Zhao et al. *Journal of Inequalities and Applications* 2014, 2014:496
http://www.journalofinequalitiesandapplications.com/content/2014/1/496

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# On non-$L^0$-linear perturbations of random isometries in random normed modules

Shien Zhao[1*], Yuan Zhao[2] and Minghua Yao[3]

*Correspondence:
zsefh@cnu.edu.cn
[1]Elementary Educational College, Capital Normal University, Beijing, 100048, P.R. China
Full list of author information is available at the end of the article

## Abstract

The purpose of this paper is to study non-$L^0$-linear perturbations of random isometries in random normed modules. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $K$ the scalar field $R$ of real numbers or $C$ of complex numbers, $L^0(\mathcal{F}, K)$ the equivalence classes of $K$-valued $\mathcal{F}$-measurable random variables on $\Omega$, $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ random normed modules over $K$ with base $(\Omega, \mathcal{F}, P)$. In this paper, we first establish the Mazur-Ulam theorem in random normed modules. Making use of this theorem and the relations between random normed modules and classical normed spaces, we show that if $f: E_1 \to E_2$ is a surjective random $\varepsilon$-isometry with $f(0) = 0$ and has the local property, where $\varepsilon \in L^0(\mathcal{F}, R)$ and $\varepsilon \geq 0$, then there is a surjective $L^0$-linear random isometry $U: E_1 \to E_2$ such that $\|f(x) - U(x)\|_2 \leq 4\varepsilon$, for all $x \in E_1$. We do not obtain a sharp estimate as the classical result, since random normed modules have a complicated stratification structure, which is the essential difference between random normed modules and classical normed spaces.

**MSC:** 46A22; 46A25; 46H25

**Keywords:** random normed module; random $\varepsilon$-isometry; $(\varepsilon, \lambda)$-topology

## 1 Introduction

Random metric theory originated from the theory of probabilistic metric spaces [1]. The random distance between two points in an original random metric space (briefly, an RM space) is a nonnegative random variable defined on some probability space, similarly, the random norm of a vector in an original random normed space (briefly, an RN space) is a nonnegative random variable defined on some probability space. The development of RN spaces in the direction of functional analysis led Guo to present a new version of RM and RN spaces in [2], where the random distances or random norms are defined to be the equivalence classes of nonnegative random variables according to the new versions. Based on the new version of an RN space, Guo presented a definitive definition of the random conjugate space for an RN space. Along with the deep development of the theory of random conjugate spaces, Guo established the notion of a random normed module (briefly, an RN module) in [3]. In the past ten years, as the central part of random metric theory, random normed modules and random locally convex modules (briefly, RLC modules) together with their random conjugate spaces have been deeply studied under the $(\varepsilon, \lambda)$-topology in the direction of functional analysis, *cf.* [4–19] and the related references in these papers.

The purpose of this paper is to study non-$L^0$-linear perturbations of random isometries in random normed modules. For the readers' convenience, let us first recall some classical results as follows.

Let $X$, $Y$ be two Banach spaces and $\varepsilon$ a nonnegative real number. A mapping $f : X \to Y$ is said to be an $\varepsilon$-isometry provided

$$\left| \left\| f(x) - f(y) \right\| - \|x - y\| \right| \leq \varepsilon \quad \text{for all } x, y \in X.$$

The study of surjective $\varepsilon$-isometry has been divided into two cases:

(1) $f$ is surjective and $\varepsilon = 0$;

(2) $f$ is surjective and $\varepsilon \neq 0$.

A celebrated result, known as the Mazur-Ulam theorem [20], is a perfect answer to case (1).

**Theorem 1.1** (Mazur-Ulam) *Let $X$ and $Y$ be two Banach spaces, $f : X \to Y$ a surjective isometry with $f(0) = 0$. Then $f$ is linear.*

For case (2), after many efforts of a number of mathematicians, the following sharp estimate was finally obtained by Omladič-Šemrl [21].

**Theorem 1.2** (Omladič-Šemrl) *Let $X$ and $Y$ be two Banach spaces, $f : X \to Y$ a surjective $\varepsilon$-isometry with $f(0) = 0$. Then there is a surjective linear isometry $U : X \to Y$ such that*

$$\left\| f(x) - U(x) \right\| \leq 2\varepsilon \quad \textit{for all } x \in X.$$

In order to introduce the main results of this paper, we need some notation and terminology as follows:

$K$: the scalar field $R$ of real numbers or $C$ of complex numbers.

