Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On products of multivalent close-to-star functions

Muhammad Arif[1], Jacek Dziok[2*], Mohsan Raza[3] and Janusz Sokół[4]

*Correspondence: jdziok@univ.rzeszow.pl
[2]Faculty of Mathematics and Natural Sciences, University of Rzeszów, Rzeszów, 35-310, Poland
Full list of author information is available at the end of the article

Abstract
In the present paper we define a class of products of multivalent close-to-star functions and determine the set of pairs $(|a|, r)$, $|a| < r \leq 1 - |a|$, such that every function from the class maps the disk $\mathcal{D}(a, r) := \{z : |z - a| < r\}$ onto a domain starlike with respect to the origin. Some consequences of the obtained result are also considered.

MSC: 30C45; 30C50; 30C55

Keywords: analytic functions; close-to-star functions; generalized starlikeness; radius of starlikeness

1 Introduction

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{D} = \mathcal{D}(0,1)$, where

$$\mathcal{D}(a,r) = \{z : |z - a| < r\}$$

and let $\mathcal{A}_p$ denote the class of functions $f \in \mathcal{A}$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad \left(z \in \mathcal{D}; p \in \mathbb{N}_0 = \{0,1,2,\ldots\}\right). \tag{1}$$

A function $f \in \mathcal{A}_p$ is said to be *starlike of order* α in $\mathcal{D}(r) := \mathcal{D}(0,r)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \left(z \in \mathcal{D}(r); 0 \leq \alpha < p\right).$$

A function $f \in \mathcal{A}_1$ is said to be *convex of order* α in $\mathcal{D}$ if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathcal{D}; 0 \leq \alpha < 1).$$

We denote by $\mathcal{S}^c(\alpha)$ the class of all functions $f \in \mathcal{A}_p$, which are convex of order α in $\mathcal{D}$ and by $\mathcal{S}_p^*(\alpha)$ we denote the class of all functions $f \in \mathcal{A}_p$, which are starlike of order α in $\mathcal{D}$. We also set $\mathcal{S}^*(\alpha) = \mathcal{S}_1^*(0)$.

Let $\mathcal{H}$ be a subclass of the class $\mathcal{A}_p$. We define *the radius of starlikeness* of the class $\mathcal{H}$ by

$$R^*(\mathcal{H}) = \inf_{f \in \mathcal{H}}\left(\sup\{r \in (0,1] : f \text{ is starlike of order } 0 \text{ in } \mathcal{D}(r)\}\right).$$

We denote by $\mathcal{P}(\beta)$, $0 < \beta \le 1$, the class of functions $h \in \mathcal{A}$ such that $h(0) = 1$ and

$$h(\mathcal{D}) \subset \Pi_\beta := \left\{ w \in \mathbb{C} \setminus \{0\} : |\operatorname{Arg} w| < \beta \frac{\pi}{2} \right\},$$

where $\operatorname{Arg} w$ denote the principal argument of the complex number w (*i.e.* from the interval $(-\pi, \pi]$). The class $\mathcal{P} := \mathcal{P}(1)$ is the well-known class of Carathéodory functions.

We say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{CS}_p^*(\alpha, \beta)$ if there exists a function $g \in \mathcal{S}_p^*(\alpha)$ such that

$$\frac{f}{g} \in \mathcal{P}(\beta).$$

In particular, we denote

$$\mathcal{CS}_p^*(\alpha) = \mathcal{CS}_p^*(\alpha, 1), \qquad \mathcal{CS}^*(\alpha) = \mathcal{CS}_1^*(\alpha), \qquad \mathcal{CS}^* = \mathcal{CS}^*(0).$$

The class $\mathcal{CS}^*$ is the well-known class of close-to-star functions with argument 0.

Silverman [1] introduced the class of functions F given by the formula

$$F(z) = z \prod_{j=1}^{n} \left(\frac{f_j(z)}{z} \right)^{a_j} \prod_{j=1}^{n} \left(g_j'(z) \right)^{b_j} \quad \left(f_j \in \mathcal{S}^*(\alpha), g_j \in \mathcal{S}^c(\beta)\ (j = 1, 2, \ldots, n) \right),$$

where a_j, b_j $(j = 1, 2, \ldots, n)$ are positive real numbers satisfying the following conditions:

$$\sum_{j=1}^{n} a_j = a, \qquad \sum_{j=1}^{n} b_j = b.$$

Dimkov [2] studied the class of functions F given by the formula

$$F(z) = z \prod_{j=1}^{n} \left(\frac{f_j(z)}{z} \right)^{a_j} \quad \left(f_j \in \mathcal{S}^*(\alpha_j), j = 1, 2, \ldots, n \right),$$

where a_j $(j = 1, 2, \ldots, n)$ are complex numbers satisfying the condition

$$\sum_{j=1}^{n} (1 - \alpha_j)|a_j| \le a.$$

Let p, n be positive integer and let a, m, M, N be positive real numbers, $b \in [-m, m]$. Moreover, let

$$\mathbf{a} = (a_1, a_2, \ldots, a_n), \qquad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n), \qquad \boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$$

be fixed vectors, with

$$a_j \in \mathbb{R}, \qquad 0 \le \alpha_j < p, \qquad 0 < \beta_j \le 1 \quad (j = 1, 2, \ldots, n).$$

