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Generalizations of Caristi Kirk's Theorem on Partial Metric Spaces

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Abstract

In this article, lower semi-continuous maps are used to generalize Cristi-Kirk's fixed point theorem on partial metric spaces. First, we prove such a type of fixed point theorem in compact partial metric spaces, and then generalize to complete partial metric spaces. Some more general results are also obtained in partial metric spaces.
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1. Introduction and preliminaries

In 1992, Matthews [1,2] introduced the notion of a partial metric space which is a generalization of usual metric spaces in which $d(x, x)$ are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see, e.g. [3]-[8])

Let X be a nonempty set. The mapping $p : X \times X \to [0, \infty)$ is said to be a *partial metric* on X if for any $x, y, z \in X$ the following conditions hold true:

(PM1) $p(x, y) = p(y, x)$ (symmetry)

(PM2) If $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (equality)

(PM3) $p(x, x) \leq p(x, y)$ (small self-distances)

(PM4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ (triangularity)

for all $x, y, z \in X$. The pair (X, p) is then called a *partial metric space*(see, e.g. [1,2]). We use the abbreviation PMS for the partial metric space (X, p).

Notice that for a partial metric p on X, the function $d_p : X \times X \to [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1}$$

is a (usual) metric on X. Observe that each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x, \varepsilon): x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as $B_p[x, \varepsilon] = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$

Definition 1. *(see, e.g.* [1,2,6]*)*

(*i*) *A sequence* $\{x_n\}$ *in a PMS* (X, p) *converges to* $x \in X$ *if and only if* $p(x, x) = \lim_{n\to\infty} p(x, x_n)$,

(*ii*) *a sequence* $\{x_n\}$ *in a PMS* (X, p) *is called Cauchy if and only if* $\lim_{n,m\to\infty} p(x_n, x_m)$ *exists (and finite),*

(*iii*) *A PMS* (X, p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges, with respect to* τ_p, *to a point* $x \in X$ *such that* $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
(*iv*) *A mapping* $f : X \to X$ *is said to be continuous at* $x_0 \in X$, *if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Lemma 2. *(see, e.g.* [1,2,6]*)*

(*A*) *A sequence* $\{x_n\}$ *is Cauchy in a PMS* (X, p) *if and only if* $\{x_n\}$ *is Cauchy in a metric space* (X, d_p),
(*B*) *A PMS* (X, p) *is complete if and only if a metric space* (X, d_p) *is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) \tag{1.2}$$

2. Main Results

Let (X, p) be a PMS, $c \subset X$ and $\phi : C \to \mathbb{R}^+$ a function on C. Then, the function ϕ is called a *lower semi-continuous (l.s.c)* on C whenever

$$\lim_{n\to\infty} p(x_n, x) = p(x, x) \Rightarrow \varphi(x) \le \lim_{n\to\infty} \inf \varphi(x_n) = sup_{n\ge1} inf_{m\ge n} \varphi(x_m). \tag{2.1}$$

Also, let $T : X \to X$ be an arbitrary self-mapping on X such that

$$p(x, Tx) \le \varphi(x) - \varphi(Tx) \text{ for all } x \in X. \tag{2.2}$$

where T is called a Caristi map on (X, p).

The following lemma will be used in the proof of the main theorem.

Lemma 3. *(see, e.g.* [8,7]*) Let* (X, p) *be a complete PMS. Then*

(A) *If* $p(x, y) = 0$ *then* $x = y$,
(B) *If* $x \ne y$, *then* $p(x, y) > 0$.

Proof. Proof of (A). Let $p(x, y) = 0$. By (PM3), we have $p(x, x) \le p(x, y) = 0$ and $p(y, y) \le p(x, y) = 0$. Thus, we have

$$p(x, x) = p(x, y) = p(y, y) = 0.$$

Hence, by (PM2), we have $x = y$.

Proof of (B). Suppose $x \ne y$. By definition $p(x, y) \ge 0$ for all $x, y \in X$. Assume $p(x, y) = 0$. By part (A), $x = y$ which is a contradiction. Hence, $p(x, y) > 0$ whenever $x \ne y$.
□

Lemma 4. *(see, e.g.* [8,7]*) Assume* $x_n \to z$ *as* $n \to \infty$ *in a PMS* (X, p) *such that* $p(z, z) = 0$. *Then,* $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ *for every* $y \in X$.

