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Common fixed points for self-mappings on partial metric spaces

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Abstract

In this paper, we prove some results of a common fixed point for two self-mappings on partial metric spaces. Our results generalize some interesting results of Ilić *et al.* (Appl. Math. Lett. 24:1326-1330, 2011). We conclude with a result of the existence of a fixed point for set-valued mappings in the context of 0-complete partial metric spaces.

MSC: 54H25; 47H10

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1 Introduction

In the mathematical field of domain theory, attempts were made to equip semantics domain with a notion of distance. In particular, Matthews [1] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification. Moreover, the existence of several connections between partial metrics and topological aspects of domain theory have been lately pointed out by other authors such as O'Neill [2], Bukatin and Scott [3], Bukatin and Shorina [4], Romaguera and Schellekens [5] and others.

After the definition of the concept of a partial metric space, Matthews [1] obtained a Banach type fixed point theorem on complete partial metric spaces. This result was recently generalized by Ilić *et al.* [6]. In this paper, we prove some results of a common fixed point for two self-mappings on partial metric spaces. Our results generalize some interesting results of Ilić *et al.* We conclude this paper with a new existence result of a fixed point for set-valued mappings in a partial metric space.

2 Preliminaries

First, we recall some definitions and some properties of partial metric spaces that can be found in [2, 5–27]. A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty[$ such that for all $x, y, z \in X$,

(p_1)　$x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2)　$p(x, x) \leq p(x, y)$,
(p_3)　$p(x, y) = p(y, x)$,
(p_4)　$p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

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It is clear that (p_4) implies the triangular inequality. A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that if $p(x, y) = 0$, then from (p_1) and (p_2), it follows that $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $([0, +\infty[, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, +\infty[$. Other examples of partial metric spaces, which are interesting from a computational point of view, can be found in [1].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to [0, +\infty[$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X.

Definition 1 Let (X, p) be a partial metric space.
  (i) A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$.
  (ii) A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n, m \to +\infty} p(x_n, x_m)$.
  (iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n, m \to +\infty} p(x_n, x_m)$.
  (iv) A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n, m \to +\infty} p(x_n, x_m) = 0$.

We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

On the other hand, the partial metric space $(\mathbb{Q} \cap [0, +\infty[, p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x, y) = \max\{x, y\}$, provides an example of a 0-complete partial metric space which is not complete.

It is easy to see that every closed subset of a complete partial metric space is complete.

Lemma 1 ([9]) *Let (X, p) be a partial metric space and $\{x_n\} \subset X$. If $x_n \to x \in X$ and $p(x, x) = 0$, then $\lim_{n \to +\infty} p(x_n, z) = p(x, z)$ for all $z \in X$.*

Define $p(x, A) = \inf\{p(x, a) : a \in A\}$. Then $a \in \overline{A} \Leftrightarrow p(a, A) = p(a, a)$, where $\overline{A}$ denotes the closure of A (for details see [10], Lemma 1).

Let X be a non-empty set and $T, f : X \to X$. The mappings T, f are said to be weakly compatible if they commute at their coincidence point (*i.e.*, $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called a point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$.

Lemma 2 (Proposition 1.4 of [28]) *Let X be a non-empty set and the mappings $T, f : X \to X$ have a unique point of coincidence v in X. If T and f are weakly compatible, then v is a unique common fixed point of T and f.*

3 Main results

Let (X, p) be a partial metric space and $T, f : X \to X$ be such that $TX \subset fX$. For every $x_0 \in X$, we consider a sequence $\{x_n\} \subset X$ defined by $fx_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ and we say that $\{Tx_n\}$ is a T-f-sequence of the initial point x_0 (see [29]). Set

$$\rho_f := \inf\{p(fx, fx) : x \in X\} \quad \text{and} \quad X_f = \{x \in X : p(fx, fx) = \rho_f\}.$$

It is not always $X_f \neq \emptyset$. This is true if and only if $\rho_f = \min\{p(fx, fx) : x \in X\}$. If $X = [0, +\infty[$, $p(x, y) = \max\{x, y\}$, $fx = x$, $x \neq 0$ and $f0 > 0$, then $\rho_f = 0$ and $X_f = \emptyset$.