$(\Omega, \mathcal{F}, P)$: a probability space.

$L^0(\mathcal{F}, K)$ = the algebra of equivalence classes of $K$-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$.

$L^0(\mathcal{F}) = L^0(\mathcal{F}, R)$.

$\bar{L}^0(\mathcal{F})$ = the set of equivalence classes of extended real-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$.

As usual, $\bar{L}^0(\mathcal{F})$ is partially ordered by $\xi \leq \eta$ iff $\xi^0(\omega) \leq \eta^0(\omega)$ for $P$-almost all $\omega \in \Omega$ (briefly, a.s.), where $\xi^0$ and $\eta^0$ are arbitrarily chosen representatives of $\xi$ and $\eta$, respectively. Then $(\bar{L}^0(\mathcal{F}), \leq)$ is a complete lattice, $\bigvee H$ and $\bigwedge H$ denote the supremum and infimum of a subset $H$, respectively. $(L^0(\mathcal{F}), \leq)$ is a conditionally complete lattice. Please refer to [1] or [9, p.3026] for the rich properties of the supremum and infimum of a set in $\bar{L}^0(\mathcal{F})$.

Let $\xi$ and $\eta$ be in $\bar{L}^0(\mathcal{F})$. $\xi < \eta$ is understood as usual, namely $\xi \leq \eta$ and $\xi \neq \eta$. In this paper we also use '$\xi < \eta$ (or $\xi \leq \eta$) on $A$' for '$\xi^0(\omega) < \eta^0(\omega)$ (resp., $\xi^0(\omega) \leq \eta^0(\omega)$) for $P$-almost all $\omega \in A$', where $A \in \mathcal{F}$, and $\xi^0$ and $\eta^0$ are representatives of $\xi$ and $\eta$, respectively. We have

$\bar{L}^0_+(\mathcal{F}) = \{\xi \in \bar{L}^0(\mathcal{F}) \mid \xi \geq 0\}$,

$L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi \geq 0\}$,

$\bar{L}^0_{++}(\mathcal{F}) = \{\xi \in \bar{L}^0(\mathcal{F}) \mid \xi > 0 \text{ on } \Omega\}$,

$L^0_{++}(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi > 0 \text{ on } \Omega\}$.

Besides, $\tilde{I}_A$ always denotes the equivalence class of $I_A$, where $A \in \mathcal{F}$ and $I_A$ is the characteristic function of $A$. When $\tilde{A}$ denotes the equivalence class of $A$ ($\in \mathcal{F}$), namely $\tilde{A} = \{B \in \mathcal{F} \mid P(A \triangle B) = 0\}$ (here, $A \triangle B = (A \setminus B) \cup (B \setminus A)$), we also use $I_{\tilde{A}}$ for $\tilde{I}_A$.

**Definition 1.3** Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two random normed modules over $K$ with base $(\Omega, \mathcal{F}, P)$ and $\varepsilon \in L^0_+(\mathcal{F})$. A mapping $f : E_1 \to E_2$ is said to be a random $\varepsilon$-isometry provided

$$\left| \left\| f(x) - f(y) \right\|_2 - \|x - y\|_1 \right| \leq \varepsilon \quad \text{for all } x, y \in E_1.$$

If $\varepsilon = 0$, then the mapping $f$ is called a random isometry; and it is said to be a surjective random $\varepsilon$-isometry if, in addition, $f(E_1) = E_2$.

Now, we give the main results of this paper, namely Theorems 1.4 and 1.5 below. For Theorem 1.4, it is easy to see that it has the same shape as the classical Mazur-Ulam theorem, but it is not trivial since we must make full use of the relations between random normed modules and classical normed spaces in the process of the proof. For Theorem 1.5, we do not get a sharp estimate as the classical result, namely Theorem 1.2, since the complicated stratification structure in the random setting needs to be considered, which is the essential difference between random normed modules and classical normed spaces.

