d(gy, gv)].$$

Also, suppose that

- (a) F is continuous, or
- (b) X is regular.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$; that is, g and F have a coupled coincidence point in X .

The following theorem is a consequence of Theorem 3.2.

Theorem 4.2 Let (X, \preceq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Consider the mappings $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ such that $F(X \times X) \subseteq g(X)$, F has the mixed g -monotone property on X and $G(X)$ is closed in X . Suppose that there exist $\psi \in \Psi$ and $\varphi, \theta \in \Theta$ such that (3.1), (3.2) and (4.1) are satisfied. Also, suppose that X is regular.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$; that is, g and F have a coupled coincidence point in X .

The following theorem is a consequence of Theorem 3.3.

Theorem 4.3 In addition to the hypotheses of Theorems 4.1 and 4.2, in both of the theorems, suppose that for every $(x, y), (x^*, y^*) \in X \times X$, there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$, and also the pair (g, F) is weakly compatible. Then g and F have a unique coupled common fixed point; that is, there exists a unique $(x, y) \in X \times X$ such that $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Example 4.1 Let $X = [0, \infty)$. Then (X, \preceq) is a partially ordered set with the natural ordering of real numbers. Let $d(x, y) = |x - y|$ for $x, y \in X$. Then (X, d) is a complete metric space.

Let $g: X \rightarrow X$ be given by $gx = x^2$ for all $x \in X$. Also, consider

$$F: X \times X \rightarrow X, \quad F(x, y) = \begin{cases} \frac{1}{3}(x^2 - y^2) & \text{if } x, y \in X, x \geq y, \\ 0 & \text{if } x \leq y, \end{cases}$$

which obeys the mixed g -monotone property.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} gx_n = a, \quad \lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} gy_n = b.$$

Then, obviously, $a = 0$ and $b = 0$.

Now, for all $n \geq 0$, $gx_n = x_n^2$, $gy_n = y_n^2$, while

$$F(x_n, y_n) = \begin{cases} \frac{1}{3}(x_n^2 - y_n^2) & \text{if } x_n \geq y_n, \\ 0 & \text{if } x_n \leq y_n, \end{cases} \quad F(y_n, x_n) = \begin{cases} \frac{1}{3}(y_n^2 - x_n^2) & \text{if } y_n \geq x_n, \\ 0 & \text{if } y_n \leq x_n. \end{cases}$$

Then it follows that

$$\lim_{n \rightarrow \infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0, \quad \lim_{n \rightarrow \infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0.$$

Hence, the pair (g, F) is compatible in X .

Let $x_0 = 0$ and $y_0 = c (> 0)$ be two points in X . Then

$$g(x_0) = g(0) = 0 = F(0, c) = F(x_0, y_0), \quad g(y_0) = g(c) = c^2 \geq \frac{c^2}{3} = F(c, 0) = F(y_0, x_0).$$

Let $\psi, \varphi, \theta: [0, \infty) \rightarrow [0, \infty)$ be given respectively by the formulas

$$\psi(t) = t^2, \quad \varphi(t) = \begin{cases} \frac{1}{2}[t]^2 & \text{if } 3 < t < 4, \\ \frac{4}{9}t^2 & \text{otherwise,} \end{cases} \quad \theta(t) = \begin{cases} \frac{1}{9}[t]^2 & \text{if } 3 < t < 4, \\ 0 & \text{otherwise.} \end{cases}$$

Then ψ, φ and θ have the properties mentioned in Theorem 4.1.

We now verify inequality (4.1) of Theorem 4.1.

We take $x, y, u, v \in X$ such that $gx \geq gu$ and $gy \leq gv$, that is, $x^2 \geq u^2$ and $y^2 \leq v^2$.

Let $M = d(gx, gu) + d(gy, gv) = |x^2 - u^2| + |y^2 - v^2| = (x^2 - u^2) + (v^2 - y^2)$.

The following are the four possible cases.

