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Strong convergence of a new general iterative method for variational inequality problems in Hilbert spaces

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Abstract

In this article, we introduce a new iterative scheme with Meir-Keeler contractions for strict pseudo-contractions in Hilbert spaces. We also discuss the strong convergence theorems of the new iterative scheme for variational inequality problems in Hilbert spaces. The methods in this article are interesting and are different from those given in many other articles. Our results improve and extend the corresponding results announced by many others.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let $F : C \to C$ be a nonlinear operator. The variational inequality problem is such that

$$V I(F, C) : \quad \langle Fx^*, v - x* \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

Variational inequalities were introduced and studied by Stampacchia [1] in 1964. It is now well known that variational inequalities cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance, see [1-25].

It is known that if F is a strongly monotone and Lipschitzian mapping on C, then the $VI(F, C)$ has a unique solution. It is also known that the $VI(F, C)$ is equivalent to the fixed point equation

$$x^* = P_C \left[x^* - \mu F(x^*) \right],$$

where P_C is the projection of H onto the closed convex set C and $\mu > 0$ is an arbitrarily fixed constant. So, fixed point methods can be implemented to find a solution of the $VI(F,C)$ provided F satisfies some conditions and $\mu > 0$ is chosen appropriately. A great deal of effort has gone into finding an approximate solution of the $VI(F,C)$ see [3,5,15-19].

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In 2001, Yamada [2] introduced the following hybrid iterative method for solving the variational inequality

$$x_{n+1} = Tx_n - \mu \lambda_n F(Tx_n), n \geq 0. \qquad (1.2)$$

On the other hand, Yao et al. [6] modified Mann's iterative scheme by using the so-called viscosity approximation method which was introduced by Moudafi [7]. More precisely, Yao et al. [6] introduced and studied the following iterative algorithm:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + 1(1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, n \geq 0. \end{cases} \qquad (1.3)$$

where T is a nonexpansive mapping of K into itself and f is a contraction on K. They obtained a strong convergence theorem under some mild restrictions on the parameters.

Zhou [8], Qin et al. [9] modified normal Mann's iterative process (1.3) for non-self-k-strictly pseudo-contractions to have strong convergence in Hilbert spaces. Qin et al. [9] introduced the following iterative algorithm scheme:

$$\begin{cases} x_1 = x \in K, \\ y_n = P_K[\beta_n x_n + (1 - \beta_n)Tx_n], \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n A)y_n, n \geq 1. \end{cases} \qquad (1.4)$$

where T is non-self-k-strictly pseudo-contraction, f is a contraction and A is a strong positive linear bounded operator. They prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ defined by (1.4) converges strongly to a fixed point of the k-strictly pseudo-contraction, which solves some variational inequality.

The following famous theorem is referred to as the Banach contraction principle.

Theorem 1. (Banach [10]) Let (X, d) be a complete metric space and let f be a contraction on X, i.e., there exists $r \in (0,1)$ such that $d(f(x), f(y)) \leq rd(x, y)$ for all $x, y \in X$. Then f has a unique fixed point.

Theorem 2. (Meir and Keeler [11]) Let (X,d) be a complete metric space and let φ be a Meir-Keeler contraction (MKC) on X, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \varepsilon + \delta$ implies $d(\varphi(x), \varphi(y)) < \varepsilon$ for all $x, y \in X$. Then φ has a unique fixed point.

Remark 1. Theorem 2 is one of generalizations of Theorem 1, because contractions are MKCs.

Question 1. Can Theorem 1 of Yao [6], Theorem 3.2 of Zhou [8], Theorem 2.1 of Qin [9], and so on be extended from one or finite k_i-strictly pseudo-contraction to infinite k_i-strictly pseudo-contraction?

Question 2. We know that the MKC is more general than the contraction. What happens if the contraction is replaced by the MKC?

Question 3. We know that the η-strongly monotone and L-Lipschitzian operator is more general than the strong positive linear bounded operator. What happens if the strong positive linear bounded operator is replaced by the η-strongly monotone and L-Lipschitzian operator?

Question 4. Can the restrictions imposed on the parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ in [9] be relaxed?

The purpose of this article is to give the affirmative answers to these questions mentioned above. Motivated by the above works, in this article we suggest and analyze a hybrid iterative algorithm as follows:

$$
\begin{cases}
x_1 = x \in C, \\
y_n = P_C[\beta_n x_n + (1 - \beta_n) \sum_{i=1}^{\infty} \mu_i^{(n)} T_i x_n], \\
x_{n+1} = \alpha_n \phi(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \alpha_n F)y_n, n \geq 1.
\end{cases}
\tag{1.5}
$$

where T_i is a non-self-k_i-strictly pseudo-contraction, φ is an MKC contraction and $F : C \to C$ is a L-Lipschitzian and η-strongly monotone mapping in Hilbert space. Under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\mu_i^n\}$, that $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of an infinite family of k_i-strictly pseudo-contractions, which solves some variational inequality.

2 Preliminaries

In this section, we first recall some notations. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to C$ be an operator. F is called L-Lipschitzian if there exists a positive constant L such that

$$\|Fx - Fy\| \leq L \|x - y\|,$$

for all $x, y \in C$, F is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2,$$

for all $x, y \in C$. Without loss of generality, we can assume that $\eta \in (0, 1]$ and $L \in [1, \infty)$. Under these conditions, it is well known that the variational inequality problem $VI(F, C)$ has a unique solution $x^* \in C$.

A self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$; $\forall x, y \in C$. We use Π_C to denote the collection of all contractions on C. That is, $\Pi_C = \{f | f : C \to C$ a contraction$\}$. We use $F(T)$ to denote the fixed point set of the mapping T and P_C to denote the metric projection of H onto its closed convex subset C.

A mapping T is said to be non-expansive, if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

T is said to be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [12], if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, x, y \in C.$$

It is clear that it is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - k}{2}\|(I - T)x - (I - T)y\|^2, x, y \in C,$$

or is equivalent to

$$\langle (I - T)x - (I - T)y, x - y \rangle \geq \frac{1 - k}{2}\|(I - T)x - (I - T)y\|^2, x, y \in C$$

An operator A be a strongly positive bounded linear operator on H, that is, there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \forall x \in H.$$

Remark 2. From the definition of A, we note that a strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and $\bar{\gamma}$-strongly monotone operator.

In order to prove our main results, we need the following lemmas.

Lemma 2.1 (Zhou [8]). Let H be a Hilbert space and C be a closed convex subset of H. If T is a k-strictly pseudo-contractive mapping on C, then the fixed point set $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.

Lemma 2.2 (Zhou [8]). Let H be a Hilbert space and C be a closed convex subset of H. Let $T : C \rightarrow H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Then $F(P_C T) = F(T)$.