**Theorem 1.4** *Let* $(E_1, \|\cdot\|_1)$ *and* $(E_2, \|\cdot\|_2)$ *be two complete random normed modules over* $K$ *with base* $(\Omega, \mathcal{F}, P)$, $f : E_1 \to E_2$ *a surjective random isometry. Then* $f$ *is an* $L^0$*-linear function.*

**Theorem 1.5** *Let* $(E_1, \|\cdot\|_1)$ *and* $(E_2, \|\cdot\|_2)$ *be two complete random normed modules over* $K$ *with base* $(\Omega, \mathcal{F}, P)$. *If* $f : E_1 \to E_2$ *is a surjective random* $\varepsilon$*-isometry with* $f(0) = 0$ *and has the local property. Then there is a surjective* $L^0$*-linear random isometry* $U : E_1 \to E_2$ *such that*

$$\left\| f(x) - U(x) \right\|_2 \leq 4\varepsilon \quad \textit{for all } x \in X.$$

The remainder of this paper is organized as follows: in Section 2 we will briefly collect some necessary well-known facts; in Section 3 we will give the proofs of the main results in this paper.

## 2 Preliminaries

**Definition 2.1** ([2, 9]) An ordered pair $(E, \|\cdot\|)$ is called a random normed space (briefly, an RN space) over $K$ with base $(\Omega, \mathcal{F}, P)$ if $E$ is a linear space over $K$ and $\|\cdot\|$ is a mapping from $E$ to $L^0_+(\mathcal{F})$ such that the following are satisfied:

(RN-1) $\|\alpha x\| = |\alpha| \|x\|$, $\forall \alpha \in K$ and $x \in E$;

(RN-2) $\|x\| = 0$ implies $x = \theta$ (the null element of $E$);

(RN-3) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in E$.

Here $\|\cdot\|$ is called the random norm on $E$ and $\|x\|$ the random norm of $x \in E$ (if $\|\cdot\|$ only satisfies (RN-1) and (RN-3) above, it is called a random seminorm on $E$).

Furthermore, if, in addition, $E$ is a left module over the algebra $L^0(\mathcal{F}, K)$ (briefly, an $L^0(\mathcal{F}, K)$-module) such that

(RNM-1) $\|\xi x\| = |\xi| \|x\|$, $\forall \xi \in L^0(\mathcal{F}, K)$ and $x \in E$.

Then $(E, \|\cdot\|)$ is called a random normed module (briefly, an RN module) over $K$ with base $(\Omega, \mathcal{F}, P)$, the random norm $\|\cdot\|$ with the property (RNM-1) is also called an $L^0$-norm on $E$ (a mapping only satisfying (RN-3) and (RNM-1) above is called an $L^0$-seminorm on $E$).

**Definition 2.2** ([2]) Let $(E, \|\cdot\|)$ be an RN space over $K$ with base $(\Omega, \mathcal{F}, P)$. A linear operator $f$ from $E$ to $L^0(\mathcal{F}, K)$ is said to be an a.s. bounded random linear functional if there is $\xi \in L^0_+(\mathcal{F})$ such that $\|f(x)\| \leq \xi\|x\|$, $\forall x \in E$. Denote by $E^*$ the linear space of a.s. bounded random linear functionals on $E$, define $\|\cdot\| : E^* \to L^0_+(\mathcal{F})$ by $\|f\| = \bigwedge\{\xi \in L^0_+(\mathcal{F}) \mid \|f(x)\| \leq \xi\|x\| \text{ for all } x \in E\}$ for all $f \in E^*$, then it is easy to check that $(E^*, \|\cdot\|)$ is also an RN module over $K$ with base $(\Omega, \mathcal{F}, P)$, called the random conjugate space of $E$.

**Definition 2.3** Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two RN modules over $K$ with base $(\Omega, \mathcal{F}, P)$, a module homomorphism $f : E_1 \to E_2$ is said to be $L^0$-linear.

**Example 2.4** ([2]) Let $L^0(\mathcal{F}, B)$ be the $L^0(\mathcal{F}, K)$-module of equivalence classes of $\mathcal{F}$-random variables (or, strongly $\mathcal{F}$-measurable functions) from $(\Omega, \mathcal{F}, P)$ to a normed space $(B, \|\cdot\|)$ over $K$. $\|\cdot\|$ induces an $L^0$-norm (still denoted by $\|\cdot\|$) on $L^0(\mathcal{F}, B)$ by $\|x\| :=$ the equivalence class of $\|x^0(\cdot)\|$ for all $x \in L^0(\mathcal{F}, B)$, where $x^0(\cdot)$ is a representative of $x$. Then $(L^0(\mathcal{F}, B), \|\cdot\|)$ is an RN module over $K$ with base $(\Omega, \mathcal{F}, P)$. Specially, $L^0(\mathcal{F}, K)$ is an RN module, the $L^0$-norm $\|\cdot\|$ on $L^0(\mathcal{F}, K)$ is still denoted by $|\cdot|$.