Let (X, p) be a partial metric space. We denote with $\mathcal{F}$ the family of pairs (T, f) such that:

(i) T and f are self-mappings on X with $TX \subset fX$;

(ii) for each $x, y \in X$ the following condition holds:

$$p(Tx, Ty) \leq \max\{kp(fx, fy), p(fx, fx), p(fy, fy)\}, \tag{3.1}$$

where $k \in [0, 1[$.

Remark 1 If $(T, f) \in \mathcal{F}$, then for each $x \in X$, we have

$$\rho_f \leq p(Tx, Tx) \leq p(fx, fx).$$

Indeed, because $TX \subset fX$, we have that $\{p(Tx, Tx) : x \in X\} = B \subset A = \{p(fx, fx) : x \in X\}$ from which follows $\inf A \leq \inf B$, that is, $\rho_f \leq \rho_T \leq p(Tx, Tx) \leq p(fx, fx)$.

Lemma 3 *Let (X, p) be a partial metric space with $(T, f) \in \mathcal{F}$ and $X_f \neq \emptyset$. If $u, v \in X_f$ are coincidence points for T and f, then $fu = fv$.*

Proof From

$$p(fu, fv) = p(Tu, Tv) \leq \max\{kp(fu, fv), p(fu, fu), p(fv, fv)\}$$

it follows that either $(1 - k)p(fu, fv) = 0$ or $p(fu, fv) \leq p(fu, fu) = p(fv, fv) = \rho_f$. Now, from (p_2), it follows $p(fu, fv) = \rho_f = \min\{p(fx, fx) : x \in X\}$, that is, $fu = fv$. □

Lemma 4 *Let (X, p) be a partial metric space and $(T, f) \in \mathcal{F}$. If fX is a complete subspace of X, then for each $s \in \mathbb{N}$, there is $x \in X$ such that*

$$p(fx, fx) = p(fx, Tx) < \rho_f + \frac{1}{s}.$$

Proof We fix $x_0 \in X$ and prove that each T-f-sequence $\{Tx_n\}$ of the initial point x_0 is a Cauchy sequence in TX.

From Remark 1, we deduce that $p(Tx_{n+1}, Tx_{n+1}) \leq p(fx_{n+1}, fx_{n+1}) = p(Tx_n, Tx_n)$. Hence, $\{p(Tx_n, Tx_n)\}$ is a nonincreasing sequence. Set

$$r := \inf\{p(Tx_n, Tx_n)\} = \lim_{n \to +\infty} p(Tx_n, Tx_n) \leq p(fx_0, fx_0)$$

and

$$M := \frac{1}{1-k}p(fx_0, Tx_0) + p(fx_0, fx_0).$$

We prove that

$$p(fx_0, Tx_n) \le M, \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Obviously, (3.2) is true for $n = 0, 1$. Assume that (3.2) is true for some $n \ge 1$, then

$$
\begin{aligned}
p(fx_0, Tx_{n+1}) &\le p(fx_0, Tx_0) + p(Tx_0, Tx_{n+1}) \\
&\le p(fx_0, Tx_0) + \max\{kp(fx_0, Tx_n), p(fx_0, fx_0)\} \\
&\le p(fx_0, Tx_0) + \frac{k}{1-k}p(fx_0, Tx_0) + p(fx_0, fx_0) \\
&= M.
\end{aligned}
$$