Lemma 2.3 (Browder and Petryshyn [12]). Let H be a Hilbert space, C be a closed convex subset of H, and $T : C \rightarrow H$ be a k-strictly pseudo-contractive mapping. Define a mapping $J : C \rightarrow H$ by $Jx = \alpha x + (1-\alpha)Tx$ for all $x \in C$. Then, as $\alpha \in [k, 1)$, J is a non-expansive mapping such that $F(J) = F(T)$.

Lemma 2.4. (see [13]). Let $\{x_n\}$, $\{z_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0,1]$ which satisfies the following condition: $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n$ for all $n \geq 0$ and $\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.5 (Xu [14]). Assume that $\{\alpha_n\}$ is a sequence of non-negative real numbers such that $\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n$, where γ_n is a sequence in $(0, 1)$ and δ_n is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.6 ([23] Demiclosedness Principle). Let H be a Hilbert space, K a closed convex subset of H, and $T : K \rightarrow K$ a non-expansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in K weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.

Lemma 2.7 Let F be a L-Lipschitzian and η-strongly monotone operator on a nonempty closed convex subset C of a real Hilbert space H with $0 < \eta \leq L$ and $0 < t < 2\eta/L^2$. Then $S = (I - tF) : C \rightarrow C$ is a contraction with contraction coefficient $\tau_t = 1 - t(\eta - \frac{tL^2}{2})$.

Proof. From the definition of η-strongly monotone and L-Lipschitzian operator, we have

$$\begin{aligned}
\|Sx - Sy\|^2 &\leq \|x - y - t(Fx - Fy)\|^2 \\
&\leq \|x - y\|^2 + t^2 \|Fx - Fy\|^2 - 2t \langle Fx - Fy, x - y \rangle \\
&\leq \|x - y\|^2 + t^2 L^2 \|x - y\|^2 - 2t\eta \|x - y\|^2 \\
&= [1 - t(2\eta - tL^2)] \|x - y\|^2 \\
&\leq \left[1 - t\left(\eta - \frac{tL^2}{2}\right)\right]^2 \|x - y\|^2.
\end{aligned}$$

for all $x, y \in C$. From $0 < \eta \leq L$ and $0 < t < 2\eta/L^2$, we have $0 < 1 - t(\eta - \frac{tL^2}{2}) < 1$ and

$$\|Sx - Sy\| \leq \tau_t \|x - y\|,$$

where $\tau_t = 1 - t(\eta - \frac{tL^2}{2}) \in (0, 1)$. Hence, S is a contraction with contraction coefficient τ_t.

Lemma 2.8 ([23] Lemma 2.3). Let φ be an MKC on a convex subset C of a Banach space E. Then for each $\varepsilon > 0$, there exists $r \in (0,1)$ such that $\|x - y\| \geq \varepsilon$ implies $\|\varphi x - \varphi y\| \leq r \|x - y\|$ for all $x, y \in C$.

Lemma 2.9. Let C be a closed convex subset of a Hilbert space H. Let $S : C \to C$ be a non-expansive mapping and φ be an MKC on C. Suppose $F: C \to C$ be a η-strongly monotone and L-Lipschitzian mapping with coefficient η and $\eta > \gamma > 0$. Then the sequence $\{x_t\}$ define by

$$x_t = t\gamma\phi(x_t) + (1 - tF)Sx_t$$

converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of S which solves the variational inequality:

$$\langle (F - \gamma\phi)\tilde{x}, \tilde{x} - z \rangle \leq 0, \quad \forall z \in F(S). \tag{2.1}$$

Proof. The definition of $\{x_t\}$ is well definition. Indeed, From the definition of MKC, we can see MKC is also a non-expansive mapping. Consider a mapping S_t on C defined by

$$S_t x = t\gamma\phi(x) + (I - tF)Sx, x \in C.$$

It is easy to see that S_t is a contraction, when $0 < t < \frac{2(\eta - \gamma)}{L^2}$ Indeed, by Lemmas 2.7 and 2.8, we have

$$
\begin{aligned}
\|S_t x - S_t y\| &\leq t\gamma \|\phi(x) - \phi(y)\| + \|(I - tF)Sx - (I - tF)Sy\| \\
&\leq t\gamma \|\phi(x) - \phi(y)\| + \tau_t \|Sx - Sy\| \\
&\leq t\gamma \|x - y\| + \tau_t \|x - y\| \\
&\leq \theta_t \|x - y\|.
\end{aligned}
$$

where $\theta_t = t\gamma + \tau_t \in (0,1)$. Hence, S_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation

$$x_t = t\gamma\phi(x_t) + (I - tF)Sx_t.$$

We next show the uniqueness of a solution of the variational inequality (2.1). Suppose both $\tilde{x} \in F(S)$ and $\hat{x} \in F(S)$ are solutions to (2.1). Not lost generality, we may assume there is a positive number ε such that $\|\hat{x} - \tilde{x}\| \geq \varepsilon$. Then, by Lemma 2.8, there is a number $r \in (0,1)$ such that $\|\phi\hat{x} - \phi\tilde{x}\| \leq r \|\hat{x} - \tilde{x}\|$. From (2.1), we know

$$\langle (F - \gamma\phi)\tilde{x}, \tilde{x} - \hat{x} \rangle \leq 0. \tag{2.2}$$

and

$$\langle (F - \gamma\phi)\hat{x}, \hat{x} - \tilde{x} \rangle \leq 0. \tag{2.3}$$

Adding up (2.2) and (2.3) gets

$$\langle (F - \gamma\phi)\hat{x} - (F - \gamma\phi)\tilde{x}, \hat{x} - \tilde{x} \rangle \leq 0.$$

Noticing that

$$
\begin{aligned}
\langle (F - \gamma\phi)\hat{x} - (F - \gamma\phi)\tilde{x}, \hat{x} - \tilde{x} \rangle &= \langle F\hat{x} - F\tilde{x}, \hat{x} - \tilde{x} \rangle - \gamma \langle \phi\hat{x} - \phi\tilde{x}, \hat{x} - \tilde{x} \rangle \\
&\geq \eta \|\hat{x} - \tilde{x}\|^2 - \gamma \|\phi\hat{x} - \phi\tilde{x}\| \|\hat{x} - \tilde{x}\| \\
&\geq \eta \|\hat{x} - \tilde{x}\|^2 - \gamma r \|\hat{x} - \tilde{x}\|^2 \\
&\geq (\eta - \gamma r) \|\hat{x} - \tilde{x}\|^2 \\
&\geq (\eta - \gamma r)\varepsilon^2 \\
&> 0.
\end{aligned}
$$

Therefore, $\hat{x} = \tilde{x}$ and the uniqueness is proved. Below we use $\tilde{x}$ to denote the unique solution of (2.1).