**Definition 2.5** ([2]) Let $(E, \|\cdot\|)$ be an RN space over $K$ with base $(\Omega, \mathcal{F}, P)$. For any positive numbers $\varepsilon$ and $\lambda$ with $0 < \lambda < 1$, let $N_\theta(\varepsilon, \lambda) = \{x \in E \mid P\{\omega \in \Omega \mid \|x\|(\omega) < \varepsilon\} > 1 - \lambda\}$, then $\{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1\}$ forms a local base at $\theta$ of some Hausdorff linear topology on $E$, called the $(\varepsilon, \lambda)$-topology induced by $\|\cdot\|$.

From now on, we always denote by $\mathcal{T}_{\varepsilon,\lambda}$ the $(\varepsilon, \lambda)$-topology for every RN space if there is no possible confusion. Clearly, the $(\varepsilon, \lambda)$-topology for the special class of RN modules $L^0(\mathcal{F}, B)$ is exactly the ordinary topology of convergence in measure, and $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})$ is a topological algebra over $K$. It is also easy to check that $(E, \mathcal{T}_{\varepsilon,\lambda})$ is a topological module over $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})$ when $(E, \|\cdot\|)$ is an RN module over $K$ with base $(\Omega, \mathcal{F}, P)$, namely the module multiplication operation is jointly continuous.

Let $E$ be an $L^0(\mathcal{F}, K)$-module. A sequence $\{x_n, n \in N\}$ in $E$ is countably concatenatable in $E$ with respect to a countable partition $\{A_n, n \in N\}$ of $\Omega$ to $\mathcal{F}$ if there is $x \in E$ such that $\tilde{I}_{A_n}x = \tilde{I}_{A_n}x_n$ for each $n \in N$, in which case we define $\sum_{n=1}^\infty \tilde{I}_{A_n}x_n$ as $x$. A subset $G$ of $E$ is said to have the countable concatenation property if each sequence $\{x_n, n \in N\}$ in $G$ is countably concatenatable in $E$ with respect to an arbitrary countable partition $\{A_n, n \in N\}$ of $\Omega$ to $\mathcal{F}$ and $\sum_{n=1}^\infty \tilde{I}_{A_n}x_n \in G$. It is easy to see that a complete RN module $E$ under $\mathcal{T}_{\varepsilon,\lambda}$ has the countable concatenation property.

The following definition is very important for the main results of this paper.

**Definition 2.6** ([9]) Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two RN modules over $K$ with base $(\Omega, \mathcal{F}, P)$. A mapping $f : E_1 \to E_2$ is said to have the local property if

$$\tilde{I}_A f(x) = \tilde{I}_A f(\tilde{I}_A x)$$

for any $A \in \mathcal{F}$ and $x \in E_1$.

## 3 Proofs of main results

In order to give the proof of Mazur-Ulam theorem on random normed modules, we need the following lemmas and readers can find the proofs of them in [9].

**Lemma 3.1** ([9]) *Let $E$ be a left module over the algebra $L^0(\mathcal{F}, R)$, $f : E \to L^0(\mathcal{F}, R)$ a random linear functional and $p : E \to L^0(\mathcal{F}, R)$ an $L^0$-linear function such that $f(x) \le p(x)$, $\forall x \in E$. Then $f$ is an $L^0$-linear function. If $R$ is replaced by $C$ and $p$ is an $L^0$-seminorm such that $|f(x)| \le p(x)$, $\forall x \in E$, then $f$ is also an $L^0$-linear function.*

**Lemma 3.2** ([2]) *Let $(E, \|\cdot\|)$ be an RN module over $K$ with base $(\Omega, \mathcal{F}, P)$ and $1 \le p \le +\infty$. Let $L^p(E) = \{x \in E \mid \|x\|_{L^p} < +\infty\}$, where $\|\cdot\|_{L^p} : E \to [0, +\infty]$ is defined by*

$$\|x\|_{L^p} = \begin{cases} (\int_\Omega \|x\|^p \, dP)^{\frac{1}{p}}, & \text{when } 1 \le p < +\infty; \\ \inf\{M \in [0, +\infty] \mid \|x\| \le M\}, & \text{when } p = +\infty \end{cases}$$

*for all $x \in E$.*

*Then $(L^p(E), \|\cdot\|_{L^p})$ is a normed space and $L^p(E)$ is $\mathcal{T}_{\varepsilon,\lambda}$-dense in $E$.*

**Remark 3.3** It is easy to see that if $(E, \|\cdot\|)$ is complete under the $(\varepsilon, \lambda)$-topology, then $(L^p(E), \|\cdot\|_{L^p})$ is also complete, for $1 \le p \le \infty$.