We fix $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $p(Tx_n, Tx_n) < r + \varepsilon$ for all $n \ge n_0$ and $2k^{n_0}M < r + \varepsilon$. For each $m \ge n \ge 2n_0$, we have

$$
\begin{aligned}
p(Tx_n, Tx_m) &\le \max\{kp(Tx_{n-1}, Tx_{m-1}), p(Tx_{n-1}, Tx_{n-1})\} \\
&\le \max\{k^2 p(Tx_{n-2}, Tx_{m-2}), p(Tx_{n-2}, Tx_{n-2})\} \\
&\quad \dots \\
&\le \max\{k^{n_0} p(Tx_{n-n_0}, Tx_{m-n_0}), p(Tx_{n-n_0}, Tx_{n-n_0})\} \\
&\le \max\{k^{n_0}[p(Tx_{n-n_0}, fx_0) + p(fx_0, Tx_{m-n_0})], \\
&\quad\quad p(Tx_{n-n_0}, Tx_{n-n_0})\} \\
&< r + \varepsilon.
\end{aligned}
$$

Now, from

$$r \le p(Tx_n, Tx_n) \le p(Tx_n, Tx_m) < r + \varepsilon$$

for each $m \ge n \ge 2n_0$, we obtain that

$$\lim_{n,m \to +\infty} p(Tx_n, Tx_m) = r = \lim_{n,m \to +\infty} p(fx_n, fx_m).$$

Since fX is a complete subspace of X, there is $z \in fX$ such that

$$p(z, z) = \lim_{n \to +\infty} p(z, Tx_n) = \lim_{n,m \to +\infty} p(Tx_n, Tx_m) = r.$$

Let $x \in X$ such that $fx = z$. We show that $p(fx, fx) = p(fx, Tx) = r$.

Now, from (3.1), we deduce that there exist $I_1, I_2, I_3 \subset \mathbb{N}$ such that

(i) $p(Tx_n, Tx) \le kp(fx_n, fx)$ for all $n \in I_1$;
(ii) $p(Tx_n, Tx) \le p(fx_n, fx_n)$ for all $n \in I_2$;

(iii) $p(Tx_n, Tx) \le p(fx, fx)$ for all $n \in I_3$.

Clearly, at least one of the sets I_1, I_2, I_3 is infinite.

Then, from

$$p(fx, Tx) \le p(fx, Tx_n) + p(Tx_n, Tx) - p(Tx_n, Tx_n),$$

if I_i ($i = 1, 2, 3$) is infinite, taking the limit as $n \to +\infty$ and $n \in I_i$, it follows that $p(fx, Tx) \le p(fx, fx)$ and so $p(fx, Tx) = p(fx, fx)$.

Now, if we choose $x_0 \in X$ such that $p(fx_0, fx_0) < \rho_f + 1/s$, we deduce that $p(fx, fx) \le p(fx_0, fx_0) < \rho_f + 1/s$. If $x_0 \in X_f$, also $x \in X_f$. □

Lemma 5 *Let (X, p) be a partial metric space and $(T, f) \in \mathcal{F}$. If fX is a complete subspace of X, then there exists $u \in X_f$ such that $fu = Tu$.*

Proof By Lemma 4, there exists a sequence $\{x_n\} \subset X$ such that

$$p(fx_n, fx_n) = p(fx_n, Tx_n) < \rho_f + \frac{1}{n}, \tag{3.3}$$

for all $n \in \mathbb{N}$. First, we prove that

$$\lim_{n,m \to +\infty} p(fx_n, fx_m) = \rho_f. \tag{3.4}$$

For $\varepsilon > 0$ fixed, we choose $n > 3/\varepsilon(1 - k)$. From Remark 1, it follows

$$\rho_f \le p(Tx_n, Tx_n) \le p(fx_n, fx_n) = p(fx_n, Tx_n) < \rho_f + \frac{1}{n} < \rho_f + \frac{\varepsilon}{3}(1 - k),$$

which implies

$$p(fx_n, Tx_n) - p(Tx_n, Tx_n) < \frac{\varepsilon}{3}(1 - k).$$

Now, for all $n, m > 3/\varepsilon(1 - k)$, we have

$$\begin{aligned}
p(fx_n, fx_m) &\le p(fx_n, Tx_n) + p(Tx_n, Tx_m) + p(Tx_m, fx_m) \\
&\quad - p(Tx_n, Tx_n) - p(Tx_m, Tx_m) \\
&= p(fx_n, Tx_n) - p(Tx_n, Tx_n) + p(Tx_m, fx_m) \\
&\quad - p(Tx_m, Tx_m) + p(Tx_n, Tx_m) \\
&< \frac{2}{3}\varepsilon(1 - k) + \max\{kp(fx_n, fx_m), p(fx_n, fx_n), p(fx_m, fx_m)\}.
\end{aligned}$$