We observe that $\{x_t\}$ is bounded. Indeed, We may assume $0 < t < \frac{\eta - \gamma}{L^2}$. For $\forall p \in F$ (S), fixed ε_1, for each t

Case 1. $\|x_t - p\| < \varepsilon_1$; In this case, we can see easily that $\{x_t\}$ is bounded.

Case 2. $\|x_t - p\| \geq \varepsilon_1$. In this case, by Lemma 2.8, there is a number $r_1 \in (0,1)$ such that

$$\|\phi(x_t) - \phi(p)\| \leq r_1 \|x_t - p\|.$$

$$
\begin{aligned}
\|x_t - p\| &= \|t\gamma\phi(x_t) + (I - tF)Sx_t - p\| \\
&= \|t(\gamma\phi(x_t) - Fp) + (I - tF)Sx_t - (I - tF)p\| \\
&\leq t \|\gamma\phi(x_t) - Fp\| + \tau_t \|x_t - p\| \\
&\leq t \|\gamma\phi(x_t) - \gamma\phi(p)\| + t \|\gamma\phi(p) - Fp\| + \tau_t \|x_t - p\| \\
&\leq t\gamma r_1 \|x_t - p\| + t \|\gamma\phi(p) - Fp\| + \tau_t \|x_t - p\|
\end{aligned}
$$

therefore, $\|x_t - p\| \leq \frac{2\|\gamma\phi(p) - Fp\|}{\eta - \gamma}$. This implies the $\{x_t\}$ is bounded.

Next, we prove that $x_t \to \tilde{x}$ as $t \to 0$.

Since $\{x_t\}$ is bounded and H is reflexive, there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n} \rightharpoonup x*$. By $x_t - Sx_t = t(\gamma\varphi(x_t) - FSx_t)$, we have $x_{t_n} - Sx_{t_n} \to 0$, as $t_n \to 0$. It follows from Lemma 2.6 that $x^* \in F(S)$.

We claim

$$\|x_{t_n} - x^*\| \to 0.$$

By contradiction, there is a number ε_0 and a subsequence $\{x_{t_m}\}$ of $\{x_{t_n}\}$ such that $\|x_{t_m} - x^*\| \geq \varepsilon_0$. From Lemma 2.8, there is a number $r_{\varepsilon_0} > 0$ such that $\|\phi(x_{t_m}) - \phi(x^*)\| \leq r_{\varepsilon_0} \|x_{t_m} - x^*\|$, we write

$$x_{t_m} - x^* = t_m(\gamma\phi(x_{t_m}) - Fx^*) + (I - t_mF)Sx_{t_m} - (I - t_mF)x*,$$

to derive that

$$
\begin{aligned}
\|x_{t_m} - x^*\|^2 &= t_m \langle \gamma\phi(x_{t_m}) - Fx^*, x_{t_m} - x^* \rangle + \langle (I - t_mF)Sx_{t_m} - (I - t_mF)x^*, x_{t_m} - x^* \rangle \\
&\leq t_m \langle \gamma\phi(x_{t_m}) - Fx^*, x_{t_m} - x^* \rangle + \tau_{t_m} \|x_{t_m} - x^*\|^2.
\end{aligned}
\tag{2.4}
$$

It follows that

$$\|x_{t_m} - x^*\|^2 \le \left(\eta - \frac{t_m L^2}{2}\right)^{-1} \langle \gamma\phi(x_{t_m}) - Fx^*, x_{t_m} - x^* \rangle$$

$$= \left(\eta - \frac{t_m L^2}{2}\right)^{-1} \left[\langle \gamma\phi(x_{t_m}) - \gamma\phi(x^*), x_{t_m} - x^* \rangle + \langle \gamma\phi(x^*) - Fx^*, x_{t_m} - x^* \rangle\right]$$

$$\le \left(\eta - \frac{t_m L^2}{2}\right)^{-1} \left[\gamma r_{\varepsilon_0} \|x_{t_m} - x^*\|^2 + \langle \gamma\phi(x^*) - Fx^*, x_{t_m} - x^* \rangle\right].$$

Therefore,

$$\|x_{t_m} - x^*\|^2 \le \frac{1}{\eta - \gamma\varepsilon_0 - \frac{t_m L^2}{2}} \langle \gamma\phi(x^*) - Fx^*, x_{t_m} - x^* \rangle. \tag{2.5}$$

By (2.5), we get that $x_{t_m} \to x^*$. It is a contradiction. Hence, we have $x_{t_n} \to x^*$. We next prove that x^* solves the variational inequality (2.1). Since

$$x_t = t\gamma\phi(x_t) + (I - tF)Sx_t$$

we derive that

$$(F - \gamma\phi)x_t = -\frac{1}{t}[(I - S)x_t - t(Fx_t - FSx_t)].$$

Notice

$$\langle (I - S)x_t - (I - S)z, x_t - z \rangle \ge \|x_t - z\|^2 - \|Sx_t - Sz\| \|x_t - z\|$$
$$\ge \|x_t - z\|^2 - \|x_t - z\|^2$$
$$= 0.$$

It follows that, for $\forall z \in F(S)$,

$$\langle (F - \gamma\phi)x_t, x_t - z \rangle = -\frac{1}{t}\langle (I - S)x_t - t(Fx_t - FSx_t), x_t - z \rangle$$
$$= -\frac{1}{t}\langle (I - S)x_t - (I - S)z, x_t - z \rangle + \langle Fx_t - F(Sx_t), x_t - z \rangle \tag{2.6}$$
$$\le L\|x_t - Sx_t\| \|x_t - z\|.$$

Noticing

$$x_t - Sx_t = t[\gamma\phi(x_t) - FSx_t].$$

Hence, we have

$$x_t - Sx_t \to 0, \quad as\ t \to 0.$$

Now replacing t in (2.6) with t_n and letting $n \to \infty$, noticing $(I - S)x_{t_n} \to (I - S)x^* = 0$ for $x^* \in F(S)$, we obtain $\langle (F - \gamma\varphi)x^*, x^* - z \rangle \le 0$. That is, $x^* \in F(S)$ is a solution of (2.1); Hence, $\tilde{x} = x^*$ by uniqueness. We have shown that each cluster point of x_t (at $t \to 0$) equals $\tilde{x}$. Therefore, $x_t \to \tilde{x}$ as $t \to 0$.