We denote by $\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})$ the class of functions F given by the formula

$$F(z) = z^p \prod_{j=1}^{n} \left(\frac{f_j(z)}{z^p}\right)^{a_j} \quad \bigl(f_j \in \mathcal{CS}_p^*(\alpha_j, \beta_j), j = 1, \dots, n\bigr). \tag{2}$$

By $\mathcal{G}_p^n(m,b,c)$ we denote union of all classes $\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})$ for which

$$\sum_{j=1}^{n}(p-\alpha_j)|a_j| = m, \qquad \sum_{j=1}^{n}(p-\alpha_j)\operatorname{Re} a_j = b, \qquad \sum_{j=1}^{n}\beta_j|a_j| = c. \tag{3}$$

Finally, let us denote

$$\mathcal{G}_p^n(M,N) := \bigcup_{\substack{c\in[0,N]\\ m\in[0,M]}} \bigcup_{b\in[-m,m]} \mathcal{G}_p^n(m,b,c). \tag{4}$$

It is clear that the class $\mathcal{G}_p^n(M,N)$ contains functions F given by the formula (2) for which

$$\sum_{j=1}^{n}(p-\alpha_j)|a_j| \le M, \qquad \sum_{j=1}^{n}\beta_j|a_j| \le N.$$

Aleksandrov [3] stated and solved the following problem.

Problem 1 Let $\mathcal{H}$ be the class of functions $f \in \mathcal{A}$ that are univalent in $\mathcal{D}$ and let $\Delta \subset \mathcal{D}$ be a domain starlike with respect to an inner point ω with smooth boundary given by the formula

$$z(t) = \omega + r(t)e^{it} \quad (0 \le t \le 2\pi).$$

Find conditions for the function $r(t)$ such that for each $f \in \mathcal{H}$ the image domain $f(\Delta)$ is starlike with respect to $f(\omega)$.

Świtoniak and Stankiewicz [4, 5], Dimkov and Dziok [6] (see also [7]) have investigated a similar problem of generalized starlikeness.

Problem 2 Let $\mathcal{H} \subset \mathcal{A}$. Determine the set $B^*(\mathcal{H})$ of all pairs $(a,r) \in \mathcal{D} \times \mathbb{R}$, such that

$$|a| < r \le 1 - |a|, \tag{5}$$

and every function $f \in \mathcal{H}$ maps the disk $\mathcal{D}(a,r)$ onto a domain starlike with respect to the origin. The set $B^*(\mathcal{H})$ is called the set of generalized starlikeness of the class $\mathcal{H}$.

We note that

$$R^*(\mathcal{H}) = \sup\bigl\{r : (0,r) \in B^*(\mathcal{H})\bigr\}. \tag{6}$$

In this paper we determine the sets $B^*(\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta}))$, $B^*(\mathcal{G}_p^n(m,b,c))$ and $B^*(\mathcal{G}_p^n(M,N))$. The sets of generalized starlikeness for some subclasses of the defined classes are also considered. Moreover, we obtain the radii of starlikeness of these classes of functions.

2 Main results

We start from listing some lemmas which will be useful later on.

Lemma 1 [5] *A function $f \in \mathcal{A}$ maps the disk $\mathcal{D}(a,r)$, $|a| < r \le 1 - |a|$, onto a domain starlike with respect to the origin if and only if*

$$\operatorname{Re} \frac{e^{i\theta} f'(a + re^{i\theta})}{f(a + re^{i\theta})} \ge 0 \quad (0 \le \theta \le 2\pi). \tag{7}$$

For a function $f \in \mathcal{S}_p^*(\alpha)$ it is easy to verify that

$$\left| \frac{zf'(z)}{f(z)} - \alpha - (p - \alpha) \frac{1 + |z|^2}{1 - |z|^2} \right| \le \frac{2(p - \alpha)|z|}{1 - |z|^2} \quad (z \in \mathcal{D}).$$

Thus, after some calculations we get the following lemma.

Lemma 2 *Let $f \in \mathcal{S}_p^*(\alpha)$, $a, \theta \in \mathbb{R}$, $z \in \mathcal{D}_0 := \mathcal{D} \setminus \{0\}$. Then*

$$\operatorname{Re}\left[ae^{i\theta} \left(\frac{f'(z)}{f(z)} - \frac{p}{z} \right) \right] \ge \frac{2(p - \alpha)}{1 - |z|^2} \operatorname{Re}\big(a\bar{z}e^{i\theta} - |a|\big).$$

Lemma 3 [8] *If $h \in \mathcal{P}(\beta)$, then*

$$\left| \frac{h'(z)}{h(z)} \right| \le \frac{2\beta}{1 - |z|^2} \quad (z \in \mathcal{D}).$$