With the above preparations, we can give the proof of Theorem 1.4.

*Proof of Theorem* 1.4 Since $f : E_1 \to E_2$ is a random isometry with $f(0) = 0$, we see that $f$ is random norm preserving and $f|_{L^2(E_1)}$ is a mapping from $L^2(E_1)$ to $L^2(E_2)$. It is clear that $(L^2(E_1), \|\cdot\|_{L^2})$ and $(L^2(E_2), \|\cdot\|_{L^2})$ are two Banach spaces and $f|_{L^2(E_1)} : (L^2(E_1), \|\cdot\|_{L^2}) \to (L^2(E_2), \|\cdot\|_{L^2})$ is a surjective isometry with $f(0) = 0$. By classical Mazur-Ulam theorem, we see that $f|_{L^2(E_1)}$ is linear. Since $L^2(E_1)$ is dense in $E_1$ and $f$ is continuous under $\mathcal{T}_{\varepsilon,\lambda}$, it is clear that $f$ is a random linear functional. Since $|f(x)| = \|x\|$, we see that $f$ is an $L^0$-linear function from Lemma 3.1. □

Making use of Theorem 1.4 and the relations between random normed modules and classical normed spaces, we give the proof of Theorem 1.5.

*Proof of Theorem* 1.5 Let $A = [\varepsilon = 0]$, $B_i = [2^{i-1} \le \varepsilon < 2^i]$, and $C_i = [\frac{1}{2^i} \le \varepsilon < \frac{1}{2^{i-1}}]$ for any $i \in N$. Since $\varepsilon \in L^0_+(\mathcal{F})$, it is clear that $A$, $B_i$, and $C_i$, $i \in N$, is a countable partition of $\Omega$ to $\mathcal{F}$. For any $i \in N$, let $\bar{f}_i : \tilde{I}_{B_i} \cdot E_1 \to \tilde{I}_{B_i} \cdot E_2$ be defined by $\bar{f}_i(x) = \tilde{I}_{B_i} \cdot f(x)$ for any $x \in \tilde{I}_{B_i} E_1$. For any $t \in \tilde{I}_{B_i} \cdot E_2$, since $f$ is surjective, there is $s \in E_1$ such that $f(s) = t$. It is easy to see that

$$t = f(s) = \tilde{I}_{B_i} f(s) = \tilde{I}_{B_i} f(\tilde{I}_{B_i} s) = \bar{f}_i(\tilde{I}_{B_i} s).$$

Hence, $\bar{f}_i$ is surjective from $\tilde{I}_{B_i} \cdot E_1$ to $\tilde{I}_{B_i} \cdot E_2$. Since $f$ is a random $\varepsilon$-isometry and has the local property, we see that

$$\left| \left\| \bar{f}_i(x) - \bar{f}_i(y) \right\|_2 - \|x - y\|_1 \right| \le \varepsilon \quad \text{for any } x, y \in \tilde{I}_{B_i} \cdot E_1,$$

and $\bar{f}_i|_{L^\infty(\tilde{I}_{B_i}\cdot E_1)}$ is also surjective from $L^\infty(\tilde{I}_{B_i}\cdot E_1)$ to $L^\infty(\tilde{I}_{B_i}\cdot E_2)$. On one hand, for any $x, y \in L^\infty(\tilde{I}_{B_i}\cdot E_1)$, since $\|\bar{f}_i(x) - \bar{f}_i(y)\|_2 \le \|x-y\|_1 + \varepsilon$, it is easy to see that $\|\bar{f}_i(x) - \bar{f}_i(y)\|_2 \le \|x-y\|_{L^\infty} + 2\cdot 2^{i-1}$. Thus, by Lemma 3.2, it follows that

$$\big\|\bar{f}_i(x) - \bar{f}_i(y)\big\|_{L^\infty} \le \|x-y\|_{L^\infty} + 2\cdot 2^{i-1}.$$