Lemma 2.10. Let H be a Hilbert space and C be a nonempty convex subset of H. Assume that $T_i : C \to E$ is a countable family of k_i-strict pseudo-contraction for some $0 \le k_i < 1$ and $\sup\{k_i : i \in \mathbb{N}\} < 1$ such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\{\mu_i\}$ is a positive sequence such that $\sum_{i=1}^{\infty} \mu_i = 1$. Then $\sum_{i=1}^{\infty} \mu_i T_i : C \to E$ is a k-strict pseudo-contraction with $k = \sup\{k_i : i \in \mathbb{N}\}$ and $F(\sum_{i=1}^{\infty} \mu_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$.

Proof. Let

$$G_n x = \mu_1 T_1 x + \mu_2 T_2 x + \cdots + \mu_n T_n x$$

and $\sum_{i=1}^{n} \mu_i = 1$. Then, $G_n : C \to E$ is a k_i-strict pseudo-contraction with $k = \max\{k_i : 1 \le i \le n\}$. Indeed, we can firstly see the case of $n = 2$.

$$
\begin{aligned}
&\langle (I - G_2) x - (I - G_2) y, x - y \rangle \\
&= \langle \mu_1 (I - T_1) x + \mu_2 (I - T_2)) x - \mu_1 (I - T_1) y - \mu_2 (I - T_2)) y, x - y \rangle \\
&= \mu_1 \langle (I - T_1) x - (I - T_1) y, x - y \rangle + \mu_2 \langle (I - T_2) x - (I - T_2) y, x - y \rangle \\
&\ge \mu_1 \frac{1 - k_1}{2} \| (I - T_1) x - (I - T_1) y \|^2 + \mu_2 \frac{1 - k_2}{2} \| (I - T_2) x - (I - T_2) y \|^2 \\
&\ge \frac{1 - k}{2} \left[\mu_1 \| (I - T_1) x - (I - T_1) y \|^2 + \mu_2 \| (I - T_2) x - (I - T_2) y \|^2 \right] \\
&\ge \frac{1 - k}{2} \| (I - G_2) x - (I - G_2) y \|^2,
\end{aligned}
$$

which shows that $G_2 : C \to E$ is a k-strict pseudo-contraction with $k = \max\{k_i : i = 1, 2\}$. By the same way, our proof method easily carries over to the general finite case.

Next, we prove the infinite case. From the definition of k-strict pseudo-contraction, we know

$$\langle (I - T_n) x - (I - T_n) y, x - y \rangle \ge \frac{1 - k}{2} \| (I - T_n) x - (I - T_n) y \|^2.$$

Hence, we can get

$$\| (I - T_n) x - (I - T_n) y \| \le \frac{2}{1 - k} \| x - y \|. \tag{2.7}$$

Taking $p \in F(T_n)$, from 2.7 we have

$$\| (I - T_n) x \| = \| (I - T_n) x - (I - T_n) p \| \le \frac{2}{1 - k} \| x - p \|.$$

Consequently, for $\forall x \in H$, if $\bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$, $\mu_i > 0$ and $\sum_{i=1}^{\infty} \mu_i = 1$, then $\sum_{i=1}^{\infty} \mu_i T_i$ strongly converges.

Let

$$Tx = \sum_{i=1}^{\infty} \mu_i T_i x,$$

we have

$$Tx = \sum_{i=1}^{\infty} \mu_i T_i x = \lim_{n \to \infty} \sum_{i=1}^{n} \mu_i T_i x = \lim_{n \to \infty} \frac{1}{\sum_{i=1}^{n} \mu_i} \sum_{i=1}^{n} \mu_i T_i x.$$

Hence,

$$
\begin{aligned}
&\langle (I - T) x - (I - T) y, x - y \rangle \\
&= \lim_{n \to \infty} \left\langle \left(I - \frac{1}{\sum_{i=1}^{n} \mu_i} \sum_{i=1}^{n} \mu_i T_i \right) x + \left(I - \frac{1}{\sum_{i=1}^{n} \mu_i} \sum_{i=1}^{n} \mu_i T_i \right) y, x - y \right\rangle \\
&= \lim_{n \to \infty} \frac{1}{\sum_{i=1}^{n} \mu_i} \sum_{i=1}^{n} \mu_i \langle (I - T_i) x - (I - T_i) y, x - y \rangle \\
&\ge \lim_{n \to \infty} \frac{1}{\sum_{i=1}^{n} \mu_i} \sum_{i=1}^{n} \mu_i \frac{1 - k}{2} \| (I - T_i) x - (I - T_i) y \|^2 \\
&\ge \frac{1 - k}{2} \lim_{n \to \infty} \left\| \left(I - \frac{1}{\sum_{i=1}^{n} \mu_i} \sum_{i=1}^{n} \mu_i T_i \right) x - \left(I - \frac{1}{\sum_{i=1}^{n} \mu_i} \sum_{i=1}^{n} \mu_i T_i \right) y \right\|^2 \\
&= \frac{1 - k}{2} \| (I - T) x - (I - T) y \|^2.
\end{aligned}
$$

So, we get T is k-strict pseudo-contraction.

Finally, we show $F\left(\sum_{i=1}^{\infty} \mu_i T_i\right) = \bigcap_{i=1}^{\infty} F(T_i)$. Suppose that $x = \sum_{i=1}^{\infty} \mu_i T_i x$, it is sufficient to show that $x \in \bigcap_{i=1}^{\infty} F(T_i)$. Indeed, for $p \in \bigcap_{i=1}^{\infty} F(T_i)$, we have

$$
\begin{aligned}
\|x - p\|^2 &= \langle x - p, x - p \rangle \\
&= \left\langle \sum_{i=1}^{\infty} \mu_i T_i x - p, x - p \right\rangle \\
&= \sum_{i=1}^{\infty} \mu_i \langle T_i x - p, x - p \rangle \\
&\leq \|x - p\|^2 - \frac{1-k}{2} \sum_{i=1}^{\infty} \mu_i \|x - T_i x\|^2,
\end{aligned}
$$

where $k = \sup\{k_i : i \in \mathbb{N}\}$. Hence, we get $x = T_i x$, this means that $x \in \bigcap_{i=1}^{\infty} F(T_i)$.

3 Main results

Lemma 3.1. Let C be a closed convex subset of a real Hilbert space E such that $C + C \subset C$. Let φ be a MKC on C. Suppose $F : C \to C$ be a η-strongly monotone and L-Lipschitzian operator and $0 < \gamma < \eta$ and $T_i : C \to E$ be k_i-strictly pseudo-contractive non-self-mapping such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume $k = \sup\{k_i : i \in \mathbb{N}\} < 1$. Let $\{x_n\}$ be a sequence of C generated by (1.5) with the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$, assume for each n, $\left\{\mu_i^{(n)}\right\}$ is a infinity sequence of positive number such that $\sum_{i=1}^{\infty} \mu_i^{(n)} = 1$.