Proof. First, note that $\lim_{n\to\infty} p(x_n, z) = p(z, z) = 0$. By the triangle inequality, we have

$$p(x_n, y) \le p(x_n, z) + p(z, y) - p(z, z) = p(x_n, z) + p(z, y)$$

and

$$p(z,y) \leq p(z,x_n) + p(x_n,y) - p(x_n,x_n) \leq p(x_n,z) + p(x_n,y).$$

Hence,

$$0 \leq |p(x_n,y) - p(z,y)| \leq p(x_n,z).$$

Letting $n \to \infty$ we conclude our claim. □

The following theorem is an extension of the result of Caristi ([9]; Theorem 2.1)

Theorem 5. *Let* (X, p) *be a complete PMS,* $\phi : X \to \mathbb{R}^+$ *a lower semi-continuous (l. s.c) function on X. Then, each self-mapping* $T : X \to X$ *satisfying (2.2) has a fixed point in X.*

Proof. For each $x \in X$, define

$$\begin{aligned} &S(x) = \{z \in X : p(x,z) \leq \varphi(x) - \varphi(z)\} \text{ and} \\ &\alpha(x) = \inf\{\varphi(z) : z \in S(x)\} \end{aligned} \tag{2.3}$$

Since $x \in S(x)$, then $S(x) \neq \emptyset$. From (2.3), we have $0 \leq \alpha(x) \leq \phi(x)$.

Take $x \in X$. We construct a sequence $\{x_n\}$ in the following way:

$$\begin{aligned} &x_1 := x \\ &x_{n+1} \in S(x_n) \text{ such that } \varphi(x_{n+1}) \leq \alpha(x_n) + \frac{1}{n}, \ \forall n \in \mathbb{N}. \end{aligned} \tag{2.4}$$

Thus, one can easily observe that

$$\begin{aligned} &p(x_n,x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}), \\ &\alpha(x_n) \leq \varphi(x_{n+1}) \leq \alpha(x_n) + \frac{1}{n}, \quad \forall n \in \mathbb{N} \end{aligned} \tag{2.5}$$

Note that (2.5) implies that $\{\phi(x_n)\}$ is a decreasing sequence of real numbers, and it is bounded by zero. Therefore, the sequence $\{\phi(x_n)\}$ is convergent to some positive real number, say L. Thus, regarding (2.5), we have

$$L = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} \alpha(x_n). \tag{2.6}$$

From (2.5) and (2.6), for each $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that

$$\varphi(x_n) \leq L + \frac{1}{k}, \quad \text{for all } \ n \geq N_k. \tag{2.7}$$

Regarding the monotonicity of $\{\phi(x_n)\}$, for $m \geq n \geq N_k$, we have

$$L \leq \varphi(x_m) \leq \varphi(x_n) \leq L + \frac{1}{k}. \tag{2.8}$$

Thus, we obtain

$$\varphi(x_n) - \varphi(x_m) < \frac{1}{k}, \quad \text{for all } m \geq n \geq N_k. \tag{2.9}$$

On the other hand, taking (2.5) into account, together with the triangle inequality, we observe that

$$\begin{aligned} p(x_n,x_{n+2}) &\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) - p(x_{n+1},x_{n+1}) \\ &\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) \\ &\leq \varphi(x_n) - \varphi(x_{n+1}) + \varphi(x_{n+1}) - \varphi(x_{n+2}), \\ &= \varphi(x_n) - \varphi(x_{n+2}). \end{aligned} \tag{2.10}$$

Analogously,

$$\begin{aligned} p(x_n, x_{n+3}) &\leq p(x_n, x_{n+2}) + p(x_{n+2}, x_{n+3}) - p(x_{n+2}, x_{n+2}) \\ &\leq p(x_n, x_{n+2}) + p(x_{n+2}, x_{n+3}) \\ &\leq \varphi(x_n) - \varphi(x_{n+2}) + \varphi(x_{n+2}) - \varphi(x_{n+3}), \\ &= \varphi(x_n) - \varphi(x_{n+3}). \end{aligned} \tag{2.11}$$