If $\max\{kp(fx_n, fx_m), p(fx_n, fx_n), p(fx_m, fx_m)\} = kp(fx_n, fx_m)$, then

$$p(fx_n, fx_m) < \frac{2}{3}\varepsilon(1 - k) + kp(fx_n, fx_m),$$

and so $p(fx_n, fx_m) < \frac{2}{3}\varepsilon$. This implies that $\rho_f \le \lim_{n,m \to +\infty} p(fx_n, fx_m) = 0$.

If

$$\max\{kp(fx_n, fx_m), p(fx_n, fx_n), p(fx_m, fx_m)\}$$
$$= \max\{p(fx_n, fx_n), p(fx_m, fx_m)\} < \rho_f + \frac{1}{3}\varepsilon(1-k),$$

then

$$\rho_f \le p(fx_n, fx_m) < \frac{2}{3}\varepsilon(1-k) + \rho_f + \frac{1}{3}\varepsilon(1-k).$$

This implies that

$$\lim_{n,m\to+\infty} p(fx_n, fx_m) = \rho_f.$$

In both cases, (3.4) holds. Since fX is a complete subspace of X, there is $z \in fX$ such that

$$p(z, z) = \lim_{n\to+\infty} p(z, fx_n) = \lim_{n,m\to+\infty} p(fx_n, fx_m) = \rho_f.$$

Let $u \in X$ such that $fu = z$. From $p(fu, fu) = p(z, z)$ it follows that $u \in X_f$. We prove that $p(fu, fu) = p(fu, Tu)$. As $p(fx_n, fx_n) = p(fx_n, Tx_n)$ and $\rho_f \le p(Tx_n, Tx_n)$ for all $n \in \mathbb{N}$, we get that

$$p(fu, Tu) \le p(fu, fx_n) + p(fx_n, Tx_n) + p(Tx_n, Tu)$$
$$- p(fx_n, fx_n) - p(Tx_n, Tx_n)$$
$$= p(fu, fx_n) + p(Tx_n, Tu) - p(Tx_n, Tx_n)$$
$$\le p(fu, fx_n) - \rho_f + p(Tx_n, Tu)$$
$$\le p(fu, fx_n) - \rho_f + \max\{kp(fx_n, fu), p(fx_n, fx_n), p(fu, fu)\}.$$

As $n \to +\infty$, we obtain $p(fu, Tu) = p(fu, fu) = p(Tu, Tu) = \rho_f$ and so $fu = Tu$. □

The following theorem of a common fixed point in a partial metric space is one of our main results.

Theorem 1 *Let (X, p) be a partial metric space with $(T, f) \in \mathcal{F}$. If fX is a complete subspace of X, then T and f have a unique point of coincidence $z = fu = Tu$ with $u \in X_f$. Moreover, if T and f are weakly compatible and $fX_f \subset X_f$, then T and f have a unique common fixed point $z \in X_f$.*

Proof By Lemma 5 and Lemma 3, there exists $u \in X_f$ such that $z = fu = Tu$ is a unique point of coincidence for T and f with $p(z, z) = \rho_f$. If T and f are weakly compatible and $fX_f \subset X_f$, by Lemma 3, from $Tz = Tfu = fTu = fz \in X_f$, that is $p(fz, fz) = \rho_f$, it follows that $fz = fu$. By Lemma 2, z is a unique common fixed point for T and f belonging to X_f. □