The following control conditions are satisfied

(i) $\sum_{i=1}^{\infty} \alpha_n = \infty, \quad \lim_{n \to \infty} \alpha_n = 0$

(ii) $k \leq \beta_n < 1$,

(iii) $\lim_{n \to \infty} (\beta_{n+1} - \beta_n) = 0, \quad \lim_{n \to \infty} \sum_{i=1}^{\infty} \left| \mu_i^{(n+1)} - \mu_i^{(n)} \right| = 0$

(iv) $0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1$.

Then $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

Proof. Write, for each $n \geq 0$, $B_n = \sum_{i=1}^{\infty} \mu_i^{(n)} T_i$. By Lemma 2.10, each B_n is a k-strict pseudo-contraction on C and $F(B_n) = \bigcap_{i=1}^{\infty} F(T_i)$ for all n and the algorithm (1.5) can be rewritten as

$$
\begin{cases}
x_1 = x \in C, \\
y_n = P_C [\beta_n x_n + (1 - \beta_n) B_n x_n] \\
x_{n+1} = \alpha_n \gamma \phi(x_n) + \gamma_n x_n + ((1 - \gamma_n) I - \alpha_n F) y_n, \ n \geq 1.
\end{cases} \tag{3.1}
$$

The rest of the proof will now be split into two parts.

Step 1. First, we show that sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Define a mapping

$$
L_n x : P_C [\beta_n x + (1 - \beta_n) B_n x].
$$

Then from the control condition (ii), Lemma 2.3, we obtain $L_n : C \to C$ is non-expansive. Taking a point $p \in \bigcap_{i=1}^{\infty} F(T_i)$, by Lemma 2.2, we can get $L_n p = p$. Hence, we have

$$\|y_n - p\| = \|L_n x_n - p\| \le \|x_n - p\|.$$

Not lose generality, we can assume $r_n \le b < 1$, and $0 < \alpha_n < \dfrac{(\eta - \gamma)(1 - b)}{L^2}$. From definition of MKC and Lemma 2.8, for any $\varepsilon > 0$ there is a number $r_\varepsilon \in (0, 1)$, if $\|x_t - p\| < \varepsilon$ then $\|\varphi(x_t) - \varphi(p)\| < \varepsilon$; If $\|x_t - p\| \ge \varepsilon$ then $\|\varphi(x_t) - \varphi(p)\| \le r_\varepsilon \|x_t - p\|$. It follows 3.1 and Lemma 2.7 that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \gamma \phi(x_n) + \gamma_n x_n + ((1 - \gamma_n) I - \alpha_n F) y_n - p\| \\
&= \|\alpha_n (\gamma \phi(x_n) - Fp) + \gamma_n(x_n - p) + [(1 - \gamma_n) I - \alpha_n F] y_n - [(1 - \gamma_n) I - \alpha_n F] p\| \\
&\le \left[1 - \gamma_n - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)}\right)\right] \|x_n - p\| + \gamma_n \|x_n - p\| + \alpha_n \|\gamma \phi(x_n) - Fp\| \\
&\le \left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)}\right)\right] \|x_n - p\| + \alpha_n \gamma \max\{r_\varepsilon \|x_n - p\|, \varepsilon\} + \alpha_n \|\gamma \phi(p) - Fp\| \\
&= \max\left\{\left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)}\right)\right] \|x_n - p\| + \alpha_n \gamma r_\varepsilon \|x_n - p\| + \alpha_n \|\gamma \phi(p) - Fp\|,\right. \\
&\qquad \left. \left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)}\right)\right] \|x_n - p\| + \alpha_n \gamma \varepsilon + \alpha_n \|\gamma \phi(p) - Fp\|\right\} \\
&= \max\left\{\left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} - \gamma r_\varepsilon\right)\right] \|x_n - p\| + \alpha_n \|\gamma \phi(p) - Fp\|,\right. \\
&\qquad \left. \left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)}\right)\right] \|x_n - p\| + \alpha_n \gamma \varepsilon + \alpha_n \|\gamma \phi(p) - Fp\|\right\}.
\end{aligned}
$$

By induction, we have

$$
\begin{aligned}
&\|x_{n+1} - p\| \\
&\le \max\left\{\left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} - \gamma r_\varepsilon\right)\right] \|x_n - p\| + \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} - \gamma r_\varepsilon\right) \frac{\|\gamma \phi(p) - Fp\|}{\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} - \gamma r_\varepsilon},\right. \\
&\qquad \left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)}\right)\right] \|x_n - p\| + \alpha_n \left[\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)}\right] \frac{\gamma \varepsilon + \|\gamma \phi(p) - Fp\|}{\eta - \frac{\alpha_n L^2}{2(1 - \gamma n)}}\right\} \\
&\le \max\left\{\left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} - \gamma r_\varepsilon\right)\right] \|x_n - p\| + \alpha_n \left[\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} - \gamma r_\varepsilon\right] \frac{2 \|\gamma \phi(p) - Fp\|}{\eta - \gamma},\right. \\
&\qquad \left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)}\right)\right] \|x_n - p\| + \alpha_n \left[\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)}\right] \frac{2 (\gamma \varepsilon + \|\gamma \phi(p) - Fp\|)}{\eta - \gamma}\right\}.
\end{aligned}
$$

Hence, we have

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{2 (\gamma \varepsilon + \|\gamma \phi(p) - Fp\|)}{\eta + \gamma r_\varepsilon}\right\}, \quad n \ge 0,$$

which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{L_n x_n\}$.

Step 2. In this part, we shall claim that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$. From (3.1), we get

$$x_{n+1} = \alpha_n \gamma \phi(x_n) + \gamma_n x_n + ((1 - \gamma_n) I - \alpha_n F) L_n x_n. \tag{3.2}$$

Define

$$x_{n+1} = (1 - \gamma_n) l_n + \gamma_n x_n, \quad \forall n \ge 0, \tag{3.3}$$

where

$$l_n = \frac{x_{n+1} - \gamma_n x_n}{1 - \gamma_n}.$$

It follows that

$$l_{n+1} - l_n$$

$$= \frac{\alpha_{n+1} \gamma \phi(x_{n+1}) + \gamma_{n+1} x_{n+1} + [(1 - \gamma_{n+1}) I - \alpha_{n+1} F] L_{n+1} x_{n+1} - \gamma_{n+1} x_{n+1}}{1 - \gamma_{n+1}}$$

$$- \frac{\alpha_n \gamma \phi(x_n) + \gamma_n x_n + [(1 - \gamma_n) I - \alpha_n F] L_n x_n - \gamma_n x_n}{1 - \gamma_n}$$

