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# A fixed point approach to the Hyers-Ulam stability of a functional equation in various normed spaces

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## Abstract

Using direct method, Kenary (Acta Universitatis Apulensis, to appear) proved the Hyers-Ulam stability of the following functional equation

$$f(mx+ny)=\frac{(m+n)f(x+y)}{2}+\frac{(m-n)f(x-y)}{2}$$

in non-Archimedean normed spaces and in random normed spaces, where $m$, $n$ are different integers greater than 1. In this article, using fixed point method, we prove the Hyers-Ulam stability of the above functional equation in various normed spaces.

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## 1. Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?" If the problem accepts a solution, then we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940. In the following year, Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [3] proved a generalization of Hyers' theorem for additive mappings. Furthermore, in 1994, a generalization of the Rassias' theorem was obtained by Găvruta [4] by replacing the bound $\varepsilon\,(||x||^p + ||y||^p)$ by a general control function $\varphi(x, y)$.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. In 1983, the Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. In 1984, Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group and, in 2002, Czerwik [7] proved the Hyers-Ulam stability of the quadratic functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [8-12]).

Using fixed point method, we prove the Hyers-Ulam stability of the following functional equation

$$f(mx + ny) = \frac{(m+n)f(x+y)}{2} + \frac{(m-n)f(x-y)}{2} \tag{1}$$

in various spaces, which was introduced and investigated in [13].

## 2. Preliminaries

In this section, we give some definitions and lemmas for the main results in this article.

A *valuation* is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0, \infty)$ such that, for all $r, s \in \mathbb{K}$, the following conditions hold:

(a) $|r| = 0$ if and only if $r = 0$;

(b) $|rs| = |r||s|$;

(c) $|r + s| \leq |r| + |s|$.

A field $\mathbb{K}$ is called a *valued field* if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

In 1897, Hensel [14] has introduced a normed space which does not have the Archimedean property.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r + s| \leq \max\{|r|, |s|\}$$

for all $r, s \in \mathbb{K}$ then the function $|\cdot|$ is called a *non-Archimedean valuation* and the field is called a *non-Archimedean field*. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

**Definition 2.1**. Let $X$ be a vector space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $||\cdot|| : X \to [0, \infty)$ is called a *non-Archimedean norm* if the following conditions hold:

$(a)$ $||x|| = 0$ if and only if $x = 0$ for all $x \in X$;

$(b)$ $||rx|| = |r|\ ||x||$ for all $r \in \mathbb{K}$ and $x \in X$;

$(c)$ the strong triangle inequality holds:

$$||x + y|| \leq \max\{||x||, ||y||\}$$

for all $x, y \in X$. Then $(X, ||\cdot||)$ is called a *non-Archimedean normed space*.

**Definition 2.2**. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$.

$(a)$ The sequence $\{x_n\}$ is called a *Cauchy sequence* if, for any $\varepsilon > 0$, there is a positive integer $N$ such that $||x_n - x_m|| \leq \varepsilon$ for all $n, m \geq N$.

$(b)$ The sequence $\{x_n\}$ is said to be *convergent* if, for any $\varepsilon > 0$, there are a positive integer $N$ and $x \in X$ such that $||x_n - x|| \leq \varepsilon$ for all $n \geq N$. Then the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denote by $\lim_{n\to\infty} x_n = x$.

($c$) If every Cauchy sequence in $X$ converges, then the non-Archimedean normed space $X$ is called a *non-Archimedean Banach space*.

It is noted that

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j||: m \leq j \leq n-1\}$$

for all $m, n \geq 1$ with $n > m$.

In the sequel (in random stability section), we adopt the usual terminology, notions, and conventions of the theory of random normed spaces as in [15].

Throughout this article (in random stability section), let $\Gamma^+$ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0,1]$ such that $F$ is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0$, $F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \Gamma^+ : l^- F(-\infty) = 1\}$, where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of $\Gamma^+$. The set $\Gamma^+$ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of $D^+$ is defined by

$$H_a(t) = \begin{cases} 0 \text{ if } t \leq a, \\ 1 \text{ if } t > a. \end{cases}$$

We can easily show that the maximal element in $\Gamma^+$ is the distribution function $H_0(t)$.