Theorem 1 *Let m, b, c be defined by* (3) *and set*

$$\mathcal{B}' = \left\{ (a,r) \in \mathbb{C} \times \mathbb{R} : \left\{ \begin{array}{l} 0 \le r \le r_1, |a| < r, \\ r_1 < r < r_2, |a| \le \varphi(r), \\ r_2 \le r < q, |a| \le q - r \end{array} \right\} \right\}, \tag{8}$$

$$\mathcal{B}'' = \big\{ (a,r) \in \mathbb{C} \times \mathbb{R} : |a| < r \le q - |a| \big\}, \tag{9}$$

where

$$r_1 = \frac{p}{4(m + c)}, \tag{10}$$

$$r_2 = \frac{p(m + c)}{(m + c + \sqrt{(m + c)^2 - 2bp + p^2})^2}, \tag{11}$$

$$q = \frac{p}{m + c + \sqrt{(m + c)^2 - 2bp + p^2}}, \tag{12}$$

$$\varphi(r) = \sqrt{r^2 - \frac{(1 - 2\sqrt{r(m + c)})^2}{2b - 1}}. \tag{13}$$

Moreover, set

$$\mathcal{B} = \begin{cases} \mathcal{B}' & \text{for } b > p/2, \\ \mathcal{B}'' & \text{for } b \le p/2. \end{cases} \tag{14}$$

If $(a,r) \in \mathcal{B}$, then a function $F \in \mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})$ maps the disk $\mathcal{D}(a,r)$ onto a domain starlike with respect to the origin. The result is sharp for $b \le p/2$ and for $b > p/2$ the set $\mathcal{B}$ cannot be larger than $\mathcal{B}''$. It means that

$$\mathcal{B}' \subset B^*\bigl(\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})\bigr) \subset \mathcal{B}'' \quad (b > p/2), \tag{15}$$

$$B^*\bigl(\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})\bigr) = \mathcal{B}'' \quad (b \le p/2). \tag{16}$$

Proof Let F belong to the class $\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})$ and let $z = a + re^{i\theta} \in \mathcal{D}$ satisfy (5). The functions

$$g_{j,s}(z) = e^{-is} f_j\bigl(e^{is}z\bigr) \quad (z \in \mathcal{D}; j = 1,2,\ldots,n, s \in \mathbb{R})$$

belong to the class $\mathcal{CS}_p^*(\alpha_j,\beta_j)$ together with the functions $f_j(z)$. Thus, by (2) the functions

$$G_s(z) = e^{-is} F\bigl(e^{is}z\bigr) \quad (z \in \mathcal{D}; s \in \mathbb{R})$$

belong to the class $\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})$ together with the function $F(z)$. In consequence, we have

$$(a,r) \in B^*\bigl(\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})\bigr) \iff \bigl(|a|,r\bigr) \in B^*\bigl(\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})\bigr) \quad (a \in \mathcal{D}, r \ge 0). \tag{17}$$

Therefore, without loss of generality we may assume that a is nonnegative real number. Since $f_j \in \mathcal{CS}_p^*(\alpha_j,\beta_j)$, there exist functions $g_j \in \mathcal{S}_p^*(\alpha_j)$ and $h_j \in \mathcal{P}(\beta_j)$ such that

$$\frac{f_j(z)}{g_j(z)} = h_j(z) \quad (z \in \mathcal{D})$$

or equivalently

$$f_j(z) = g_j(z)h_j(z) \quad (z \in \mathcal{D}). \tag{18}$$

After logarithmic differentiation of the equality (2) we obtain

$$\frac{F'(z)}{F(z)} = \frac{p}{z} + \sum_{j=1}^n a_j \left(\frac{f_j'(z)}{f_j(z)} - \frac{p}{z}\right) \quad (z \in \mathcal{D}).$$

Thus, using (18) we have

$$\begin{aligned}\operatorname{Re} \frac{e^{i\theta} F'(z)}{F(z)} &= \operatorname{Re} \frac{pe^{i\theta}}{z} + \sum_{j=1}^n \operatorname{Re}\left(a_j e^{i\theta}\left(\frac{g'_j(z)}{g_j(z)} - \frac{p}{z}\right)\right) + \sum_{j=1}^n \operatorname{Re}\left(a_j e^{i\theta} \frac{h'_j(z)}{h_j(z)}\right) \\ &\ge \operatorname{Re} \frac{pe^{i\theta}}{z} + \sum_{j=1}^n \operatorname{Re}\left(a_j e^{i\theta}\left(\frac{g'_j(z)}{g_j(z)} - \frac{p}{z}\right)\right) - \sum_{j=1}^n |a_j| \left|\frac{h'_j(z)}{h_j(z)}\right|.\end{aligned}$$

By Lemma 2 and Lemma 3 we obtain

$$\begin{aligned}\operatorname{Re} \frac{e^{i\theta} F'(z)}{F(z)} \ge{}& \operatorname{Re} \frac{pe^{i\theta}}{z} + \frac{2}{1-|z|^2} \sum_{j=1}^n (p-\alpha_j) a_j \operatorname{Re}\bigl(\overline{z} e^{i\theta}\bigr) \\ &- \frac{2}{1-|z|^2} \sum_{j=1}^n (p-\alpha_j)|a_j| - \frac{2}{1-|z|^2} \sum_{j=1}^n |a_j| \beta_j.\end{aligned}$$

