Moosaei *Fixed Point Theory and Applications* 2012, **2012**:164
http://www.fixedpointtheoryandapplications.com/content/2012/1/164

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Fixed point theorems in convex metric spaces

Mohammad Moosaei*

*Correspondence:
m.moosaei@basu.ac.ir
Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran

**Abstract**
In this paper, we study some fixed point theorems for self-mappings satisfying certain contraction principles on a convex complete metric space. In addition, we investigate some common fixed point theorems for a Banach operator pair under certain generalized contractions on a convex complete metric space. Finally, we also improve and extend some recent results.
**MSC:** 47H09; 47H10; 47H19; 54H25

**Keywords:** Banach operator pair; common fixed point; convex metric spaces; fixed point

## 1 Introduction

In 1970, Takahashi [1] introduced the notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. A convex metric space is a generalized space. For example, every normed space and cone Banach space is a convex metric space and convex complete metric space, respectively. Subsequently, Beg [2], Beg and Abbas [3, 4], Chang, Kim and Jin [5], Ciric [6], Shimizu and Takahashi [7], Tian [8], Ding [9], and many others studied fixed point theorems in convex metric spaces.

The purpose of this paper is to study the existence of a fixed point for self-mappings defined on a nonempty closed convex subset of a convex complete metric space that satisfies certain conditions. We also study the existence of a common fixed point for a Banach operator pair defined on a nonempty closed convex subset of a convex complete metric space that satisfies suitable conditions. Our results improve and extend some of Karapinar's results in [10] from a cone Banach space to a convex complete metric space. For instance, Karapinar proved that for a closed convex subset $C$ of a cone Banach space $X$ with the norm $\|x\|_p = d(x,0)$, if a mapping $T\colon C \to C$ satisfies the condition

$$d(x,Tx) + d(y,Ty) \leq qd(x,y)$$

for all $x,y \in C$, where $2 \leq q < 4$, then $T$ has at least one fixed point. Letting $x = y$ in the above inequality, it is easy to see that $T$ is an identity mapping. In this paper, the above result is improved and extended to a convex complete metric space.

## 2 Preliminaries

**Definition 2.1** (see [11]) Let $(X,d)$ be a metric space and $I = [0,1]$. A mapping $W\colon X \times X \times I \to X$ is said to be a convex structure on $X$ if for each $(x,y,\lambda) \in X \times X \times I$ and $u \in X$,

$$d\bigl(u, W(x,y,\lambda)\bigr) \leq \lambda d(u,x) + (1-\lambda)d(u,y).$$

A metric space $(X, d)$ together with a convex structure $W$ is called a convex metric space, which is denoted by $(X, d, W)$.

**Example 2.2** Let $(X, \| \|)$ be a normed space. The mapping $W: X \times X \times I \to X$ defined by $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ for each $x, y \in X$, $\lambda \in I$ is a convex structure on $X$.

**Definition 2.3** (see [11]) Let $(X, d, W)$ be a convex metric space. A nonempty subset $C$ of $X$ is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times I$.

**Definition 2.4** (see [3]) Let $(X, d, W)$ be a convex metric space and $C$ be a convex subset of $X$. A self-mapping $f$ on $C$ has a property (I) if $f(W(x, y, \lambda)) = W(f(x), f(y), \lambda)$ for each $x, y \in C$ and $\lambda \in I$.

**Example 2.5** If $(X, \| \|)$ is a normed space, then every affine mapping on a convex subset of $X$ has the property (I).

**Definition 2.6** Let $f, g: X \to X$. A point $x \in X$ is called
(i) a fixed point of $f$ if $f(x) = x$,
(ii) a coincidence point of a pair $(f, g)$ if $f(x) = g(x)$,
(iii) a common fixed point of a pair $(f, g)$ if $f(x) = g(x) = x$.
$F(f)$, $C(f, g)$, and $F(f, g)$ denote the set of all fixed points of $f$, coincidence points of the pair $(f, g)$, and common fixed points of the pair $(f, g)$, respectively.

**Definition 2.7** (see [12, 13]) The ordered pair $(f, g)$ of two self-maps of a metric space $(X, d)$ is called a Banach operator pair if $F(g)$ is $f$-invariant, namely $f(F(g)) \subseteq F(g)$.

**Example 2.8** (i) Let $(X, d)$ be a metric space and $k \geq 0$. If the self-maps $f$, $g$ of $X$ satisfy $d(g(f(x)), f(x)) \leq kd(g(x), x)$ for all $x \in X$, then $(f, g)$ is a Banach operator pair.