**Definition 2.3**. [15] A function $T : [0, 1]^2 \to [0, 1]$ is a *continuous triangular norm* (briefly, a $t$-norm) if $T$ satisfies the following conditions:

($a$) $T$ is commutative and associative;

($b$) $T$ is continuous;

($c$) $T(x, 1) = x$ for all $x \in [0, 1]$;

($d$) $T(x, y) \leq T(z, w)$ whenever $x \leq z$ and $y \leq w$ for all $x, y, z, w \in [0, 1]$.

Three typical examples of continuous $t$-norms are as follows: $T(x, y) = xy$, $T(x, y) = \max\{a + b - 1, 0\}$, and $T(x, y) = \min(a, b)$.

**Definition 2.4**. [16] A *random normed space* (briefly, *RN*-space) is a triple $(X, \mu, T)$, where $X$ is a vector space, $T$ is a continuous $t$-norm, and $\mu : X \to D^+$ is a mapping such that the following conditions hold:

($a$) $\mu_x(t) = H_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;

($b$) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \geq 0$;

($c$) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

**Definition 2.5**. Let $(X, \mu, T)$ be an RN-space.

(1) A sequence $\{x_n\}$ in $X$ is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$for all $t > 0$.

(2) A sequence $\{x_n\}$ in $X$ is called a *Cauchy sequence* in $X$ if $\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$ for all $t > 0$.

(3) The *RN*-space $(X, \mu, T)$ is said to be *complete* if every Cauchy sequence in $X$ is convergent.

**Theorem 2.1**. [15]*If* $(X, \mu, T)$ *is an RN-space and* $\{x_n\}$ *is a sequence such that* $x_n \to x$, *then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.

**Definition 2.6**. [17]*Let* $X$ *be a real vector space. A function* $N : X \times \mathbb{R} \to [0, 1]$ *is called a fuzzy norm on* $X$ *if for all* $x, y \in X$ *and all* $s, t \in \mathbb{R}$,

($N1$) $N(x, t) = 0$ *for* $t \leq 0$;

($N2$) $x = 0$ *if and only if* $N(x, t) = 1$ *for all* $t > 0$;

($N3$) $N(cx, t) = N(x, \frac{t}{|c|})$*if* $c \neq 0$;

($N4$) $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$;

($N5$) $N(x,.)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x, t) = 1$;

($N6$) *for* $x \neq 0$, $N(x,.)$ *is continuous on* $\mathbb{R}$.

The pair $(X, N)$ is called a fuzzy normed vector space. The properties of fuzzy normed vector space are given in [18].

**Example 2.1**. *Let* $(X, ||\cdot||)$ *be a normed linear space and* $\alpha, \beta > 0$. *Then*

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

*is a fuzzy norm on X.*

**Definition 2.7**. [17]*Let* $(X, N)$ *be a fuzzy normed vector space. A sequence* $\{x_n\}$ *in X is said to be convergent or converge if there exists an* $x \in X$ *such that* $\lim_{t\to\infty} N(x_n - x, t) = 1$ *for all* $t > 0$. *In this case, x is called the limit of the sequence* $\{x_n\}$ *in X and we denote it by* $N - \lim_{t\to\infty} x_n = x$.

**Definition 2.8**. [17]*Let* $(X, N)$ *be a fuzzy normed vector space. A sequence* $\{x_n\}$ *in X is called Cauchy if for each* $\varepsilon > 0$ *and each* $t > 0$ *there exists an* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *and all* $p > 0$, *we have* $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

**Example 2.2**. *Let* $N : \mathbb{R} \times \mathbb{R} \to [0, 1]$ *be a fuzzy norm on* $\mathbb{R}$ *defined by*

$$N(x,t) = \begin{cases} \frac{t}{t+|x|} & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

*Then* $(\mathbb{R}, N)$ *is a fuzzy Banach space.*

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on $X$ [19].