$$= \frac{\alpha_{n+1} [\gamma \phi(x_{n+1}) - F L_{n+1} x_{n+1}]}{1 - \gamma_{n+1}} - \frac{\alpha_n [\gamma \phi(x_n) - F L_n x_n]}{1 - \gamma_n} + L_{n+1} x_{n+1} - L_n x_n$$

which yields that

$$\|l_{n+1} - l_n\|$$

$$\leq \frac{\alpha_{n+1} \|\gamma \phi(x_{n+1}) - F L_{n+1} x_{n+1}\|}{1 - \gamma_{n+1}} + \frac{\alpha_n \|\gamma \phi(x_n) - F L_n x_n\|}{1 - \gamma_n} + \|L_{n+1} x_{n+1} - L_n x_n\|$$

$$\leq \frac{\alpha_{n+1} \|\gamma \phi(x_{n+1}) - F L_{n+1} x_{n+1}\|}{1 - \gamma_{n+1}} + \frac{\alpha_n \|\gamma \phi(x_n) - F L_n x_n\|}{1 - \gamma_n} + \|L_{n+1} x_{n+1} - L_{n+1} x_n\| \tag{3.4}$$

$$+ \|L_{n+1} x_n - L_n x_n\|$$

$$\leq \frac{\alpha_{n+1} \|\gamma \phi(x_{n+1}) - F L_{n+1} x_{n+1}\|}{1 - \gamma_{n+1}} + \frac{\alpha_n \|\gamma \phi(x_n) - F L_n x_n\|}{1 - \gamma_n} + \|x_{n+1} - x_n\|$$

$$+ \|L_{n+1} x_n - L_n x_n\|.$$

Next, we estimate $\|L_{n+1} x_n - L_n x_n\|$. Notice that

$$\|L_{n+1} x_n - L_n x_n\| = \|P_C [\beta_{n+1} x_n + (1 - \beta_{n+1}) B_{n+1} x_n] - P_C [\beta_n x_n + (1 - \beta_n) B_n x_n]\|$$

$$\leq \|[\beta_{n+1} x_n + (1 - \beta_{n+1}) B_{n+1} x_n] - [\beta_n x_n + (1 - \beta_n) B_n x_n]\|$$

$$\leq \|x_n - B_{n+1} x_n\| |\beta_{n+1} - \beta_n| + (1 - \beta_n) \|B_{n+1} x_n - B_n x_n\| \tag{3.5}$$

$$\leq \|x_n - B_{n+1} x_n\| |\beta_{n+1} - \beta_n| + (1 - \beta_n) \sum_{i=1}^{\infty} \left| \mu_i^{(n+1)} - \mu_i^{(n)} \right| \|T_i x_n\|.$$

Substituting (3.5) into (3.4), we have

$$\|l_{n+1} - l_n\| \leq \frac{\alpha_{n+1} \|\gamma \phi(x_{n+1}) - F L_{n+1} x_{n+1}\|}{1 - \gamma_{n+1}} + \frac{\alpha_n \|\gamma \phi(x_n) - F L_n x_n\|}{1 - \gamma_n} + \|x_{n+1} - x_n\|$$

$$+ \|x_n - B_{n+1} x_n\| |\beta_{n+1} - \beta_n| + (1 - \beta_n) \sum_{i=1}^{\infty} \left| \mu_i^{(n+1)} - \mu_i^{(n)} \right| \|T_i x_n\|.$$

Hence, we have

$$\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|$$

$$\leq \frac{\alpha_{n+1} \|\gamma \phi(x_{n+1}) - F L_{n+1} x_{n+1}\|}{1 - \gamma_{n+1}} + \frac{\alpha_n \|\gamma \phi(x_n) - F L_n x_n\|}{1 - \gamma_n} + \|x_n - B_{n+1} x_n\| |\beta_{n+1} - \beta_n|$$

$$+ (1 - \beta_n) \sum_{i=1}^{\infty} \left| \mu_i^{(n+1)} - \mu_i^{(n)} \right| \|T_i x_n\|.$$

Observing conditions (i), (iii), (iv), and the boundedness of $\{x_n\}$ it follows that

$$\limsup_{n \to \infty} \{\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|\} \leq 0.$$

Thus by Lemma 2.4, we have $\lim_{n \to \infty} \|l_n - x_n\| = 0$.

From (3.3), we have

$$x_{n+1} - x_n = (1 - \gamma_n)(l_n - x_n).$$

Therefore,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Theorem 3.2.. Let H be a real Hilbert space and C is a closed convex subset of H such that $C + C \subseteq C$. Let φ be an MKC on C. Suppose $F: C \to C$ is η-strongly monotone and L-Lipschitzian operator and $\eta > \gamma > 0$. Let $T_i : C \to E$ be k_i-strictly pseudo-contractive non-self-mapping such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume $k = \sup\{k_i : i \in \mathbb{N}\} < 1$. Let $\{x_n\}$ be a sequence of C generated by (1.5) with the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$. Assume for each n, $\sum_{i=1}^{\infty} \mu_i^{(n)} = 1$ for all n and $\mu_i^{(n)} > 0$ for all $i \in \mathbb{N}$. They satisfy the conditions (i), (ii), (iii), (iv) of Lemma 3.1 and (v) $\lim_{n \to \infty} \beta_n = \alpha$, $\lim_{n \to \infty} \sum_{i=1}^{\infty} |\mu_i^n - \mu_i| = 0$ and $\sum_{i=1}^{\infty} \mu_i = 1 \, (\mu_i > 0)$. Then $\{x_n\}$ converges strongly to $\tilde{x} \in F$, which also solves the following variational inequality

$$\langle \gamma \phi(\tilde{x}) - F\tilde{x}, p - \tilde{x} \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{\infty} F(T_i).$$

Proof. From (3.1), we obtain

$$
\begin{aligned}
\|L_n x_n - x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - L_n x_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n \gamma \phi(x_n) + \gamma_n(x_n - L_n x_n) - \alpha_n F L_n x_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n(\|\gamma \phi(x_n)\| + \|F L_n x_n\|) + \gamma_n \|x_n - L_n x_n\|.
\end{aligned}
$$