(ii) It is obvious that a commuting pair $(f, g)$ of self-maps on $X$ (namely $fg(x) = gf(x)$ for all $x \in X$) is a Banach operator pair, but the converse is generally not true. For example, let $X = \mathbb{R}$ with the usual norm, and let $f(x) = x^2 - 2x$, $g(x) = x^2 - x - 3$ for all $x \in X$, then $F(g) = \{-1, 3\}$. Moreover, $f(F(g)) \subseteq F(g)$ implies that $(f, g)$ is a Banach operator pair, but the pair $(f, g)$ does not commute.

In [10], Karapinar obtained the following theorems.

**Theorem 2.9** (see Theorem 2.4 of [10]) *Let $C$ be a closed and convex subset of a cone Banach space $X$ with the norm $\|x\|_p = d(x, 0)$, and $T: C \to C$ be a mapping which satisfies the condition*

$$d(x, Tx) + d(y, Ty) \leq qd(x, y)$$

*for all $x, y \in C$, where $2 \leq q < 4$. Then, $T$ has at least one fixed point.*

**Theorem 2.10** (see Theorem 2.6 of [10]) *Let $C$ be a closed and convex subset of a cone Banach space $X$ with the norm $\|x\|_p = d(x, 0)$, and $T: C \to C$ be a mapping which satisfies the condition*

$$d(Tx, Ty) + d(x, Tx) + d(y, Ty) \leq rd(x, y)$$

*for all $x, y \in C$, where $2 \leq r < 5$. Then, $T$ has at least one fixed point.*

## 3 Main results

To prove the next theorem, we need the following lemma.

**Lemma 3.1** *Let $(X, d, W)$ be a convex metric space, then the following statements hold*:

(i) $d(x,y) = d(x, W(x,y,\lambda)) + d(y, W(x,y,\lambda))$ *for all* $(x,y,\lambda) \in X \times X \times I$.

(ii) $d(x, W(x,y,\frac{1}{2})) = d(y, W(x,y,\frac{1}{2})) = \frac{1}{2}d(x,y)$ *for all* $x, y \in X$.

*Proof* (i) For any $(x,y,\lambda) \in X \times X \times I$, we have

$$\begin{aligned} d(x,y) &\leq d\big(x, W(x,y,\lambda)\big) + d\big(y, W(x,y,\lambda)\big) \\ &\leq (1-\lambda)d(x,y) + \lambda d(x,y) \\ &= d(x,y). \end{aligned}$$

Therefore, $d(x,y) = d(x, W(x,y,\lambda)) + d(y, W(x,y,\lambda))$ holds.

(ii) Let $x, y \in X$. By the definition of $W$ and using (i), we have

$$d\left(x, W\left(x,y,\frac{1}{2}\right)\right) \leq \frac{1}{2}d(x,y) = \frac{1}{2}d\left(x, W\left(x,y,\frac{1}{2}\right)\right) + \frac{1}{2}d\left(y, W\left(x,y,\frac{1}{2}\right)\right).$$

Therefore,

$$\frac{1}{2}d\left(x, W\left(x,y,\frac{1}{2}\right)\right) \leq \frac{1}{2}d\left(y, W\left(x,y,\frac{1}{2}\right)\right).$$

Similarly,

$$\frac{1}{2}d\left(y, W\left(x,y,\frac{1}{2}\right)\right) \leq \frac{1}{2}d\left(x, W\left(x,y,\frac{1}{2}\right)\right).$$

Therefore, $d(x, W(x,y,\frac{1}{2})) = d(y, W(x,y,\frac{1}{2}))$. Now, from (i), we obtain

$$d\left(x, W\left(x,y,\frac{1}{2}\right)\right) = d\left(y, W\left(x,y,\frac{1}{2}\right)\right) = \frac{1}{2}d(x,y)$$

for all $x, y \in C$, and the proof of the lemma is complete. □

The following theorem improves and extends Theorem 2.6 in [10].