On the other hand, it is easy to see that

$$\big\|\bar{f}_i(x) - \bar{f}_i(y)\big\|_{L^\infty} \ge \|x-y\|_{L^\infty} - 2\cdot 2^{i-1}.$$

Hence, we can see that $\bar{f}_i|_{L^\infty(\tilde{I}_{B_i}\cdot E_1)} : L^\infty(\tilde{I}_{B_i}\cdot E_1) \to L^\infty(\tilde{I}_{B_i}\cdot E_2)$ is surjective with

$$\bar{f}_i|_{L^\infty(\tilde{I}_{B_i}\cdot E_1)}(0) = 0$$

and

$$\big|\big\|\bar{f}_i|_{L^\infty(\tilde{I}_{B_i}\cdot E_1)}(x) - \bar{f}_i|_{L^\infty(\tilde{I}_{B_i}\cdot E_1)}(y)\big\|_{L^\infty} - \|x-y\|_{L^\infty}\big| \le 2\cdot 2^{i-1}.$$

By Theorem 1.1, we see that there exists a surjective linear isometry $\bar{g}_i : L^\infty(\tilde{I}_{B_i}\cdot E_1) \to L^\infty(\tilde{I}_{B_i}\cdot E_2)$ such that

$$\big\|\bar{f}_i|_{L^\infty(\tilde{I}_{B_i}\cdot E)}(x) - \bar{g}_i(x)\big\|_{L^\infty} \le 4\cdot 2^{i-1} \quad \text{for any } x \in L^\infty(\tilde{I}_{B_i}\cdot E_1).$$

Next, we prove $\tilde{I}_{G^c}\bar{g}_i(\tilde{I}_G x) = 0$ for any $x \in L^\infty(\tilde{I}_{B_i}\cdot E_1)$ and $G \in \mathcal{F}$ with $G \subset B_i$ and $P(G) > 0$. By Lemma 3.2, it is clear that

$$\big\|\bar{f}_i|_{L^\infty(\tilde{I}_{B_i}\cdot E)}(x) - \bar{g}_i(x)\big\|_2 \le \big\|\bar{f}_i|_{L^\infty(\tilde{I}_{B_i}\cdot E)}(x) - \bar{g}_i(x)\big\|_{L^\infty} \le 4\cdot 2^{i-1}.$$

Thus, we see that

$$\bar{g}_i(x) \le \bar{f}_i|_{L^\infty(\tilde{I}_{B_i}\cdot E)}(x) + 4\cdot 2^{i-1}$$

and for any $G \in \mathcal{F}$ with $G \subset B_i$ and $P(G) > 0$,

$$\tilde{I}_{G^c}\bar{g}_i(\tilde{I}_G x) \le \tilde{I}_{G^c}\bar{f}_i|_{L^\infty(\tilde{I}_{B_i}\cdot E)}(\tilde{I}_G x) + 4\cdot 2^{i-1}\tilde{I}_{G^c}.$$

Since $f$ has the local property, it is easy to see that

$$\tilde{I}_{G^c}\bar{g}_i(\tilde{I}_G x) \le 4\cdot 2^{i-1}\tilde{I}_{G^c}.$$

Since $x$ is an arbitrary element in $L^\infty(\tilde{I}_{B_i}\cdot E_1)$ and $\bar{g}_i$ is linear, we see that

$$\tilde{I}_{G^c}\bar{g}_i(\tilde{I}_G x) = 0$$

for any $x \in L^\infty(\tilde{I}_{B_i}\cdot E_1)$ and $G \in \mathcal{F}$ with $G \subset B_i$ and $P(G) > 0$. Since $\bar{g}_i(x) = \bar{g}_i(\tilde{I}_G x + \tilde{I}_{G^c}x)$, it is easy to check that

$$\tilde{I}_G\bar{g}_i(x) = \tilde{I}_G\bar{g}_i(\tilde{I}_G x) = \bar{g}_i(\tilde{I}_G x).$$