Setting $z = a + re^{i\theta}$ and using (3) the above inequality yields

$$\operatorname{Re} \frac{e^{i\theta} F'(a + re^{i\theta})}{F(a + re^{i\theta})} \geq \operatorname{Re} \frac{pe^{i\theta}}{a + re^{i\theta}} + 2 \frac{\operatorname{Re}(r + ae^{-i\theta})b - m - c}{1 - |a + re^{i\theta}|^2}.$$

By Lemma 1 it is sufficient to show that the right-hand side of the last inequality is non-negative, that is,

$$\operatorname{Re} \frac{p}{r + ae^{-i\theta}} + 2 \frac{\operatorname{Re}(r + ae^{-i\theta})b - m - c}{1 - |r + ae^{-i\theta}|^2} \geq 0. \tag{19}$$

If we put

$$r + ae^{-i\theta} = x + yi,$$

then we obtain

$$\frac{px}{x^2 + y^2} + 2 \frac{bx - m - c}{1 - x^2 - y^2} \geq 0.$$

Thus, using the equality

$$(x - r)^2 + y^2 = a^2, \tag{20}$$

we obtain

$$\begin{aligned} w(x) = {} & 2r(2b - p)x^2 - \bigl((2b - p)\bigl(r^2 - a^2\bigr) + 4r(m + c) - p\bigr)x \\ & + 2(m + c)\bigl(r^2 - a^2\bigr) \geq 0. \end{aligned} \tag{21}$$

The discriminant Δ of $w(x)$ is given by

$$\begin{aligned} \Delta = {} & \bigl((2b - p)\bigl(r^2 - a^2\bigr) + 4r(m + c) - p\bigr)^2 \\ & - 16r(2b - p)(m + c)\bigl(r^2 - a^2\bigr) = A_1 A_2, \end{aligned} \tag{22}$$

where

$$A_1 = (p - 2b)\bigl(r^2 - a^2\bigr) + p + 4r(m + c) + 4\sqrt{rp(m + c)}, \tag{23}$$

$$A_2 = (p - 2b)\bigl(r^2 - a^2\bigr) + p + 4r(m + c) - 4\sqrt{rp(m + c)}. \tag{24}$$

Let

$$D = \bigl\{(a, r) \in \mathbb{R}^2 : 0 \leq a < r \leq 1 - a\bigr\}. \tag{25}$$

First, we discuss the case $b > p/2$. Thus, the inequality (21) is satisfied for every $x \in [r - a, r + a]$ if one of the following conditions is fulfilled:

1° $\Delta \leq 0$,
2° $\Delta > 0$, $w(r - a) \geq 0$ and $x_0 \leq r - a$,
3° $\Delta > 0$, $w(r + a) \geq 0$ and $x_0 \geq r + a$,

where

$$x_0 = \frac{(2b-p)(r^2-a^2)+4r(m+c)-p}{4(2b-p)r}. \tag{26}$$

Ad 1°. Since $A_1 > 0$, by (22), the condition $\Delta \le 0$ is equivalent to the inequality $A_2 \le 0$. Then

$$\mathcal{B}_1 := \{(a,r) \in D : \Delta \le 0\} = \{(a,r) \in D : A_2 \le 0\} = \{(a,r) \in D : a \le \varphi(r)\},$$

where φ is defined by (13). Let

$$\gamma = \{(a,r) \in \overline{D} : a = \varphi(r)\}.$$

Then γ is the curve which is tangent to the straight lines $a = r$ and $a = q - r$ at the points

$$S_1 = (r_1, r_1) \quad \text{and} \quad S_2 = (q - r_2, r_2), \tag{27}$$

where r_1, r_2, q are defined by (10), (11), (12), respectively.

Moreover, γ cuts the straight line $a = 0$ at the points

$$r_3 = p\Bigl(\sqrt{m+c+\sqrt{p(2b-p)}}+\sqrt{m+c}\Bigr)^{-2},$$
$$r_4 = p\Bigl(\sqrt{m+c-\sqrt{p(2b-p)}}+\sqrt{m+c}\Bigr)^{-2}.$$

Since

$$0 < r_3 < r_1 < r_2 < r_4 < q,$$

we have

$$\gamma = \{(a,r) \in \mathbb{R}^2 : r_3 \le r \le r_4, a = \varphi(r)\},$$

and consequently

$$\mathcal{B}_1 = \{(a,r) \in \mathbb{R}^2 : r_3 \le r \le r_4, 0 \le a \le \varphi(r)\}, \tag{28}$$

where φ is defined by (13) (see Figure 1).