Throughout this article, assume that $X$ is a vector space and that $(Y, N)$ is a fuzzy Banach space.

**Definition 2.9**. Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies the following conditions:

($a$) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

($b$) $d(x, y) = d(y, x)$ for all $x, y \in X$;

($c$) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 2.2**. [20,21]*Let (X, d) be a complete generalized metric space and* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then, for all* $x \in X$, *either*

$$d(J^n x, J^{n+1} x) = \infty$$

*for all non-negative integers n or there exists a positive integer* $n_0$ *such that*

($a$) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n_0 \geq n_0$;

($b$) *the sequence* $\{J^n x\}$ *converges to a fixed point* $y^*$ *of J*;

($c$) $y^*$ *is the unique fixed point of J in the set* $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;

(*d*) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

## 3. Non-Archimedean stability of the functional equation (1)

In this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in non-Archimedean normed spaces.

Throughout this section, let $X$ be a non-Archimedean normed space and $Y$ a complete non-Archimedean normed space. Assume that $|m| \neq 1$.

**Lemma 3.1**. *Let $X$ and $Y$ be linear normed spaces and $f : X \to Y$ a mapping satisfying (1). Then $f$ is an additive mapping.*

*Proof.* Letting $y = 0$ in (1), we obtain

$$f(mx) = mf(x)$$

for all $x \in X$. So one can show that

$$f(m^n x) = m^n f(x)$$

for all $x \in X$ and all $n \in \mathbb{N}$.□

**Theorem 3.1**. *Let $\zeta : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$|m|\zeta(x, y) \leq L\zeta(mx, my)$$

*for all $x, y \in X$. If $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and the inequality*

$$\left\| f(mx + ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2} \right\| \leq \zeta(x, y) \tag{2}$$

*for all $x, y \in X$, then there is a unique additive mapping $A : X \to Y$ such that*

$$\| f(x) - A(x) \| \leq \frac{L\zeta(x, 0)}{|m| - |m|L}. \tag{3}$$

*Proof.* Putting $y = 0$ and replacing $x$ by $\frac{x}{m}$ in (2), we have

$$\left\| mf\left(\frac{x}{m}\right) - f(x) \right\| \leq \zeta\left(\frac{x}{m}, 0\right) \leq \frac{L}{|m|}\zeta(x, 0) \tag{4}$$

for all $x \in X$. Consider the set

$$S := \{g : X \to Y; g(0) = 0\}$$

and the generalized metric $d$ in $S$ defined by

$$d(f, g) = \inf\left\{\mu \in \mathbb{R}^+ : \| g(x) - h(x) \| \leq \mu\zeta(x, 0), \quad \forall x \in X\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that $(S, d)$ is complete (see [[22], Lemma 2.1]). Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) := mh\left(\frac{x}{m}\right)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \varepsilon$. Then we have

$$\| g(x) - h(x) \| \leq \varepsilon\zeta(x, 0)$$

for all $x \in X$ and so

$$\| Jg(x) - Jh(x) \| = \left\| mg\left(\frac{x}{m}\right) - mh\left(\frac{x}{m}\right) \right\| \leq |m|\varepsilon\zeta\left(\frac{x}{m}, 0\right) \leq |m|\varepsilon\frac{L}{|m|}\zeta(x, 0)$$

for all $x \in X$. Thus $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$. It follows from (4) that

$$d(f, Jf) \leq \frac{L}{|m|}.$$

By Theorem 2.2, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, that is,

$$A\left(\frac{x}{m}\right) = \frac{1}{m}A(x) \tag{5}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that $A$ is a unique mapping satisfying (5) such that there exists $\mu \in (0, \infty)$ satisfying

$$\| f(x) - A(x) \| \leq \mu\zeta(x, 0)$$

for all $x \in X$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f, Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, A) \leq \frac{L}{|m| - |m|L}.$$

This implies that the inequality (3) holds. By (2), we have

$$\begin{aligned}&\left\| m^n f\left(\frac{mx+ny}{m^n}\right) - \frac{m^n(m+n)f(\frac{x+y}{m^n})}{2} - \frac{m^n(m-n)f(\frac{x-y}{m^n})}{2} \right\| \\ &\qquad \leq |m|^n \zeta\left(\frac{x}{m^n}, \frac{y}{m^n}\right) \leq |m|^n \cdot \frac{L^n}{|m|^n}\zeta(x, y)\end{aligned}$$

for all $x, y \in X$ and $n \geq 1$ and so

$$\left\| A(mx+ny) - \frac{(m+n)A(x+y)}{2} - \frac{(m-n)A(x-y)}{2} \right\| = 0$$

for all $x, y \in X$.