Case-1: $x \geq y$ and $u \geq v$. Then

$$\begin{aligned} & d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &= d\left(\frac{x^2 - y^2}{3}, \frac{u^2 - v^2}{3}\right) + d(0, 0) = \left| \frac{(x^2 - y^2) - (u^2 - v^2)}{3} \right| \\ &= \left| \frac{(x^2 - u^2) + (v^2 - y^2)}{3} \right| = \frac{(x^2 - u^2) + (v^2 - y^2)}{3} = \frac{1}{3}M. \end{aligned}$$

Case-2: $x < y$ and $u < v$. Then

$$\begin{aligned} & d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &= d(0, 0) + d\left(\frac{y^2 - x^2}{3}, \frac{v^2 - u^2}{3}\right) = \left| \frac{(y^2 - x^2) - (v^2 - u^2)}{3} \right| \\ &= \left| \frac{(x^2 - u^2) + (v^2 - y^2)}{3} \right| = \frac{(x^2 - u^2) + (v^2 - y^2)}{3} = \frac{1}{3}M. \end{aligned}$$

Case-3: $x \geq y$ and $u \leq v$. Then

$$\begin{aligned} & d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &= d\left(\frac{x^2 - y^2}{3}, 0\right) + d\left(0, \frac{v^2 - u^2}{3}\right) \\ &= \frac{x^2 - y^2}{3} + \frac{v^2 - u^2}{3} = \frac{(x^2 - u^2) + (v^2 - y^2)}{3} = \frac{1}{3}M. \end{aligned}$$

Case-4: The case ' $x < y$ and $u > v$ ' is not possible. Under this condition, $x^2 < y^2$ and $u^2 > v^2$. Then by the condition $y^2 \leq v^2$, we have $x^2 < y^2 \leq v^2 < u^2$, which contradicts that $x^2 \geq u^2$.

In all the above cases, it can be verified that (4.1) is satisfied. Hence the required conditions of Theorem 4.1 are satisfied, and it is seen that $(0, 0)$ is a coupled coincidence point of g and F in X . Also, the conditions of Theorem 4.3 are satisfied, and it is seen that $(0, 0)$ is the unique coupled common fixed point of g and F in X .

Remark 4.2 In the above example, the pair (g, F) is compatible but not commuting so that Corollary 4.1 is not applicable to this example, and hence Theorem 4.1 properly contains its Corollary 4.1.

Remark 4.3 As discussed in Remarks 3.2-3.5, Theorem 4.1 properly contains its Corollaries 4.2-4.6.

Remark 4.4 Theorem 4.1 properly contains its Corollary 4.6, which is an extension of Theorem 3 of Berinde [9], and Theorems 2.1 and 2.2 of Bhaskar and Lakshmikantham [8]. Therefore, Theorem 4.1 is an actual extension over Theorem 3 of Berinde [9] and Theorems 2.1 and 2.2 of Bhaskar and Lakshmikantham [8].

Example 4.2 Let $X = [0, 1]$. Then (X, \leq) is a partially ordered set with the natural ordering of real numbers. Let $d(x, y) = |x - y|$ for $x, y \in X$. Then (X, d) is a metric space with the required properties of Theorem 4.2.

Let $F: X \times X \rightarrow X$, $F(x, y) = 1$ for $(x, y) \in X \times X$. Let $g: X \rightarrow X$ be given by the formula

$$gx = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ \frac{1}{2} & \text{if } x \text{ is irrational.} \end{cases}$$

Then F and g have the properties mentioned in Theorems 4.2.

Let $\psi, \varphi, \theta: [0, \infty) \rightarrow [0, \infty)$ be given respectively by the formulas

$$\psi(t) = t^2, \quad \varphi(t) = \begin{cases} \frac{1}{2}[t]^2 & \text{if } 3 < t < 4, \\ \frac{4}{9}t^2 & \text{otherwise,} \end{cases} \quad \theta(t) = \begin{cases} \frac{1}{9}[t]^2 & \text{if } 3 < t < 4, \\ 0 & \text{otherwise.} \end{cases}$$

Then ψ , φ and θ have the properties mentioned in Theorem 4.2.

All the required conditions of Theorem 4.2 are satisfied. It is seen that every $(x, y) \in X \times X$, where both x and y are rational, is a coupled coincidence point of g and F in X . Also, the conditions of Theorem 4.3 are satisfied and it is seen that $(1, 1)$ is the unique coupled common fixed point of g and F in X .

Remark 4.5 In the above example, the function g is not continuous. Therefore, Theorem 4.1 is not applicable to the above example.

Remark 4.6 In some recent papers [30, 31] it has been proved that some of the contractive conditions involving continuous control functions are equivalent. Here two of our control functions are discontinuous. Therefore, the contraction we use here is not included in the class of contractions addressed by Aydi *et al.* [30] and Jachymski [31].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors completed the paper together. All authors read and approved the final manuscript.

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