Ad 2°. Let

$$\mathcal{B}_2 := \{(a,r) \in D : \Delta > 0 \wedge w(r-a) \ge 0 \wedge x_0 \le r-a\}.$$

It is easy to verify that

$$\begin{aligned} w(r-a) &= (r-a)\bigl((2b-1)(r-a)^2 - 2(m+c)(r-a)+1\bigr) \\ &= (2b-1)(r-a)\bigl(r-a-q'\bigr)(r-a-q), \end{aligned}$$

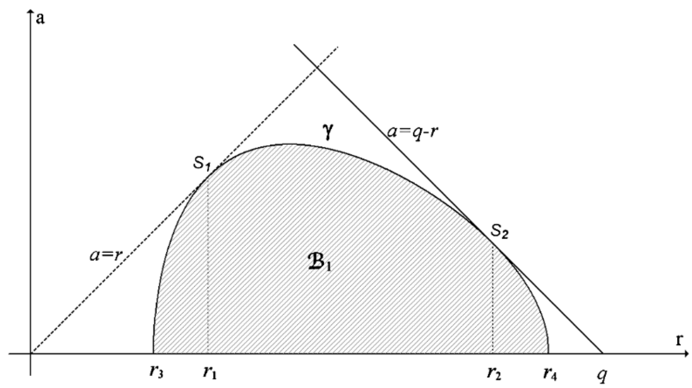

Figure 1 The set $\mathcal{B}_1$.

where q is defined by (12) and

$$q' = p\left(m + c - \sqrt{(m+c)^2 - 2bp + p^2}\right)^{-1}. \tag{29}$$

Since

$$0 < q < 1 < q' \quad \left(p/2 < b \le m, (a,r) \in D\right), \tag{30}$$

we see that

$$(r-a)\left(r - a - q'\right) < 0 \quad \left((a,r) \in D\right).$$

Thus, the inequality $w(r-a) \ge 0$ is true if $a \ge r - q$. The inequality $x_0 \le r - a$ may be written in the form

$$(2b-p)a^2 + 3(2b-p)r^2 - 4(m+c)r - 4(2b-p)ar + p \ge 0. \tag{31}$$

The hyperbola h_1, which is the boundary of the set of all pairs $(a,r) \in \mathbb{R}^2$ satisfying (31), cuts the boundary of the set D at the point S_1 defined by (27) and at the point $(r_5, 0)$, where

$$r_5 = p\left(2(m+c) + \sqrt{4(m+c)^2 - 3p(2b-p)}\right)^{-1}. \tag{32}$$

It is easy to verify that

$$r_3 < r_5 < r_4 < q.$$

Thus we determine the set

$$\mathcal{B}_2 = \left\{ (a,r) \in \mathbb{R}^2 : \left\{ \begin{array}{l} 0 \le r \le r_3, 0 \le a < r, \\ r_3 < r < r_1, \varphi(r) < a < r \end{array} \right\} \right\}, \tag{33}$$

where φ is defined by (13) (see Figure 2).

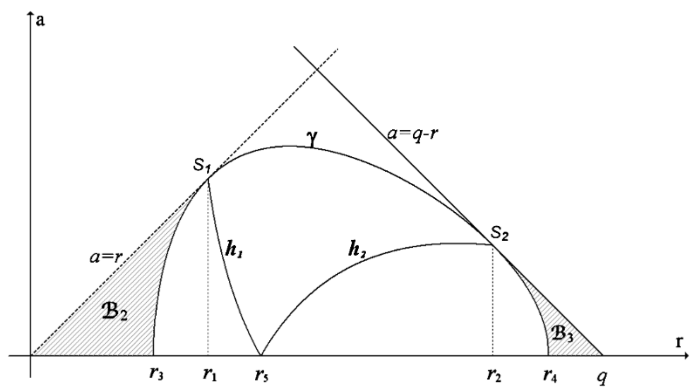

Figure 2 The sets $\mathcal{B}_2$ and $\mathcal{B}_3$.

Ad 3°. Let

$$\mathcal{B}_3 := \{(a,r) \in D : \Delta > 0 \wedge w(r+a) \geq 0 \wedge x_0 \geq r+a\}$$

and let q and q' be defined by (12) and (29), respectively. Then

$$\begin{aligned} w(a+r) &= (r+a)\left[(2b-p)(r+a)^2 - 2(m+c)(r+a) + p\right] \\ &= (2b-p)(r+a)\left(r+a-q'\right)(r+a-q). \end{aligned}$$

Moreover, by (30) we have

$$(r+a)\left(r+a-q'\right) < 0 \quad \left((a,r) \in D\right).$$

Thus, we conclude that the inequality $w(r+a) \geq 0$ is true if $a \leq q-r$. The inequality $x_0 \geq r+a$ may be written in the form

$$(2b-p)a^2 + 3(2b-p)r^2 - 4(m+c)r + 4(2b-p)ar + p \leq 0. \tag{34}$$

The hyperbola h_2, which is the boundary of the set of all pairs $(a,r) \in \mathbb{R}^2$ satisfying (34), cuts the boundary of the set D at the point S_2 defined by (27) and at the point $(r_5, 0)$, where r_5 is defined by (32). Thus, we describe the set

$$\mathcal{B}_3 = \left\{(a,r) \in \mathbb{R}^2 : \left\{\begin{matrix} r_2 < r < r_4, \varphi(r) < a \leq q-r, \\ r_4 < r < q, 0 \leq a \leq q-r \end{matrix}\right\}\right\}, \tag{35}$$

where φ is defined by (13) (see Figure 2). The union of the sets $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ defined by (28), (33), and (35) gives the set

$$\widetilde{\mathcal{B}}' = \left\{(a,r) \in \mathbb{R}^2 : \left\{\begin{matrix} 0 \leq r \leq r_1, 0 \leq a < r, \\ r_1 < r < r_2, 0 \leq a \leq \varphi(r), \\ r_2 \leq r < q, 0 \leq a \leq q-r \end{matrix}\right\}\right\}.$$