On the other hand

$$mA\left(\frac{x}{m}\right) - A(x) = \lim_{n\to\infty} m^{n+1} f\left(\frac{x}{m^{n+1}}\right) - \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right) = 0.$$

Therefore, the mapping $A : X \to Y$ is additive. This completes the proof. □

**Corollary 3.1**. *Let* $\theta \geq 0$ *and* $p$ *be a real number with* $0 < p < 1$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and the inequality*

$$\left\| f(mx+ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2} \right\| \leq \theta(||x||^p + ||y||^p) \tag{6}$$

*for all $x, y \in X$. Then, for all $x \in X$,*

$$A(x) = \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$$

*exists and $A : X \to Y$ is a unique additive mapping such that*

$$\| f(x) - A(x) \| \leq \frac{|m|\theta||x||^p}{|m|^{p+1} - |m|^2} \tag{7}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.1 if we take

$$\zeta(x,y) = \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. In fact, if we choose $L = |m|^{1-p}$, then we get the desired result. □

**Theorem 3.2**. *Let $\zeta : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\frac{\zeta(mx, my)}{|m|} \leq L\zeta(x,y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2). Then there is a unique additive mapping $A : X \to Y$ such that*

$$\| f(x) - A(x) \| \leq \frac{\zeta(x,0)}{|m| - |m|L}.$$

*Proof.* The proof is similar to the proof of Theorem 3.1. □

**Corollary 3.2**. *Let $\theta \geq 0$ and $p$ be a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (6). Then, for all $x \in X$*

$$A(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^n}$$

*exists and $A : X \to Y$ is a unique additive mapping such that*

$$\| f(x) - A(x) \| \leq \frac{\theta||x||^p}{|m| - |m|^p}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.2 if we take

$$\zeta(x,y) = \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. In fact, if we choose $L = |2m|^{p-1}$, then we get the desired result. □

**Example 3.1**. *Let $Y$ be a complete non-Archimedean normed space. Let $f : Y \to Y$ be a mapping defined by*

$$f(z) = \begin{cases} z, \ z \in \{mx+ny : \| mx+ny \| < 1\} \cap \{x-y : \| x-y \| < 1\} \\ 0, \ otherwise \end{cases}.$$

*Then one can easily show that $f : Y \to Y$ satisfies* (3.5) *for the case $p = 1$ and that there does not exist an additive mapping satisfying* (3.6).

## 4. Random stability of the functional equation (1)

In this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in random normed spaces.

**Theorem 4.1**. *Let $X$ be a linear space, $(Y, \mu, T)$ a complete RN-space and $\Phi$ a mapping from $X^2$ to $D^+$ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that there exists* $0 < \alpha < \frac{1}{m}$ *such that*

$$\Phi_{mx,my}\left(\frac{t}{\alpha}\right) \leq \Phi_{x,y}(t) \tag{8}$$

*for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\mu_{f(mx+ny)-\frac{(m+n)f(x+y)}{2}-\frac{(m-n)f(x-y)}{2}}(t) \geq \Phi_{x,y}(t) \tag{9}$$

*for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$*

$$A(x) := \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$$

*exists and $A : X \to Y$ is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,0}\left(\frac{(1-m\alpha)t}{\alpha}\right) \tag{10}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* Putting $y = 0$ in (9) and replacing $x$ by $\frac{x}{m}$, we have

$$\mu_{mf\left(\frac{x}{m}\right)-f(x)}(t) \geq \Phi_{\frac{x}{m},0}(t) \tag{11}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S^* := \{g : X \to Y; g(0) = 0\}$$

and the generalized metric $d^*$ in $S^*$ defined by

$$d^*(f, g) = \inf_{u\in(0,+\infty)} \{\mu_{g(x)-h(x)}(ut) \geq \Phi_{x,0}(t),\ \ \forall x \in X, t > 0\},$$

where $\inf \varnothing = +\infty$. It is easy to show that $(S^*, d^*)$ is complete (see [[22], Lemma 2.1]).