By induction, we obtain that

$$p(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m) \quad \text{for all } m \geq n, \tag{2.12}$$

and taking (2.9) into account, (2.12) turns into

$$p(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m) < \frac{1}{k}, \quad \text{for all } m \geq n \geq N_k. \tag{2.13}$$

Since the sequence $\{\phi(x_n)\}$ is convergent which implies that the right-hand side of (2.13) tends to zero. By definition,

$$\begin{aligned} d_p(x_n, x_m) &= 2p(x_n, x_m) - p(x_m, x_m) - p(x_n, x_n), \\ &\leq 2p(x_n, x_m). \end{aligned} \tag{2.14}$$

Since $p(x_n, x_m)$ tends to zero as n, $m \to \infty$, then (2.14) yields that $\{x_n\}$ is Cauchy in (X, d_p). Since (X, p) is complete, by Lemma 2, (X, d_p) is complete, and thus the sequence $\{x_n\}$ is convergent in X, say $z \in X$. Again by Lemma 2,

$$p(z, z) = \lim_{n \to \infty} p(x_n, z) = \lim_{n,m \to \infty} p(x_n, x_m) \tag{2.15}$$

Since $\lim_{n,m \to \infty} p(x_n, x_m) = 0$, then by (2.15), we have $p(z, z) = 0$.

Because ϕ is l.s.c together with (2.13)

$$\begin{aligned} \varphi(z) &\leq \lim_{m \to \infty} \inf \varphi(x_m) \\ &\leq \lim_{m \to \infty} \inf[\varphi(x_n) - p(x_n, x_m)] = \varphi(x_n) - p(x_n, z) \end{aligned} \tag{2.16}$$

and thus

$$p(x_n, z) \leq \varphi(x_n) - \varphi(z).$$

By definition, $z \in S(x_n)$ for all $n \in \mathbb{N}$ and thus $\alpha(x_n) \leq \phi(z)$. Taking (2.6) into account, we obtain $L \leq \phi$ (z). Moreover, by l.s.c of ϕ and (2.6), we have ϕ (z) $\lim_{n \to \infty} \phi$ (x_n) = L. Hence, ϕ $(z) = L$.

Since $z \in S(x_n)$ for each $n \in \mathbb{N}$ and (2.2), then $Tz \in S(z)$ and by triangle inequality

$$\begin{aligned} p(x_n, Tz) &\leq p(x_n, z) + p(z, Tz) - p(z, z) \\ &\leq p(x_n, z) + p(z, Tz) \\ &\leq \varphi(x_n) - \varphi(z) + \varphi(z) - \varphi(Tz) = \varphi(x_n) - \varphi(Tz). \end{aligned}$$

is obtained. Hence, $Tz \in S(x_n)$ for all $n \in \mathbb{N}$ which yields that $\alpha(x_n) \leq \phi(Tz)$ for all $n \in \mathbb{N}$.

From (2.6), the inequality $\phi(Tz) \geq L$ is obtained. By ϕ $(Tz) \leq \phi$ (z), observed by (2.2), and by the observation ϕ $(z) = L$, we achieve as follows:

$$\varphi(z) = L \leq \varphi(Tz) \leq \varphi(z)$$

Hence, $\phi(Tz) = \phi(z)$. Finally, by (2.2), we have $p(Tz, z) = 0$. Regarding Lemma 3, $Tz = z$.

□

The following theorem is a generalization of the result in [10]

Theorem 6. *Let $\phi : X \to \mathbb{R}^+$ be a l.s.c function on a complete PMS. If ϕ is bounded below, then there exits $z \in X$ such that*

$$\varphi(z) < \varphi(x) + p(z, x) \textit{ forall } x \in X \textit{ with } x \neq z.$$