Thus, by (17) we have

$$\mathcal{B}' \subset B^*\big(\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})\big) \quad (b > p/2), \tag{36}$$

where $\mathcal{B}'$ is defined by (8).

Now, let $b < p/2$. Then the inequality (21) is satisfied for every $x \in [r-a, r+a]$ if

$$w(r-a) \geq 0 \quad \text{and} \quad w(r+a) \geq 0. \tag{37}$$

We see that

$$w(a+r) = (2b-p)(r+a)\big(r+a-q'\big)(r+a-q),$$
$$w(r-a) = (2b-p)(r-a)\big(r-a-q'\big)(r-a-q),$$

where q and q' are defined by (12) and (29), respectively. Since

$$q' < 0 < q < 1 \quad (b < p/2),$$

the condition (37) is satisfied if $(a, r) \in D$ and

$$a \leq q - r. \tag{38}$$

Let $b = 1/2$. Then by (21) we obtain

$$\big(p - 4r(m+c)\big)x + 2(m+c)\big(r^2 - a^2\big) \geq 0.$$

The above inequality holds for every $x \in [r-a, r+a]$ if $(a, r) \in D$ and

$$r - a \leq \frac{p}{2(m+c)}$$

or equivalently (38). Thus, by (17) we have

$$\mathcal{B}'' \subset B^*\big(\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})\big) \quad (b \leq p/2), \tag{39}$$

where $\mathcal{B}''$ is defined by (9). Because the function

$$F(z) = z^p \prod_1^n \left(\frac{1}{(1+\operatorname{sgn}(a_j)z)^{2(p-\alpha)}} \left(\frac{1-z}{1+z}\right)^{\beta_j \operatorname{sgn}(a_j)} \right)^{a_j} \quad (z \in \mathcal{D}) \tag{40}$$

belongs to the class $\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})$, and for $z = a + r$, $\theta = 0$, $a + r > q$ we have

$$\operatorname{Re} \frac{e^{i\theta} F'(z)}{F(z)} = \frac{p - 2(m+c)(a+r) + (2b-p)(a+r)^2}{(a+r)(1-(a+r)^2)} < 0.$$

Lemma 1 yields

$$B^*\big(\mathcal{H}_p^n(\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta})\big) \subset \mathcal{B}''. \tag{41}$$

From (36) and (41) we have (15), while (39) and (41) give (16), which completes the proof. □

Since the set $\mathcal{B}$ defined by (14) is dependent only of m, b, c, the following result is an immediate consequence of Theorem 1.

Theorem 2 *Let $\mathcal{B}$ be defined by* (14). *If $(a,r) \in \mathcal{B}$, then a function $F \in \mathcal{G}_p^n(m,b,c)$ maps the disk $\mathcal{D}(a,r)$ onto a domain starlike with respect to the origin. The obtained result is sharp for $b \le p/2$ and for $b > p/2$ the set $\mathcal{B}$ cannot be larger than $\mathcal{B}''$, where $\mathcal{B}''$ is defined by* (8). *It means that*

$$\mathcal{B} \subset B^*\big(\mathcal{G}_p^n(m,b,c)\big) \subset \mathcal{B}'' \quad (b > p/2),$$
$$B^*\big(\mathcal{G}_p^n(m,b,c)\big) = \mathcal{B} \quad (b \le p/2).$$

The functions described by (40), *with* (3) *are the extremal functions.*

Theorem 3

$$B^*\big(\mathcal{G}^n(M,N)\big) = \big\{(a,r) \in \mathbb{C} \times \mathbb{R} : |a| < r \le q_1 - |a|\big\}, \tag{42}$$

where

$$q_1 = \frac{p}{M + N + \sqrt{(M+N)^2 + 2Mp + p^2}}.$$

The equality in (42) *is realized by the function F of the form*

$$F(z) = z^p \frac{(1-z)^{2M+N}}{(1+z)^N} \quad (z \in \mathcal{D}). \tag{43}$$

Proof Let M, N be positive real numbers and let $\mathcal{B}' = \mathcal{B}'(m,b,c)$, $\mathcal{B}'' = \mathcal{B}''(m,b,c)$, $q = q(m,b,c)$ and $\varphi(r) = \varphi(r;m,b,c)$ be defined by (8), (9), (12), and (13), respectively.