So $\|L_n x_n - x_n\| \leq \frac{1}{1-\gamma_n}(\|x_n - x_{n+1}\| + \alpha_n(\|\gamma \phi(x_n)\| + \|F L_n x_n\|)$, which together with the condition (i), (iv) and Lemma 3.1 implies

$$\lim_{n \to \infty} \|L_n x_n - x_n\| = 0. \tag{3.7}$$

Define $B = \sum_{i=1}^{\infty} \mu_i T_i$, then $B : C \to E$ is a k-strict pseudo-contraction such that $F(B) = \bigcap_{i=1}^{\infty} F(T_i)$ by Lemma 2.10, furthermore $B_n x \to Bx$ as $n \to \infty$ for all $x \in C$ by (v). Defines $T: C \to E$ by

$$Tx = \alpha x + (1 - \alpha)Bx.$$

Then, T is non-expansive with $F(T) = F(B)$ by Lemma 2.3. It follows from Lemma 2.2 that $F(P_C T) = F(T)$. Notice that

$$
\begin{aligned}
\|P_C T x_n - x_n\| &\leq \|x_n - L_n x_n\| + \|L_n x_n - P_C T x_n\| \\
&\leq \|x_n - L_n x_n\| + \|\beta_n x_n + (1 - \beta_n)B_n x_n - [\alpha x_n + (1 - \alpha)Bx_n]\| \\
&\leq \|x_n - L_n x_n\| + \|(\beta_n - \alpha)(x_n - B_n x_n) + (1 - \alpha)(B_n x_n - Bx_n)\| \\
&\leq \|x_n - L_n x_n\| + (\beta_n - a)\|x_n - B_n x_n\| + (1 - \alpha)\|B_n x_n - Bx_n\|
\end{aligned}
$$

which combines with (3.7) yielding that

$$\lim_{n \to \infty} \|P_C T x_n - x_n\| = 0. \tag{3.8}$$

Next, we show that

$$\limsup_{n\to\infty} \langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_n - \tilde{x} \rangle \leq 0, \tag{3.9}$$

where $\tilde{x} = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto t\gamma\phi(x) + (1 - tF)P_C Tx.$$

To see this, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_n - \tilde{x} \rangle = \lim_{k\to\infty} \langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n_k} - \tilde{x} \rangle.$$

We may also assume that $x_{n_k} \rightharpoonup q$. Note that $q \in F(T)$ in virtue of Lemmas 2.6, 2.2, and (3.8). It follows from Lemma 2.9, we can get that

$$\limsup_{n\to\infty} \langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_n - \tilde{x} \rangle = \lim_{k\to\infty} \langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n_k} - \tilde{x} \rangle = \langle \gamma\phi(\tilde{x}) - F\tilde{x}, q - \tilde{x} \rangle \leq 0.$$

Finally, we show $\|x_n - \tilde{x}\| \to 0$. By contradiction, there is a number ε_0 such that

$$\limsup_{n\to\infty} \|x_n - \tilde{x}\| \geq \varepsilon_0.$$

Case 1. Fixed ε_1 ($\varepsilon_1 < \varepsilon_0$), if for some $n \geq N \in \mathbb{N}$ such that $\|x_n - \tilde{x}\| \geq \varepsilon_0 - \varepsilon_1$, and for the other $n \geq N \in \mathbb{N}$ such that $\|x_n - \tilde{x}\| < \varepsilon_0 - \varepsilon_1$.

Let

$$M_n = \frac{2 \langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n+1} - \tilde{x} \rangle}{(\varepsilon_0 - \varepsilon_1)^2}.$$

From 3.9, we know $\limsup_{n\to\infty} M_n \leq 0$. Hence, there are two numbers h and N, when $n > N$ we have $M_n \leq h$, where $h = \min \left\{ \eta - \frac{\alpha_n L^2}{2(1-\gamma_n)} - \gamma \right\}$. From the above introduction, we can extract a number $n_0 > N$ satisfying $\|x_{n_0} - \tilde{x}\| < \varepsilon_0 - \varepsilon_1$, then we estimate $\|x_{n_0+1} - \tilde{x}\|$. From Lemma 2.7 and (3.1), we have

$$\|x_{n_0+1} - \tilde{x}\|^2$$
$$= \|\alpha_{n_0}\gamma\phi(x_{n_0}) + \gamma_{n_0}x_{n_0} + [(1-\gamma_{n_0})I - \alpha_{n_0}F]y_{n_0} - \tilde{x}\|^2$$
$$= \|[(1-\gamma_{n_0})I - \alpha_{n_0}F]y_{n_0} - [(1-\gamma_{n_0})I - \alpha_{n_0}F]\tilde{x} + \alpha_{n_0}[\gamma\phi(x_{n_0}) - F\tilde{x}] + \gamma_{n_0}(x_{n_0} - \tilde{x})\|^2$$
$$= \langle [(1-\gamma_{n_0})I - \alpha_{n_0}F]y_{n_0} - [(1-\gamma_{n_0})I - \alpha_{n_0}F]\tilde{x} + \alpha_{n_0}[\gamma\phi(x_{n_0}) - F\tilde{x}] + \gamma_{n_0}(x_{n_0} - \tilde{x}), x_{n_0+1} - \tilde{x} \rangle$$
$$= \langle [(1-\gamma_{n_0})I - \alpha_{n_0}F]y_{n_0} - [(1-\gamma_{n_0})I - \alpha_{n_0}F]\tilde{x}, x_{n_0+1} - \tilde{x} \rangle + \langle \alpha_{n_0}[\gamma\phi(x_{n_0}) - F\tilde{x}], x_{n_0+1} - \tilde{x} \rangle$$
$$\quad + \langle \gamma_{n_0}(x_{n_0} - \tilde{x}), x_{n_0+1} - \tilde{x} \rangle$$
$$= \langle [(1-\gamma_{n_0})I - \alpha_{n_0}F]y_{n_0} - [(1-\gamma_{n_0})I - \alpha_{n_0}F]\tilde{x}, x_{n_0+1} - \tilde{x} \rangle + \alpha_{n_0}\langle \gamma\phi(x_{n_0}) - \gamma\phi(\tilde{x}), x_{n_0+1} - \tilde{x} \rangle$$
$$\quad + \alpha_{n_0}\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n_0+1} - \tilde{x} \rangle + \gamma_{n_0}\langle x_{n_0} - \tilde{x}, x_{n_0+1} - \tilde{x} \rangle$$
$$\leq \|[(1-\gamma_{n_0})I - \alpha_{n_0}F]y_{n_0} - [(1-\gamma_{n_0})I - \alpha_{n_0}F]\tilde{x}\| \|x_{n_0+1} - \tilde{x}\| + \alpha_{n_0}\gamma\|x_{n_0} - \tilde{x}\| \|x_{n_0+1} - \tilde{x}\|$$
$$\quad + \alpha_{n_0}\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n_0+1} - \tilde{x} \rangle + \gamma_{n_0}\|x_{n_0} - \tilde{x}\| \|x_{n_0+1} - \tilde{x}\|$$
$$\leq \left[1 - \gamma_{n_0} - \alpha_{n_0}\left(\eta - \frac{\alpha_{n_0}L^2}{2(1-\gamma_{n_0})} \right) \right] \|x_{n_0} - \tilde{x}\| \|x_{n_0+1} - \tilde{x}\| + \alpha_{n_0}\gamma\|x_{n_0} - \tilde{x}\| \|x_{n_0+1} - \tilde{x}\|$$
$$\quad + \alpha_{n_0}\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n_0+1} - \tilde{x} \rangle + \gamma_{n_0}\|x_{n_0} - \tilde{x}\| \|x_{n_0+1} - \tilde{x}\|$$
$$= \left[1 - \alpha_{n_0}\left(\eta - \frac{\alpha_{n_0}L^2}{2(1-\gamma_{n_0})} - \gamma \right) \right] \|x_{n_0} - \tilde{x}\| \|x_{n_0+1} - \tilde{x}\| + \alpha_{n_0}\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n_0+1} - \tilde{x} \rangle$$
$$\leq \frac{1}{2}\left[1 - \alpha_{n_0}\left(\eta - \frac{\alpha_{n_0}L^2}{2(1-\gamma_n)} - \gamma \right) \right] \|x_n - \tilde{x}\|^2 + \frac{1}{2}\|x_{n_0+1} - \tilde{x}\|^2 + \alpha_{n_0}\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n_0+1} - \tilde{x} \rangle$$
$$\leq \frac{1}{2}\left[1 - \alpha_{n_0}\left(\eta - \frac{\alpha_{n_0}L^2}{2(1-\gamma_n)} - \gamma \right) \right] (\varepsilon_0 - \varepsilon)^2 + \frac{1}{2}\|x_{n_0+1} - \tilde{x}\|^2 + \alpha_{n_0}\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n_0+1} - \tilde{x} \rangle$$