Now, we consider a linear mapping $J : S^* \to S^*$ such that

$$Jh(x) := mh\left(\frac{x}{m}\right)$$

for all $x \in X$.

First, we prove that $J$ is a strictly contractive mapping with the Lipschitz constant $m\alpha$. In fact, let $g, h \in S^*$ be such that $d^*(g, h) < \varepsilon$. Then we have

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \Phi_{x,0}(t)$$

for all $x \in X$ and $t > 0$ and so

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}(m\alpha\varepsilon t) = \mu_{mg\left(\frac{x}{m}\right)-mh\left(\frac{x}{m}\right)}(m\alpha\varepsilon t) &= \mu_{g\left(\frac{x}{m}\right)-h\left(\frac{x}{m}\right)}(\alpha\varepsilon t)\\ &\geq \Phi_{\frac{x}{m},0}(\alpha t)\\ &\geq \Phi_{x,0}(t)\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d^*(g, h) < \varepsilon$ implies that $d^*(Jg, Jh) < m\alpha\varepsilon$. This means that

$$d^*(Jg, Jh) \leq m\alpha d(g, h)$$

for all $g, h \in S^*$. It follows from (11) that

$$d^*(f, Jf) \leq \alpha.$$

By Theorem 2.2, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, that is,

$$A\left(\frac{x}{m}\right) = \frac{1}{m}A(x) \tag{12}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set

$$\Omega = \{h \in S^* : d^*(g, h) < \infty\}.$$

This implies that $A$ is a unique mapping satisfying (12) such that there exists $u \in (0, \infty)$ satisfying

$$\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,0}(t)$$

for all $x \in X$ and $t > 0$.

(2) $d^*(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right) = A(x)$$

for all $x \in X$.

(3) $d^*(f, A) \leq \frac{d^*(f, Jf)}{1-m\alpha}$ with $f \in \Omega$, which implies the inequality

$$d^*(f, A) \leq \frac{\alpha}{1 - m\alpha}$$

and so

$$\mu_{f(x)-A(x)}\left(\frac{\alpha t}{1 - m\alpha}\right) \geq \Phi_{x,0}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (10) holds.

On the other hand

$$\mu_{m^n f\left(\frac{mx+ny}{m^n}\right) - \frac{m^n(m+n)f\left(\frac{x+y}{m^n}\right)}{2} - \frac{m^n(m-n)f\left(\frac{x-y}{m^n}\right)}{2}}(t) \geq \Phi_{\frac{x}{m^n}, \frac{y}{m^n}}\left(\frac{t}{m^n}\right)$$

for all $x, y \in X$, $t > 0$ and $n \geq 1$ and so, from (8), it follows that

$$\Phi_{\frac{x}{m^n}, \frac{y}{m^n}}\left(\frac{t}{m^n}\right) \geq \Phi_{x,y}\left(\frac{t}{m^n\alpha^n}\right).$$

Since

$$\lim_{n\to\infty} \Phi_{x,y}\left(\frac{t}{m^n\alpha^n}\right) = 1$$

for all $x$, $y \in X$ and $t > 0$, we have

$$\mu_{A(mx+ny)-\frac{(m+n)A(x+y)}{2}-\frac{(m-n)A(x-y)}{2}}(t) = 1$$

for all $x$, $y \in X$ and $t > 0$.