**Theorem 3.2** *Let $C$ be a nonempty closed convex subset of a convex complete metric space $(X, d, W)$ and $f$ be a self-mapping of $C$. If there exist $a$, $b$, $c$, $k$ such that*

$$2b - |c| \leq k < 2(a+b+c) - |c|, \tag{3.1}$$

$$ad\big(x, f(x)\big) + bd\big(y, f(y)\big) + cd\big(f(x), f(y)\big) \leq kd(x,y) \tag{3.2}$$

*for all $x, y \in C$, then $f$ has at least one fixed point.*

*Proof* Suppose $x_0 \in C$ is arbitrary. We define a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$x_n = W\left(x_{n-1}, f(x_{n-1}), \frac{1}{2}\right), \quad n = 1, \ldots. \tag{3.3}$$

As $C$ is convex, $x_n \in C$ for all $n \in \mathbb{N}$. By Lemma 3.1(ii) and (3.3), we have

$$d\big(x_n, f(x_n)\big) = 2d(x_n, x_{n+1}), \tag{3.4}$$

$$d\big(x_n, f(x_{n-1})\big) = d(x_n, x_{n-1}) \tag{3.5}$$

for all $n \in \mathbb{N}$. Now, by substituting $x$ with $x_n$ and $y$ with $x_{n-1}$ in (3.2), we get

$$ad\big(x_n, f(x_n)\big) + bd\big(x_{n-1}, f(x_{n-1})\big) + cd\big(f(x_n), f(x_{n-1})\big) \leq kd(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$. Therefore, from (3.4) and (3.5), it follows that

$$2ad(x_n, x_{n+1}) + 2bd(x_n, x_{n-1}) + cd\big(f(x_n), f(x_{n-1})\big) \leq kd(x_n, x_{n-1}) \tag{3.6}$$

for all $n \in \mathbb{N}$. Let $c$ be a nonnegative number. Using the triangle inequality, (3.4) and (3.5), we obtain

$$2cd(x_n, x_{n+1}) - cd(x_n, x_{n-1}) \leq cd\big(f(x_n), f(x_{n-1})\big)$$

for all $n \in \mathbb{N}$. Similarly, for the case $c < 0$, we have

$$2cd(x_n, x_{n+1}) + cd(x_n, x_{n-1}) \leq cd\big(f(x_n), f(x_{n-1})\big)$$

for all $n \in \mathbb{N}$. Therefore, for each case we have

$$2cd(x_n, x_{n+1}) - |c|d(x_n, x_{n-1}) \leq cd\big(f(x_n), f(x_{n-1})\big) \tag{3.7}$$

for all $n \in \mathbb{N}$. Now, from (3.6) and (3.7), it follows that

$$2ad(x_n, x_{n+1}) + 2bd(x_n, x_{n-1}) + 2cd(x_n, x_{n+1}) - |c|d(x_n, x_{n-1}) \leq kd(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$. This implies

$$d(x_n, x_{n+1}) \leq \frac{k - 2b + |c|}{2(a + c)} d(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$. From (3.1), $\frac{k-2b+|c|}{2(a+c)} \in [0, 1)$, and hence, $\{x_n\}_{n=1}^{\infty}$ is a contraction sequence in $C$. Therefore, it is a Cauchy sequence. Since $C$ is a closed subset of a complete space, there exists $v \in C$ such that $\lim_{n\to\infty} x_n = v$. Therefore, the triangle inequality and (3.4) imply $\lim_{n\to\infty} f(x_n) = v$. Now, by substituting $x$ with $v$ and $y$ with $x_n$ in (3.2), we obtain

$$ad\big(v, f(v)\big) + bd\big(x_n, f(x_n)\big) + cd\big(f(v), f(x_n)\big) \leq kd(v, x_n)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, it follows that

$$(a + c)d\big(v, f(v)\big) \leq 0.$$

Since $a + c$ is positive from (3.1), it follows that $d(v, f(v)) = 0$. Therefore, $f(v) = v$ and the proof of the theorem is complete. □

The following corollary improves and extends Theorem 2.4 in [10].

**Corollary 3.3** *Let $(X, d, W)$ be a convex complete metric space and $C$ be a nonempty closed convex subset of $X$. Suppose that $f$ is a self-map of $C$. If there exist $a$, $b$, $k$ such that*

$$2b \leq k < 2(a + b),$$

$$ad\big(x, f(x)\big) + bd\big(y, f(y)\big) \leq kd(x, y)$$

*for all $x, y \in C$, then $F(f)$ is a nonempty set.*

*Proof* Set $c = 0$ in Theorem 3.2. □

**Theorem 3.4** *Let $(X, d, W)$ be a convex complete metric space and $C$ be a nonempty subset of $X$. Suppose that $f$, $g$ are self-mappings of $C$, and there exist $a$, $b$, $c$, $k$ such that*

$$2b - |c| \leq k < 2(a + b + c) - |c|, \tag{3.8}$$

$$ad\big(g(x), f(x)\big) + bd\big(g(y), f(y)\big) + cd\big(f(x), f(y)\big) \leq kd\big(g(x), g(y)\big) \tag{3.9}$$