It is easy to verify that

$$\varphi(r;m,b,c) \ge q(m,p/2,c) - r \quad \big(1/\big(2q(m,p/2,c)\big) \le r \le q(m,p/2,c), p/2 < b \le m\big).$$

Moreover, the function $q = q(m,b,c)$ is decreasing with respect to m and c, and increasing with respect to b. Thus, from Theorems 1 and 2 we have (see Figure 3)

$$B^*\big(\mathcal{G}^n(m,p/2,c)\big) = \mathcal{B}''(m,p/2,c) \subset \mathcal{B}'(m,b,c) \subset B^*\big(\mathcal{G}^n(m,b,c)\big)$$
$$(m \in [0,M], c \in [0,N], b \in (p/2,m])$$

and

$$B^*\big(\mathcal{G}^n(M,-M,N)\big) \subset B^*\big(\mathcal{G}^n(m,b,c)\big) \subset B^*\big(\mathcal{G}^n(m,p/2,c)\big)$$
$$\big(m \in [0,M], c \in [0,N], b \in [-m,p/2]\big).$$

Therefore, by (4) we obtain

$$B^*\big(\mathcal{G}^n(M,N)\big) = B^*\big(\mathcal{G}^n(M,-M,N)\big) \tag{44}$$

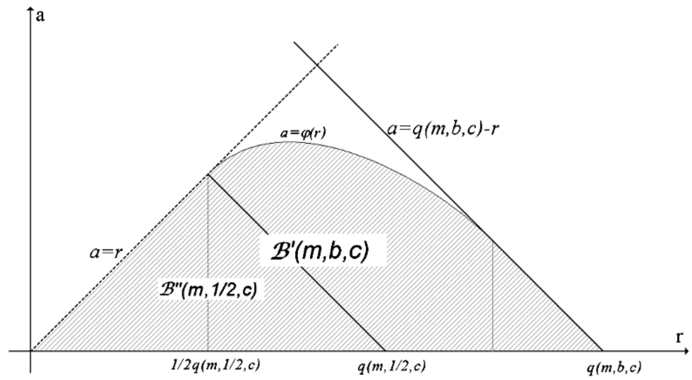

Figure 3 The sets $\mathcal{B}'$ and $\mathcal{B}''$.

and by Theorem 2 we get (42). Putting $m = M$, $b = -M$ in (3) we see that $a_1, a_2, \ldots, a_n$ are negative real numbers. Thus, the extremal function (40) has the form

$$F(z) = z^p \prod_{j=1}^{n} \left(\frac{1}{(1-z)^{2(1-\alpha_j)}} \left(\frac{1+z}{1-z} \right)^{\beta_j} \right)^{a_j} \quad (z \in \mathcal{D})$$

or equivalently

$$F(z) = \frac{z^p}{(1-z)^{-2\sum_{j=1}^{n}(1-\alpha_j)|a_j|}} \left(\frac{1+z}{1-z} \right)^{-\sum_{j=1}^{n}\beta_j|a_j|} \quad (z \in \mathcal{D}).$$

Consequently, using (3) we obtain

$$F(z) = \frac{z^p}{(1-z)^{-2M}} \left(\frac{1+z}{1-z} \right)^{-N} \quad (z \in \mathcal{D}),$$

that is, we have the function (43) and the proof is completed. □

Since $\mathcal{H}_p^a((1),(\alpha),(\beta)) = \mathcal{CS}_p^*(\alpha,\beta)$, by Theorem 1 we obtain the following theorem.

Theorem 4 *Let* $0 \le \alpha < p$, $0 < \beta \le 1$, *and*

$$\mathcal{B}' = \left\{ (a,r) \in \mathbb{C} \times \mathbb{R} : \left\{ \begin{array}{l} 0 \le r \le r_1, |a| < r, \\ r_1 < r < r_2, |a| \le \varphi(r), \\ r_2 \le r < q, |a| \le q - r \end{array} \right\} \right\},$$

$$\mathcal{B}'' = \{ (a,r) \in \mathbb{C} \times \mathbb{R} : |a| < r \le q - |a| \},$$

where

$$r_1 = \frac{1}{4(\beta - \alpha + p)},$$

$$r_2 = \frac{p(\beta + p - \alpha)}{(\beta + p - \alpha + \sqrt{(\beta-\alpha)^2 + 2\beta p})^2},$$

$$q = \frac{p}{\beta + p - \alpha + \sqrt{(\beta - \alpha)^2 + 2\beta p}},$$

$$\varphi(r) = \sqrt{r^2 - \frac{(1 - 2\sqrt{r(\beta - \alpha + p)})^2}{2p - 2\alpha - 1}}.$$

Moreover, let us put

$$\mathcal{B} = \begin{cases} \mathcal{B}' & \text{for } \alpha < p/2, \\ \mathcal{B}'' & \text{for } \alpha \geq p/2. \end{cases}$$

If $(|a|, r) \in \mathcal{B}$, *then the function* $f \in \mathcal{CS}_p^*(\alpha, \beta)$ *maps the disk* $\mathcal{D}(a, r)$ *onto a domain starlike with respect to the origin. The obtained result is sharp for* $\alpha \geq p/2$ *and for* $\alpha < p/2$ *the set* $\mathcal{B}$ *cannot be larger then* $\mathcal{B}''$. *It means that*

$$\mathcal{B} \subset B^*(\mathcal{CS}_p^*(\alpha)) \subset \mathcal{B}'' \quad (\alpha < p/2),$$

$$B^*(\mathcal{CS}_p^*(\alpha)) = \mathcal{B} \quad (\alpha \geq p/2).$$

The function F of the form

$$F(z) = z^p \frac{(1 + z)^\beta}{(1 - z)^{2p - 2\alpha + \beta}} \quad (z \in \mathcal{D})$$

is the extremal function.