On the other hand

$$\begin{aligned} A(mx) - mA(x) &= \lim_{n\to\infty} m^n f\left(\frac{x}{m^{n-1}}\right) - m \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right) \\ &= m\left[\lim_{n\to\infty} m^{n-1} f\left(\frac{x}{m^{n-1}}\right) - \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)\right] \\ &= 0. \end{aligned}$$

Thus the mapping $A : X \to Y$ is additive. This completes the proof. □

**Corollary 4.1**. *Let X be a real normed space, $\theta \geq 0$ and let p be a real number with p >1. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\mu_{f(mx+ny)-\frac{(m+n)f(x+y)}{2}-\frac{(m-n)f(x-y)}{2}}(t) \geq \frac{t}{t + \theta\left(||x||^p + ||y||^p\right)} \tag{13}$$

*for all $x$, $y \in X$ and $t > 0$. Then, for all $x \in X$,*

$$A(x) = \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$$

*exists and $A : X \to Y$ is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \geq \frac{m^p(1-m^{1-p})t}{m^p(1-m^{1-p})t + \theta||x||^p} \tag{14}$$

*for all $x \in X$ and $t > 0$.*

*Proof*. The proof follows from Theorem 4.1 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta\left(||x||^p + ||y||^p\right)}$$

for all $x$, $y \in X$ and $t > 0$. In fact, if we choose $\alpha = m^{-p}$, then we get the desired result. □

**Theorem 4.2**. *Let X be a linear space, $(Y, \mu, T)$ a complete RN-space and $\Phi$ a mapping from $X^2$ to $D^+$ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that for some $0 < \alpha < m$*

$$\Phi_{\frac{x}{m},\frac{y}{m}}(t) \leq \Phi_{x,y}(\alpha t)$$

*for all $x$, $y \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\mu_{f(mx+ny)-\frac{(m+n)f(x+y)}{2}-\frac{(m-n)f(x-y)}{2}}(t) \geq \Phi_{x,y}(t)$$

*for all $x$, $y \in X$ and $t > 0$. Then, for all $x \in X$,*

$$A(x) := \lim_{n\to\infty} \frac{f(m^n x)}{m^n}$$

*exists and $A : X \to Y$ is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,0}((m-\alpha)t) \tag{15}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* The proof is similar to the proof of Theorem 4.1. □

**Corollary 4.2**. *Let $X$ be a real normed space, $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\mu_{f(mx+ny)-\frac{(m+n)f(x+y)}{2}-\frac{(m-n)f(x-y)}{2}}(t) \geq \frac{t}{t+\theta\left(||x||^p+||y||^p\right)}$$

*for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$,*

$$A(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^n}$$

*exists and $A : X \to Y$ is a unique additive mapping such that*

$$\mu_{f(x)-A(x)}(t) \geq \frac{(m-m^p)t}{(m-m^p)t+\theta||x||^p}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* The proof follows from Theorem 4.2 if we take

$$\Phi_{x,y}(t) = \frac{t}{t+\theta\left(||x||^p+||y||^p\right)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = m^p$, then we get the desired result. □

**Example 4.1**. *Let $(Y, \mu, T)$ be a normed complete RN-space. Let $f : Y \to Y$ be a mapping defined by*

$$f(z) = \begin{cases} z, \ z \in \{mx+ny : \ ||mx+ny|| < 1\} \cap \{x-y : \ ||x-y|| < 1\} \\ 0, \ otherwise \end{cases}.$$

*Then one can easily show that $f : Y \to Y$ satisfies (4.6) for the case $p = 1$ and that there does not exist an additive mapping satisfying (4.7).*

## 5. Fuzzy stability of the functional equation (1)

Throughout this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in fuzzy normed spaces.

In the rest of the article, assume that $X$ is a vector space and that $(Y, N)$ is a fuzzy Banach space.