which implies that

$$
\begin{aligned}
&\left\| x_{n_0+1} - \tilde{x} \right\|^2 \\
&< \left[1 - \alpha_{n_0} \left(\eta - \frac{\alpha_{n_0} L^2}{2(1 - \gamma_{n_0})} - \gamma \right) \right] (\varepsilon_0 - \varepsilon)^2 + 2\alpha_{n_0} \left\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n_0+1} - \tilde{x} \right\rangle \\
&= \left[1 - \alpha_n \left(\eta - \frac{\alpha_{n_0} L^2}{2(1 - \gamma_{n_0})} - \gamma - M_{n_0} \right) \right] (\varepsilon_0 - \varepsilon_1)^2 \\
&\leq (\varepsilon_0 - \varepsilon_1)^2
\end{aligned}
$$

Hence, we have

$$
\left\| x_{n_0+1} - \tilde{x} \right\| < \varepsilon_0 - \varepsilon_1.
$$

In the same way, we can get

$$
\left\| x_n - \tilde{x} \right\| < \varepsilon_0 - \varepsilon_1, \quad \forall n \geq n_0.
$$

It contradict the $\limsup_{n\to\infty} \left\| x_n - \tilde{x} \right\| \geq \varepsilon_0$.

Case 2. Fixed ε_1 ($\varepsilon_1 < \varepsilon_0$), if $\left\| x_n - \tilde{x} \right\| \geq \varepsilon_0 - \varepsilon_1$ for all $n \geq N \in \mathbb{N}$. In this case from Lemma 2.8, there is a number $r \in (0,1)$, such that

$$
\left\| \phi(x_n) - \phi(\tilde{x}) \right\| \leq r \left\| x_n - \tilde{x} \right\|, \quad n \geq N.
$$

It follow 3.1 that

$$
\begin{aligned}
&\left\| x_{n+1} - \tilde{x} \right\|^2 \\
&= \left\langle [(1 - \gamma_n) I - \alpha_n F] y_n - [(1 - \gamma_n) I - \alpha_n F] \tilde{x}, x_{n+1} - \tilde{x} \right\rangle + \alpha_n \left\langle \gamma\phi(x_n) - \gamma\phi(\tilde{x}), x_{n+1} - \tilde{x} \right\rangle \\
&\quad + \alpha_n \left\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n+1} - \tilde{x} \right\rangle + \gamma_n \left\langle x_n - \tilde{x}, x_{n+1} - \tilde{x} \right\rangle \\
&\leq \left[1 - \gamma_n - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} \right) \right] \left\| x_n - \tilde{x} \right\| \left\| x_{n+1} - \tilde{x} \right\| + \alpha_n \gamma r \left\| x_n - \tilde{x} \right\| \left\| x_{n+1} - \tilde{x} \right\| \\
&\quad + \alpha_n \left\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n+1} - \tilde{x} \right\rangle + \gamma_n \left\| x_n - \tilde{x} \right\| \left\| x_{n+1} - \tilde{x} \right\| \\
&\leq \frac{1}{2} \left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} - \gamma r \right) \right] \left\| x_n - \tilde{x} \right\|^2 + \frac{1}{2} \left\| x_{n+1} - \tilde{x} \right\|^2 + \frac{1}{2} \left\| x_{n+1} - \tilde{x} \right\|^2 + \alpha_n \left\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n+1} - \tilde{x} \right\rangle
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left\| x_{n+1} - \tilde{x} \right\|^2 \\
&\leq \left[1 - \alpha_n \left(\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} - \gamma r \right) \right] \left\| x_n - \tilde{x} \right\|^2 + \alpha_n \left[\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} - \gamma r \right] \frac{2 \left\langle \gamma\phi(\tilde{x}) - F\tilde{x}, x_{n+1} - \tilde{x} \right\rangle}{\eta - \frac{\alpha_n L^2}{2(1 - \gamma_n)} - \gamma r}. \quad (3.10)
\end{aligned}
$$

Apply Lemma 2.5 to (3.10) to conclude $x_n \to \tilde{x}$ as $n \to \infty$. It contradict the $\left\| x_n - \tilde{x} \right\| \geq \varepsilon_0 - \varepsilon_1$. This completes the proof.

Remark 3. We conclude the article with the following observations.

(i) Theorem 3.2 improve and extend Theorem 3.4 of Marino and Xu [24], Theorem 3.2 of Zhou [8], Theorem 2.1 of Qin [9] and includes those results as special cases. Especially, our results extends above results form contractions to more general MKC. Our iterative scheme studied in this article can be viewed as a refinement and modification of the iterative methods in [8,9,24]. On the other hand, our iterative schemes concern an infinite countable family of k_i-strict pseudo-contractions mappings, in this respect, they can be viewed as an another improvement.

(ii) Our results extend above results form strong positive linear bounded operator to η-strongly monotone and L-Lipschitzian operator.

(iii) The advantage of the results in this article is that less restrictions on the parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\eta_i^n\}$ are imposed. Our results unify many recent results including the results in [8,9,24].

(iv) It is worth noting that we obtained two strong convergence results concerning an infinite countable family of λ_i-strict pseudo-contractions mappings. Our result is new and the proofs are simple and different from those in [6,8,9,24,25].

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Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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