Now, we prove that $\|\bar{g}_i(x)\|_2 = \|x\|_1$ for any $x \in L^\infty(\tilde{I}_{B_i} \cdot E_1)$. Assume by way of contradiction that $\|\bar{g}_i(x)\|_2 \neq \|x\|_1$. Then $P([\|\bar{g}_i(x)\|_2 \neq \|x\|_1]) > 0$. Let, without loss generality, $H = [\|\bar{g}_i(x)\|_2 > \|x\|_1]$, and $P(H) > 0$. It is clear that $H \subset B_i$ and $\|\tilde{I}_H\bar{g}_i(x)\|_{L^\infty} > \|\tilde{I}_H x\|_{L^\infty}$ on $H$. Then we see that

$$\left\|\bar{g}_i(\tilde{I}_H x)\right\|_{L^\infty} = \left\|\tilde{I}_H\bar{g}_i(x)\right\|_{L^\infty} > \|\tilde{I}_H x\|_{L^\infty}.$$

It is a contradiction, because $\bar{g}_i$ is an isometry from $L^\infty(\tilde{I}_{B_i} \cdot E_1)$ to $L^\infty(\tilde{I}_{B_i} \cdot E_2)$. Therefore, we see that $\|\bar{g}_i(x)\|_2 = \|x\|_1$ and $\bar{g}_i$ is continuous under the $(\varepsilon,\lambda)$-topology. Since $L^\infty(\tilde{I}_{B_i} \cdot E_1)$ is dense in $\tilde{I}_{B_i} \cdot E_1$ under the $(\varepsilon,\lambda)$-topology, thus we can define $g_i : \tilde{I}_{B_i} \cdot E_1 \to \tilde{I}_{B_i} \cdot E_2$ by

$$g_i(x) = \lim_{n\to\infty} \bar{g}_i(x_n)$$

for any $x \in \tilde{I}_{B_i} \cdot E_1$, where $\{x_n, n \in N\}$ is a sequence in $L^\infty(\tilde{I}_{B_i} \cdot E_1)$ and converges to $x$ under the $(\varepsilon,\lambda)$-topology. From Theorem 1.4, it is easy to see that $g_i$ is a surjective $L^0$-linear random isometry from $\tilde{I}_{B_i} \cdot E_1$ to $\tilde{I}_{B_i} \cdot E_2$ and

$$\left\|\bar{f}_i(x) - g_i(x)\right\|_2 \leq 4 \cdot 2^{i-1} \leq 4\varepsilon$$

for any $x \in \tilde{I}_{B_i} \cdot E_1$.

For any $i \in N$, let $\bar{\bar{f}}_i : \tilde{I}_{C_i} \cdot E_1 \to \tilde{I}_{C_i} \cdot E_2$ be defined by $\bar{\bar{f}}_i(x) = \tilde{I}_{C_i} \cdot f(x)$ for any $x \in \tilde{I}_{C_i}E_1$. By the same method as above, we can prove that for any $i \in N$, there exists $h_i : \tilde{I}_{C_i} \cdot E_1 \to \tilde{I}_{C_i} \cdot E_2$ such that $h_i$ is a surjective $L^0$-linear random isometry from $\tilde{I}_{C_i} \cdot E_1$ to $\tilde{I}_{C_i} \cdot E_2$ and

$$\left\|\bar{\bar{f}}_i(x) - h_i(x)\right\|_2 \leq 4 \cdot \frac{1}{2^i} \leq 4\varepsilon$$

for any $x \in \tilde{I}_{C_i} \cdot E_1$. Let $U : E_1 \to E_2$ be defined by

$$U(x) = f(\tilde{I}_A x) + \Sigma_{i=1}^{\infty} g_i(\tilde{I}_{B_i} x) + \Sigma_{i=1}^{\infty} h_i(\tilde{I}_{C_i} x).$$

Then we see that $U$ is a surjective random isometry from $E_1$ to $E_2$ with $U(0) = 0$ and

$$\left\|f(x) - U(x)\right\|_2 \leq 4\varepsilon$$

for any $x \in E_1$. By Theorem 1.4, $U$ is a surjective $L^0$-linear random isometry. It completes the proof. □

**Remark 3.4** In Theorem 1.5, we do not obtain a sharp estimate as the classical result, namely Theorem 1.2, since random normed modules have a complicated stratification structure, which is the essential difference between random normed modules and classical normed spaces.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**

SZ used the skills in random metric theory to give the proofs of the main results. YZ helped to draft the manuscript. All authors read and approved the final manuscript.