**Theorem 5.1**. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{m}, \frac{y}{m}\right) \leq \frac{L}{m}\varphi(x,y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(f(mx+ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}, t\right) \geq \frac{t}{t+\varphi(x,y)} \tag{16}$$

*for all $x, y \in X$ and all $t > 0$. Then the limit*

$$A(x) := N\text{-}\lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$$

*exists for each $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that*

$$N(f(x) - A(x), t) \geq \frac{(m - mL)t}{(m - mL)t + L\varphi(x, 0)}.$$

*for all $x, y \in X$ and all $t > 0$.*

*Proof.* Putting $y = 0$ in (16) and replacing $x$ by $\frac{x}{m}$, we have

$$N\left(mf\left(\frac{x}{m}\right) - f(x), t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, 0\right)}$$

for all $x \in X$ and $t > 0$. Consider the set

$$s^{**} := \{g : X \to Y, g(0) = 0\}$$

and the generalized metric $d^{**}$ in $S^{**}$ defined by

$$d^{**}(f, g) = \inf\left\{\mu \in \mathbb{R}^+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}, \quad \forall x \in X,\ t > 0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that $(S^{**}, d^{**})$ is complete (see [[22], Lemma 2.1]).

Now, we consider a linear mapping $J : S^{**} \to S^{**}$ such that

$$Jg(x) := mg\left(\frac{x}{m}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 4.1. □

**Corollary 5.1**. *Let $\theta \geq 0$ and let $p$ be a real number with $p > 1$. Let $X$ be a normed vector space with norm $||\cdot||$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(f(mx + ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}, t\right) \geq \frac{t}{t + \theta\left(||x||^p + ||y||^p\right)} \tag{17}$$

*for all $x, y \in X$ and all $t > 0$. Then*

$$A(x) := N\text{-}\lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$$

*exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N(f(x) - A(x), t) \geq \frac{(m^{p+1} - m^2)t}{(m^{p+1} - m^2)t + m\theta||x||^p}. \tag{18}$$

*Proof.* The proof follows from Theorem 5.1 by taking

$$\varphi(x, y) := \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then we can choose $L = m^{1-p}$ and we get the desired result. □

**Theorem 5.2**. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(mx, my) \leq mL\varphi(x, y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(f(mx + ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}, t\right) \geq \frac{t}{t + \phi(x, y)}$$

*for all* $x, y \in X$ *and all* $t > 0$. *Then the limit*

$$R(x) := N\text{-}\lim_{n\to\infty} \frac{f(m^n x)}{m^n}$$

*exists for each* $x \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \geq \frac{(m - mL)t}{(m - mL)t + \varphi(x, 0)}$$

*for all* $x, y \in X$ *and all* $t > 0$.

*Proof.* The proof is similar to that of the proofs of Theorems 4.1 and 5.1. □

**Corollary 5.2**. *Let* $\theta \geq 0$ *and let* $p$ *be a real number with* $0 < p < 1$. *Let* $X$ *be a normed vector space with norm* $||\cdot||$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$N\left(f(mx + ny) - \frac{(m + n)f(x + y)}{2} - \frac{(m - n)f(x - y)}{2}, t\right) \geq \frac{t}{t + \theta\left(||x||^p + ||y||^p\right)}$$

*for all* $x, y \in X$ *and all* $t > 0$. *Then the limit*

$$A(x) := N\text{-}\lim_{n\to\infty} \frac{f(m^n x)}{m^n}$$

*exists for each* $x \in X$ *and defines a unique additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \geq \frac{(m - m^p)t}{(m - m^p)t + \theta||x||^p}.$$

*Proof.* The proof follows from Theorem 5.2 by taking

$$\varphi(x, y) := \theta(||x||^p + ||y||^p)$$

for all $x, y, z \in X$. Then we can choose $L = m^{p-1}$ and we get the desired result. □

**Example 5.1**. *Let* $(Y, N)$ *be a normed fuzzy Banach space. Let* $f : Y \to Y$ *be a mapping defined by*

$$f(z) = \begin{cases} z, \ z \in \{mx + ny : \| mx + ny \| < 1\} \cap \{x - y : \| x - y \| < 1\} \\ 0, \ otherwise \end{cases}.$$

*Then one can easily show that* $f : Y \to Y$ *satisfies* (5.2) *for the case* $p = 1$ *and that there does not exist an additive mapping satisfying* (5.3).