Proof. It is enough to show that the point z, obtained in the Theorem 5, satisfies the statement of the theorem. Following the same notation in the proof of Theorem 5, it is needed to show that $x \notin S(z)$ for $x \neq z$. Assume the contrary, that is, for some $w \neq z$, we have $w \in S(z)$. Then, $0 < p(z, w) \leq \phi(z) - \phi(w)$ implies $\phi(w) < \phi(z) = L$. By triangular inequality,

$$\begin{aligned} p(x_n, w) &\leq p(x_n, z) + p(z, w) - p(z, z) \\ &\leq p(x_n, z) + p(z, w) \\ &\leq \varphi(x_n) - \varphi(z) + \varphi(z) - \varphi(w) \\ &= \varphi(x_n) - \varphi(w), \end{aligned}$$

which implies that $w \in S(x_n)$ and thus $\alpha(x_n) \leq \phi(w)$ for all $n \in \mathbb{N}$. Taking the limit when n tends to infinity, one can easily obtain $L \leq \phi(w)$, which is in contradiction with $\phi(w) < \phi(z) = L$. Thus, for any $x \in X$, $x \neq z$ implies $x \notin S(z)$ that is,

$$x \neq z \Rightarrow p(z, x) > \varphi(z) - \varphi(x).$$

□

Theorem 7. *Let X and Y be complete partial metric spaces and $T : X \to X$ an self-mapping. Assume that $R : X \to Y$ is a closed mapping, $\phi : X \to \mathbb{R}^+$ is a l.c.s, and a constant $k > 0$ such that*

$$max\{p(x, Tx), kp(Rx, RTx)\} \leq \varphi(Rx) - \varphi(RTx), \textit{ forall } x \in X. \tag{2.17}$$

Then, T has a fixed point.

Proof. For each $x \in X$, we define

$$\begin{aligned} &S(x) = \{y \in X : \max\{p(x, y), kp(Rx, Ry)\} \leq \varphi(Rx) - \varphi(Ry)\} \text{ and} \\ &\alpha(x) = \inf\{\varphi(Ry) : y \in S(x)\} \end{aligned} \tag{2.18}$$

For $x \in X$ set $x_1 := x$ and construct a sequesnce $x_1, x_2, x_3, ..., x_n, ...$ as in the proof of Theorem 5:

$x_{n+1} \in S(x_n)$ such the $\varphi(Rx_{n+1}) \leq \alpha(x_n) + \dfrac{1}{n}$ for each $n \in \mathbb{N}$.

As in Theorem 5, one can easily get that $\{x_n\}$ is convergent to $z \in X$. Analogously, $\{Rx_n\}$ is Cauchy sequence in Y and convergent to some t. Since R is closed mapping, $Rz = t$. Then, as in the proof of Theorem 5, we have

$$\varphi(t) = \varphi(Rz) = L = \lim_{n\to\infty} \alpha(x_n).$$

As in the proof of Theorem 6, we get that $x \neq z$ implies $x \notin S(z)$. From (2.17), $Tz \in S(z)$, we have $Tz = z$.

□

Define $p_x : X \to R^+$ such that $p_x(y) = p(x, y)$.

Theorem 8. *Let* (X, p) *be a complete PMS. Assume for each* $x \in X$, *the function* p_x *defined above is continuous on X, and* $\mathcal{F}$ *is a family of mappings* $f : X \to X$. *If there exists a l.s.c function* $\phi : X \to \mathbb{R}^+$ *such that*

$$p(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \quad \text{for all } x \in X \text{ and for all } f \in \mathcal{F}, \tag{2.19}$$

then, for each $x \in X$, *there is a common fixed point z of* $\mathcal{F}$ *such that*

$$p(x, z) \leq \varphi(x) - s, \quad \text{where } s = \inf\{\varphi(x) : x \in X\}.$$

Proof. Let $S(x): = \{y \in X : p(x, y) \leq \phi(x) - \phi(y)\}$ and $\alpha(x): = \inf\{\phi(y): y \in S(x)\}$ for all $x \in X$. Note that $x \in S(x)$, and so $S(x) \neq \emptyset$ as well as $0 \leq \alpha(x) \leq \phi(x)$.

For $x \in X$, set $x_1 := x$ and construct a sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ as in the proof of Theorem 5: $x_{n+1} \in S(x_n)$ such that $\varphi(x_{n+1}) \leq \alpha(x_n) + \frac{1}{n}$ for each $n \in \mathbb{N}$. Thus, one can observe that for each n,

(*i*) $p(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1})$.
(*ii*) $\alpha(x_n) \leq \varphi(x_{n+1}) \leq \alpha(x_n) + \frac{1}{n}$.