*for all $x, y \in C$. If $(f, g)$ is a Banach operator pair, $g$ has the property (I) and $F(g)$ is a nonempty closed subset of $C$, then $F(f, g)$ is nonempty.*

*Proof* From (3.9), we obtain

$$ad\big(x, f(x)\big) + bd\big(y, f(y)\big) + cd\big(f(x), f(y)\big) \leq kd(x, y) \tag{3.10}$$

for all $x, y \in F(g)$. $F(g)$ is convex because $g$ has the property (I). It follows from Theorem 3.2 that $F(f, g)$ is nonempty. □

**Theorem 3.5** *Let $(X, d, W)$ be a convex complete metric space and $C$ be a nonempty subset of $X$. Suppose that $f$, $g$ are self-mappings of $C$, $F(g)$ is a nonempty closed subset of $C$, and there exist $a$, $b$, $c$, $k$ such that*

$$2b - |c| \leq k < 2(a + b + c) - |c|, \tag{3.11}$$

$$ad\big(g(x), g\big(f(x)\big)\big) + bd\big(g(y), g\big(f(y)\big)\big) + cd\big(g\big(f(x)\big), g\big(f(y)\big)\big) \leq kd\big(g(x), g(y)\big) \tag{3.12}$$

*for all $x, y \in C$. If $(f, g)$ is a Banach operator pair and $g$ has the property (I), then $F(f, g)$ is nonempty.*

*Proof* Since $(f, g)$ is a Banach operator pair from (3.12), we have

$$ad\big(x, f(x)\big) + bd\big(y, f(y)\big) + cd\big(f(x), f(y)\big) \leq kd(x, y)$$

for all $x, y \in F(g)$. Because $g$ has the property (I) and $F(g)$ is closed, Theorem 3.2 guaranties that $F(f, g)$ is nonempty. □

**Competing interests**
The author declares that he has no competing interests.

**Acknowledgements**
The author is grateful to Bu-Ali Sina University for supporting this research.

Received: 28 February 2012 Accepted: 30 August 2012 Published: 25 September 2012

**References**

1. Takahashi, T: A convexity in metric spaces and nonexpansive mapping I. Kodai Math. Semin. Rep. **22**, 142-149 (1970)
2. Beg, I: An iteration scheme for asymptotically nonexpansive mappings on uniformly convex metric spaces. Nonlinear Anal. Forum **6**(1), 27-34 (2001)
3. Beg, I, Abbas, M: Common fixed points and best approximation in convex metric spaces. Soochow J. Math. **33**(4), 729-738 (2007)
4. Beg, I, Abbas, M: Fixed-point theorem for weakly inward multivalued maps on a convex metric space. Demonstr. Math. **39**(1), 149-160 (2006)
5. Chang, SS, Kim, JK, Jin, DS: Iterative sequences with errors for asymptotically quasi nonexpansive mappings in convex metric spaces. Arch. Inequal. Appl. **2**, 365-374 (2004)
6. Ciric, L: On some discontinuous fixed point theorems in convex metric spaces. Czechoslov. Math. J. **43**(188), 319-326 (1993)
7. Shimizu, T, Takahashi, W: Fixed point theorems in certain convex metric spaces. Math. Jpn. **37**, 855-859 (1992)
8. Tian, YX: Convergence of an Ishikawa type iterative scheme for asymptotically quasi nonexpansive mappings. Comput. Math. Appl. **49**, 1905-1912 (2005)
9. Ding, XP: Iteration processes for nonlinear mappings in convex metric spaces. J. Math. Anal. Appl. **132**, 114-122 (1998)
10. Karapinar, E: Fixed point theorems in cone Banach spaces. Fixed Point Theory Appl. **2009**, Article ID 609281 (2009). doi:10.1155/2009/609281
11. Agarwal, RP, O'Regan, D, Sahu, DR: Fixed Point Theory for Lipschitzian-Type Mappings with Applications. Springer, Heidelberg (2009)
12. Chen, J, Li, Z: Common fixed-points for Banach operator pairs in best approximation. J. Math. Anal. Appl. **336**, 1466-1475 (2007)
13. Pathak, HK, Hussain, N: Common fixed points for Banach operator pairs with applications. Nonlinear Anal. **69**, 2788-2802 (2008)

doi:10.1186/1687-1812-2012-164
**Cite this article as:** Moosaei: **Fixed point theorems in convex metric spaces.** *Fixed Point Theory and Applications* 2012 **2012**:164.