## 6. Conclusion

We linked here five different disciplines, namely, the random normed spaces, non-Archimedean normed spaces, fuzzy normed spaces, functional equations, and fixed point theory. We established the Hyers-Ulam stability of the functional equation (1) in various normed spaces by using fixed point method.

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**Authors' contributions**
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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The authors declare that they have no competing interests.

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**References**

1. Ulam, SM: Problems in Modern Mathematics. John Wiley and Sons, Science (1964)
2. Hyers, DH: On the stability of the linear functional equation. Proc Natl Acad Sci USA. **27**, 222–224 (1941). doi:10.1073/pnas.27.4.222
3. Rassias, ThM: On the stability of the linear mapping in Banach spaces. Proc Am Math Soc. **72**, 297–300 (1978). doi:10.1090/S0002-9939-1978-0507327-1
4. Găvruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J Math Anal Appl. **184**, 431–436 (1994). doi:10.1006/jmaa.1994.1211
5. Skof, F: Local properties and approximation of operators. Rend Sem Mat Fis Milano. **53**, 113–129 (1983). doi:10.1007/BF02924890
6. Cholewa, PW: Remarks on the stability of functional equations. Aequationes Math. **27**, 76–86 (1984). doi:10.1007/BF02192660
7. Czerwik, S: Functional Equations and Inequalities in Several Variables. World Scientific, River Edge, NJ (2002)
8. Kenary, HA: The probabilistic stability of a Pexiderial functional equation in random normed spaces. Rend Del Circolo Math Di Palermo. (to appear)
9. Kenary, HA, Shafaat, Kh, Shafei, M, Takbiri, G: Hyers-Ulam-Rassias stability of the Appollonius quadratic mapping in RN-spaces. J Nonlinear Sci Appl. **4**, 110–119 (2011)
10. Park, C, Moradlou, F: Stability of homomorphisms and derivations on $C^*$-ternary rings. Taiwan J Math. **13**, 1985–1999 (2009)
11. Saadati, R, Vaezpour, M, Cho, Y: A note to paper "On the stability of cubic mappings and quartic mappings in random normed spaces". J Inequal Appl **2009** (2009). Article ID 214530
12. Saadati, R, Zohdi, MM, Vaezpour, SM: Nonlinear $\mathcal{L}$-random stability of an ACQ functional equation. J Inequal Appl **2011** (2011). Article ID 194394
13. Kenary, HA: On the Hyers-Ulam-Rassias stability of a functional equation in non-Archimedean and random normed spaces. Acta Universitatis Apulensis. (to appear)
14. Hensel, K: Ubereine news Begrundung der Theorie der algebraischen Zahlen. Jahresber Deutsch Math Verein. **6**, 83–88 (1897)
15. Schewizer, B, Sklar, A: Probabilistic Metric Spaces. North-Holland Series in Probability and Applied Mathematics, North-Holland, New York (1983)
16. Sherstnev, AN: On the notion of a random normed space. Dokl Akad Nauk SSSR **149**, 280–283 (1963). (in Russian)
17. Bag, T, Samanta, SK: Finite dimensional fuzzy normed linear spaces. J Fuzzy Math. **11**, 687–705 (2003)
18. Mirmostafaee, AK, Mirzavaziri, M, Moslehian, MS: Fuzzy stability of the Jensen functional equation. Fuzzy Sets Syst. **159**, 730–738 (2008). doi:10.1016/j.fss.2007.07.011
19. Bag, T, Samanta, SK: Fuzzy bounded linear operators. Fuzzy Sets Syst. **151**, 513–547 (2005). doi:10.1016/j.fss.2004.05.004
20. Diaz, J, Margolis, B: A fixed point theorem of the alternative for contractions on a generalized complete metric space. Bull Am Math Soc. **74**, 305–309 (1968). doi:10.1090/S0002-9904-1968-11933-0
21. Cădariu, L, Radu, V: Fixed points and the stability of Jensen's functional equation. J Inequal Pure Appl Math **4** (2003). Art ID 4
22. Mihet, D, Radu, V: On the stability of the additive Cauchy functional equation in random normed spaces. J Math Anal Appl. **343**, 567–572 (2008)

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