Similar to the proof of Theorem 5, (*ii*) implies that

$$L = \lim_{n\to\infty} \alpha(x_n) = \lim_{n\to\infty} \varphi(x_n). \tag{2.20}$$

Also, using the same method as in the proof of Theorem 5, it can be shown that $\{x_n\}$ is a Cauchy sequence and converges to some $z \in X$ and $\phi(z) = L$.

We shall show that $f(z) = z$ for all $f \in \mathcal{F}$. Assume on the contrary that there is $f \in \mathcal{F}$ such that $f(z) \neq z$. Replace $x = z$ in (2.19); then we get $\phi(f(z)) < \phi(z) = L$:

Thus, by definition of L, there is $n \in \mathbb{N}$ such that $\phi(f(z)) < \alpha(x_n)$. Since $z \in S(x_n)$, we have

$$\begin{aligned} p(x_n, f(z)) &\leq p(x_n, z) + p(z, f(z)) - p(z, z) \\ &\leq p(x_n, z) + p(z, f(z)) \\ &\leq [\varphi(x_n) - \varphi(z)] + [\varphi(z) - \varphi(f(z))] \\ &= \varphi(x_n) - \varphi(f(z)), \end{aligned}$$

which implies that $f(z) \in S(x_n)$. Hence, $\alpha(x_n) \leq \phi(f(z))$ which is in a contradiction with $\phi(f(z)) < \alpha(x_n)$. Thus, $f(z) = z$ for all $f \in \mathcal{F}$.

Since $z \in S(x_n)$, we have

$$\begin{aligned} p(x_n, z) &\leq \varphi(x_n) - \varphi(z) \\ &\leq \varphi(x_n) - \inf\{\varphi(y) : y \in X\} \\ &= \varphi(x) - s \end{aligned}$$

is obtained. □

The following theorem is a generalization of ([11]; Theorem 2.2).

Theorem 9. *Let A be a set,* (X, p) *as in Theorem 8,* $g : A \to X$ *a surjective mapping and* $\mathcal{F} = \{f\}$ *a family of arbitrary mappings* $f : A \to X$. *If there exists a l.c.s: function* $\phi : X \to [0, \infty)$ *such that*

$$p(g(a), f(a)) \leq \varphi(g(a)) - \varphi(f(a)), \quad \text{for all } f \in \mathcal{F} \tag{2.21}$$

and each $a \in A$, then g and $\mathcal{F}$ have a common coincidence point, that is, for some b $\in A$; $g(b) = f(b)$ for all $f \in \mathcal{F}$.

Proof. Let x be arbitrary and $z \in X$ as in Theorem 8. Since g is surjective, for each $x \in X$ there is some $a = a(x)$ such that $g(a) = x$. Let $f \in \mathcal{F}$ be a fixed mapping. Define by f a mapping $h = h(f)$ of X into itself such that $h(x) = f(a)$, where $a = a(x)$, that is, $g(a) = x$. Let $\mathcal{H}$ be a family of all mappings $h = h(f)$. Then, (2.21) yields that

$$p(x, h(x)) \leq \varphi(x) - \varphi(h(x)), \quad \text{for all} h \in \mathcal{H}.$$

Thus, by Theorem 8, $z = h(z)$ for all $h \in \mathcal{H}$. Hence $g(b) = f(b)$ for all $f \in \mathcal{F}$, where $b = b(z)$ is such that $g(b) = z$.

Example 10. *Let* $X = \mathbb{R}^+$ *and* $p(x, y) = \max\{x, y\}$; *then* (X, p) *is a PMS (see, e.g.* [6].*) Suppose* $T : X \to X$ *such that* $Tx = \frac{x}{8}$ *for all* $x \in X$ *and* $\varphi(t)$: $[0, \infty) \to [0, \infty)$ *such that* $\varphi(t) = 2t$. *Then*

$$[p(x, Tx) = \max\{x, \frac{x}{8}\} = x \quad \text{and} \quad \phi(x) - \phi(Tx) = \frac{7x}{4}$$

Thus, it satisfies all conditions of Theorem 5. it guarantees that T has a fixed point; indeed $x = 0$ *is the required point.*

3. Competing interests

The authors declare that they have no competing interests.

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