Theorem 2 *Let (X, p) be a partial metric space and $T, f : X \to X$. Suppose that the following condition holds:*

$$p(Tx, Ty) \le \max\left\{kp(fx, fy), \frac{p(fx, fx) + p(fy, fy)}{2}\right\}, \tag{3.5}$$

for each $x, y \in X$, where $k \in [0, 1[$. *If fX is a complete subspace of X, and T and f are weakly compatible, then T and f have a unique common fixed point $z \in X$.*

Proof Clearly, $(T, f) \in \mathcal{F}$, by Lemma 5, there exists $u \in X_f$ such that $z = fu = Tu$. If $v \in X$ is such that $fv = Tv$ and $p(fv, fv) = p(fu, fu)$, then $fu = fv$ by Lemma 3. If $p(fv, fv) > p(fu, fu)$, from (3.5) we obtain either $p(fu, fv) = p(Tu, Tv) \leq kp(fu, fv)$ or $p(fu, fv) = p(Tu, Tv) < p(fv, fv)$. In each of these cases, we deduce that $fu = fv$, and so T and f have a unique point of coincidence. By Lemma 2, since T and f are weakly compatible, T and f have a unique common fixed point $z \in X$. □

If in Theorems 1 and 2 we choose $f(x) = x$, we obtain Theorems 3.1 and 3.3 of Ilić *et al.*

Example 1 Let $X = [0, 6]$ and $p : X \times X \to \mathbb{R}$ be defined by $p(x, y) = \max\{x, y\}$. Then (X, p) is a complete partial metric space. Let $T, f : X \to X$ be defined by

$$
Tx = \begin{cases} 2x - 1 & \text{if } x \in [1, 3], \\ 3 & \text{if } x \in]3, 6] \end{cases}
$$

and

$$
fx = \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x \in]1, 4/3[, \\ 2x & \text{if } x \in [4/3, 3], \\ 6 & \text{if } x \in]3, 6], \end{cases}
$$

respectively.

In order to show that T and f satisfy the contractive condition (3.5) in Theorem 2 with $k = 5/6$, we consider the following cases.

Case 1. $x = y = 1$. We have

$$
p(T1, T1) = 1 = \frac{p(f1, f1) + p(f1, f1)}{2}
$$
$$
= \max\left\{ kp(f1, f1), \frac{p(f1, f1) + p(f1, f1)}{2} \right\}.
$$

Case 2. $x, y \in [1, 4/3[$ and $x < y$. We have

$$
p(Tx, Ty) = 2y - 1 \leq \frac{5}{6} 2 \leq \max\left\{ kp(fx, fy), \frac{p(fx, fx) + p(fy, fy)}{2} \right\}.
$$

Case 3. $x \in [1, 3]$, $y \in [4/3, 3]$ and $x < y$. We have

$$
p(Tx, Ty) = 2y - 1 \leq \frac{5}{6} 2y \leq \max\left\{ kp(fx, fy), \frac{p(fx, fx) + p(fy, fy)}{2} \right\}.
$$

Case 4. $x \in [1, 3]$ and $y \in]3, 6]$. We have

$$
p(Tx, Ty) = \max\{3, 2x - 1\} \leq \frac{5}{6} 6 \leq \max\left\{ kp(fx, fy), \frac{p(fx, fx) + p(fy, fy)}{2} \right\}.
$$

Case 5. $x, y \in\ ]3, 6]$. We have

$$p(Tx, Ty) = 3 \leq \frac{5}{6}6 \leq \max\left\{kp(fx, fy), \frac{p(fx, fx) + p(fy, fy)}{2}\right\}.$$

Since fX is a complete subspace of X, and T and f are weakly compatible, by Theorem 2, T and f have a unique common fixed point $z = 1$.

4 Fixed points for set-valued mappings

Investigations of the existence of fixed points of set-valued contractions in metric spaces were initiated by Nadler [30]. The following theorem is motivated by Nadler's results and also generalizes the well-known Banach contraction theorem in several ways.

Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty[\to [0, +\infty[$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Lemma 6 *For every function $\psi \in \Psi$, the following holds: $\psi(t) < t$ for each $t > 0$.*

Definition 2 Let (X, p) be a partial metric space and let $T : X \to C(X)$, where $C(X)$ is the family of nonempty closed subsets of X. T is ψ-contractive if there exists $\psi \in \Psi$ such that, for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$, there is $y_2 \in Tx_2$ with

$$p(y_1, y_2) \leq \psi\big(p(x_1, x_2)\big).$$

Theorem 3 *Let (X, p) be a 0-complete partial metric space and let $T : X \to C(X)$ be a ψ-contractive mapping. Then there exists $z \in X$ such that $z \in Tz$, i.e., z is a fixed point of T, and $p(z, z) = 0$.*

Proof Fix $x_0 \in X$ and let $x_1 \in Tx_0$. If $p(x_0, x_1) = 0$, then $x_1 = x_0$ and x_0 is a fixed point of T. Assume, hence, $p(x_0, x_1) > 0$; then there exists $x_2 \in Tx_1$ such that $p(x_1, x_2) \leq \psi(p(x_0, x_1))$. Proceeding in this way, we have a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and $p(x_n, x_{n+1}) \leq \psi(p(x_{n-1}, x_n))$ for every $n \in \mathbb{N}$. Consequently,

$$p(x_n, x_{n+1}) \leq \psi^n\big(p(x_0, x_1)\big), \quad \text{for all } n \in \mathbb{N}.$$

Since, the series $\sum_{n=1}^{+\infty} \psi^n(p(x_0, x_1))$ converges, we get that $\sum_{n=1}^{+\infty} p(x_n, x_{n+1})$ converges too. It follows, for $m > n$,

$$p(x_m, x_n) \leq \sum_{k=n}^{m-1} p(x_k, x_{k+1}) \leq \sum_{k=n}^{+\infty} p(x_k, x_{k+1}) \to 0, \quad \text{as } n \to +\infty. \tag{4.1}$$

Now, from (4.1), we deduce that $\lim_{n,m \to +\infty} p(x_n, x_m) = 0$ and hence $\{x_n\}$ is a 0-Cauchy sequence in X. Since X is a 0-complete space, there exists $z \in X$ such that $x_n \to z$ and $p(z, z) = 0$. For $x_n \in Tx_{n-1}$ there is $y_n \in Tz$ such that $p(x_n, y_n) \leq \psi(p(x_{n-1}, z))$. From

$$p(z, Tz) \leq p(z, y_n) \leq p(z, x_n) + p(x_n, y_n)$$

$$\leq p(z, x_n) + \psi\big(p(x_{n-1}, z)\big)$$

$$\leq p(z, x_n) + p(x_{n-1}, z)$$

as $n \to +\infty$, we obtain that $p(z, Tz) \leq 0$; since Tz is closed, we have $z \in Tz$. $\qquad\square$

Example 2 Let $X = \mathbb{Q} \cap [0, +\infty[$ and $p : X \times X \to \mathbb{R}$ be defined by $p(x,y) = \frac{1}{4}|x - y| + \frac{1}{2}\max\{x, y\}$. Then (X, p) is a 0-complete partial metric space. Let $T : X \to C(X)$ be defined by

$$Tx = \mathbb{Q} \cap \left[\frac{x}{4}, \frac{x}{2}\right].$$

Note that Tx is closed and bounded for all $x \in X$ under the given partial metric p.

We show that T is a ψ-contractive mapping with respect to $\psi : [0, +\infty[\to [0, +\infty[$ defined by $\psi(t) = t/2$ for all $t \in [0, +\infty[$. In fact, for all $x_1, x_2 \in X$ and $y_1 \in Tx_1$, if $y_1 = kx_1/2$, $k \in \mathbb{Q} \cap [1/2, 1]$, then we choose $y_2 = kx_2/2$. It implies that

$$p(y_1, y_2) = k\left[\frac{1}{8}|x_1 - x_2| + \frac{1}{4}|x_1 - x_2|\right] = k\psi\big(p(x_1, x_2)\big) \leq \psi\big(p(x_1, x_2)\big).$$

By Theorem 3, T has a fixed point $z = 0$.

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