Using (6) and Theorems 1-4, we obtain the radii of starlikeness for the classes $\mathcal{H}_p^n(\mathbf{a}, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $\mathcal{G}_p^n(m, b, c)$, $\mathcal{G}_p^n(M, N)$, $\mathcal{CS}_p^*(\alpha, \beta)$.

Corollary 1 *The radius of starlikeness of the classes* $\mathcal{G}_p^n(m, b, c)$ *and* $\mathcal{H}_p^n(\mathbf{a}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *is given by*

$$R^*(\mathcal{G}_p^n(m, b, c)) = R^*(\mathcal{H}_p^n(\mathbf{a}, \boldsymbol{\alpha}, \boldsymbol{\beta})) = \frac{p}{m + c + \sqrt{(m + c)^2 - 2bp + p^2}}.$$

Corollary 2 *The radius of starlikeness of the class* $\mathcal{G}_p^n(M, N)$ *is given by*

$$R^*(\mathcal{G}_p^n(M, N)) = \frac{p}{M + N + \sqrt{(M + N)^2 + 2Mp + p^2}}.$$

Corollary 3 *The radius of starlikeness of the class* $\mathcal{CS}_p^*(\alpha, \beta)$ *is given by*

$$R^*(\mathcal{CS}_p^*(\alpha, \beta)) = \frac{p}{\beta + p - \alpha + \sqrt{(\beta - \alpha)^2 + 2\beta p}}.$$

Remark 1 Putting $\beta = 1$ in Corollary 3 we get the radius of starlikeness of the class $\mathcal{CS}_p^*(\alpha) = \mathcal{CS}_p^*(\alpha, 1)$ obtained by Dziok [7]. Putting $p = \beta = 1$ we get the radius of starlikeness of the class $\mathcal{CS}^*(\alpha) = \mathcal{CS}_1^*(\alpha, 1)$ obtained by Ratti [9]. Putting, moreover, $\alpha = 0$ we get the radius of starlikeness of the class $\mathcal{CS}^* = \mathcal{CS}_1^*(0, 1)$ obtained by MacGregor [10].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

Author details
[1]Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan. [2]Faculty of Mathematics and Natural Sciences, University of Rzeszów, Rzeszów, 35-310, Poland. [3]Department of Mathematics, GC University Faisalabad, Faisalabad, Pakistan. [4]Department of Mathematics, Rzeszów University of Technology, Rzeszów, 35-959, Poland.

Acknowledgements
This work is partially supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge, University of Rzeszów.

Received: 11 October 2014 Accepted: 3 December 2014 Published: 06 Jan 2015

References
1. Silverman, H: Products of starlike and convex functions. Ann. Univ. Mariae Curie-Skłodowska, Sect. A **29**, 109-116 (1975)
2. Dimkov, G: On products of starlike functions. I. Ann. Pol. Math. **55**, 75-79 (1991)
3. Aleksandrov, IA: On the star-shaped character of the mappings of a domain by functions regular and univalent in the circle. Izv. Vysš. Učebn. Zaved., Mat. **4**(11), 9-15 (1959)
4. Stankiewicz, J, Świtoniak, B: Generalized problems of convexity and starlikeness. In: Complex Analysis and Applications '85 (Varna, 1985), pp. 670-675. Publ. House Bulgar. Acad. Sci., Sofia (1986)
5. Świtoniak, B: On a starlikeness problem in the class of functions S^*_α. Folia Sci. Univ. Tech. Resov. **14**, 17-27 (1984)
6. Dimkov, G, Dziok, J: Generalized problem of starlikeness for products of *p*-valent starlike functions. Serdica Math. J. **24**, 339-344 (1998)
7. Dziok, J: Generalized problem of starlikeness for products of close-to-star functions. Ann. Pol. Math. **107**, 109-121 (2013)
8. Nunokawa, M, Causey, WM: On certain analytic functions bounded argument. Sci. Rep. Fac. Educ., Gunma Univ. **34**, 1-3 (1985)
9. Ratti, JS: The radius of univalence of certain analytic functions. Math. Z. **107**, 241-248 (1968)
10. MacGregor, TH: The radius of univalence of certain analytic functions. Proc. Am. Math. Soc. **14**, 514-520 (1963)

10.1186/1029-242X-2015-5
Cite this article as: Arif et al.: On products of multivalent close-to-star functions. *Journal of Inequalities and Applications* 2015, 2015:5