**Author details**

[1]Elementary Educational College, Capital Normal University, Beijing, 100048, P.R. China. [2]Department of Basic Sciences, Hebei Finance University, Baoding, 071051, P.R. China. [3]Department of Mathematics Physics and Information Engineering, Jiaxing University Nanhu College, Jiaxing, 314001, P.R. China.

**Acknowledgements**

This work was supported by National Natural Science Foundation of China, grant 11401399, and Beijing Natural Science Foundation, grant 1144008.

Received: 3 September 2014 Accepted: 26 November 2014 Published: 12 Dec 2014

**References**

1. Schweizer, B, Sklar, A: Probabilistic Metric Spaces. North-Holland, New York (1983); reissued by Dover, New York (2005)
2. Guo, TX: Some basic theories of random normed linear spaces and random inner product spaces. Acta Anal. Funct. Appl. **1**(2), 160-184 (1999)
3. Guo, TX: Extension theorems of continuous random linear operators on random domains. J. Math. Anal. Appl. **193**(1), 15-27 (1995)
4. Wu, MZ: The Bishop-Phelps theorem in compete random normed modules endowed with the $(\varepsilon, \lambda)$-topology. J. Math. Anal. Appl. **391**, 648-652 (2012)
5. Wu, MZ: Farkas' lemma in random locally convex modules and Minkowski-Weyl type results in $L^0(\mathcal{F}, R^n)$. J. Math. Anal. Appl. **404**, 300-309 (2013)
6. Zhao, SE, Zhao, Y: A minimax theorem for $L^0$-valued functions on random normed modules. J. Funct. Spaces **2013**, Article ID 704251 (2013)
7. Guo, TX: Representation theorems of the dual of Lebesgue-Bochner function spaces. Sci. China Math. **43**, 234-243 (2000)
8. Guo, TX: The relation of Banach-Alaoglu theorem and Banach-Bourbaki-Kakutani-Šmulian theorem in complete random normed modules to stratification structure. Sci. China Math. **51**, 1651-1663 (2008)
9. Guo, TX: Relations between some basic results derived from two kinds of topologies for a random locally convex module. J. Funct. Anal. **258**, 3024-3047 (2010)
10. Guo, TX: Recent progress in random metric theory and its applications to conditional risk measures. Sci. China Math. **54**(4), 633-660 (2011)
11. Guo, TX, Zeng, XL: Random strict convexity and random uniform convexity in random normed modules. Nonlinear Anal., Theory Methods Appl. **73**(5), 1239-1263 (2010)
12. Guo, TX, Zeng, XL: An $L^0$-valued function's intermediate value theorem and its applications to random uniform convexity. Acta Math. Sin. Engl. Ser. **28**(5), 909-924 (2012)
13. Guo, TX, Zhang, X: Stone's representation theorem of a group of random unitary operators on complete complex random inner product modules. Sci. Sin., Math. **42**(3), 181-202 (2012)
14. Guo, TX, Yang, YJ: Ekeland's variational principle for an $\bar{L}^0$-valued function on a complete random metric space. J. Math. Anal. Appl. **389**, 1-14 (2012)
15. Zhang, X: On mean ergodic semigroups of random linear operators. Proc. Jpn. Acad., Ser. A **88**(4), 53-58 (2012)
16. Tang, YH: A new version of the Gleason-Kahane-Zelazko theorem in complete random normed algebras. J. Inequal. Appl. **2012**, 85 (2012)
17. Tang, YH: The Wintner theorem in unital complete random normed algebras. Bull. Korean Math. Soc. **50**(6), 1973-1979 (2013)
18. Tang, YH: Random spectral theorems of self-adjoint random linear operators on complete complex random inner product modules. Linear Multilinear Algebra **61**(3), 409-416 (2013)
19. Tang, YH, Guo, TX: Complete random normed algebras. Southeast Asian Bull. Math. **37**(6), 931-940 (2013)
20. Mazur, S, Ulam, S: Sur les transformations isométriques d'espaces vectoriel normes. C. R. Acad. Sci. Paris **194**, 946-948 (1932)
21. Omladič, M, Šemrl, P: On non-linear perturbations of isometries. Math. Ann. **303**, 617-628 (1995)

10.1186/1029-242X-2014-496
**Cite this article as:** Zhao et al.: **On non-$L^0$-linear perturbations of random isometries in random normed modules.** *Journal of Inequalities and Applications* 2014, 2014:496