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Polynomial operators for one-sided L_p-approximation to Riemann integrable functions

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Abstract

We present some operators for one-sided approximation of Riemann integrable functions on $[0,1]$ by algebraic polynomials in L_p-spaces. The estimates for the error of approximation are given with an explicit constant.

Keywords: one-sided approximation; operators for one-sided approximation; Riemann integrable functions

1 Introduction

Let $L_p[a,b]$ $(1 \leq p < \infty)$ be the space of all real-valued Lebesgue measurable functions, $f : [a,b] \to \mathbb{R}$, such that

$$\|f\|_{p,[a,b]} = \left(\int_a^b \left|f(t)\right|^p dt \right)^{1/p} < \infty$$

and let $C[0,1]$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}$ with the sup norm $\|f\|_\infty = \sup\{|f(t)| : x \in [0,1]\}$. We simply write $\|f\|_p = \|f\|_{p,[0,1]}$. Let $\mathcal{R}[0,1]$ be the set of all Riemann integrable functions on $[0,1]$ (recall that Riemann integrable functions on $[0,1]$ are bounded). As usual, we denote by $W_p^1[0,1]$ the space of all absolutely continuous functions $f : [0,1] \to \mathbb{R}$ such that $f' \in L_p[0,1]$. We also denote by $\mathbb{P}_n$ the family of all algebraic polynomials of degree not greater than n.

The local modulus of continuity of a function $g : [0,1] \to \mathbb{R}$ at a point x is defined by

$$\omega(g,x,t) = \sup\left\{\left|g(v) - g(w)\right| : v,w \in [x-t,x+t] \cap [0,1]\right\}, \quad t \geq 0.$$

For $p \geq 1$, the average modulus of continuity is defined by

$$\tau(g,t)_p = \left\|\omega(g,\cdot,t)\right\|_p.$$

This modulus is well defined whenever g is a bounded measurable function.

For a bounded function $f \in L_p[0,1]$ and $n \in \mathbb{N}$, the best one-sided approximation is defined by

$$\widetilde{E}_n(f)_p = \inf\left\{\|P-Q\|_p : P,Q \in \mathbb{P}_n, Q(x) \leq f(x) \leq P(x), x \in [0,1]\right\}. \tag{1}$$

It was shown in [1], with $\Delta_n = n^{-2} + n^{-1}\sqrt{1-t^2}$, that there exists a constant C such that, for any bounded measurable function $f : [0,1] \to \mathbb{R}$,

$$\widetilde{E}_n(f)_p \leq C\tau(f, \Delta_n)_p. \tag{2}$$

The analogous result for a trigonometric approximation was given in [2]. It is well known that $\lim_{t\to 0+} \tau(g,t)_1 = 0$ if and only if $g \in \mathcal{R}[0,1]$ (see [3]). That is the reason why we only consider Riemann integrable functions.

In this paper, we present some sequences of polynomial operators for one-sided L_p-approximation which realize the rate of convergence given in (2). Our construction also provides a specific constant. We point out that a one-sided approximation cannot be realized with polynomial linear operators.

Let us consider the step function

$$G(t) = \begin{cases} 0, & \text{if } -1 \leq t \leq 0, \\ 1, & \text{if } 0 < t \leq 1, \end{cases} \tag{3}$$

and fix two sequences of polynomials $\{P_n\}$ and $\{Q_n\}$ ($P_n, Q_n \in \mathbb{P}_n$) such that

$$P_n(t) \leq G(t) \leq Q_n(t), \quad t \in [-1,1]$$

and

$$\|Q_n - P_n\|_{1,[-1,1]} \to 0, \quad \text{as } n \to \infty. \tag{4}$$

The existence of such sequences of polynomials P_n and Q_n satisfying (4) is well known (see, for example, [4]). Probably, the first construction of the optimal solution for (4) is due to Markoff or Stieltjes (*cf.* Szegö [5, Section 3.411, p.50]).

Fix $p \in [1,\infty)$. In [4] we constructed a sequence of polynomial operators as follows. For $n \in \mathbb{N}$, $f \in W_p^1[0,1]$, and $x \in [0,1]$, define

$$\lambda_n(f,x) = f(0) + \int_0^1 P_n(t-x)(f')_+(t)\,dt - \int_0^1 Q_n(t-x)(f')_-(t)\,dt \tag{5}$$

and

$$\Lambda_n(f,x) = f(0) + \int_0^1 Q_n(t-x)(f')_+(t)\,dt - \int_0^1 P_n(t-x)(f')_-(t)\,dt, \tag{6}$$

where, as usual,

$$g_+(x) = \max\{0, g(x)\} \quad \text{and} \quad g_-(x) = \max\{0, -g(x)\}.$$

Also in [4], it is proved that $\lambda_n(f), \Lambda_n(f) \in \mathbb{P}_n$,

$$\lambda_n(f,x) \leq f(x) \leq \Lambda_n(f,x), \quad x \in [0,1] \tag{7}$$

and

$$\max\{\|f-\lambda_n(f)\|_p, \|f-\Lambda_n(f)\|_p\} \le \alpha_n \|f'\|_p, \tag{8}$$

where

$$\alpha_n = \|Q_n - P_n\|_{1,[-1,1]}. \tag{9}$$

For a function $f \in \mathcal{R}[0,1]$, $h \in (0,1)$ and $x \in [0,1]$, set

$$L_h(f,x) = \int_0^1 \bigl[f\bigl((1-h)x+hs\bigr) - \omega\bigl(f,(1-h)x+hs,h\bigr)\bigr]\,ds \tag{10}$$

and

$$M_h(f,x) = \int_0^1 \bigl[f\bigl((1-h)x+hs\bigr) + \omega\bigl(f,(1-h)x+hs,h\bigr)\bigr]\,ds. \tag{11}$$

It turns out (see Section 2) that $L_h(f), M_h(f) \in W_p^1[0,1]$, for $p \ge 1$, and therefore we can define

$$A_{n,h}(f,x) = \lambda_n\bigl(L_h(f),x\bigr) \quad \text{and} \quad B_{n,h}(f,x) = \Lambda_n\bigl(M_h(f),x\bigr), \tag{12}$$

where λ_n and Λ_n are given by (5) and (6), respectively. We will prove that

$$A_{n,h}(f,x) \le f(x) \le B_{n,h}(f,x), \quad x \in [0,1],$$

and present upper estimates for the error $f - A_{n,h}(f)$ and $B_{n,h}(f) - f$ in $L_p[0,1]$ in terms of the average modulus of continuity.

In the last years, there has been interest in studying open problems related to one-sided approximations (see [4, 6–8] and [9]). We point out that other operators for the one-sided approximation have been constructed in [10, 11] and [12]. In particular, the operators presented in [12] yield the non-optimal rate $O(\tau(f,1/\sqrt{n})_1)$ whereas the ones considered in [10, 11] give the optimal rate, but without an explicit constant.

The paper is organized as follows. In Section 2 we present some properties of the Steklov type functions (10) and (11). Finally, in Section 3 we consider an approximation by means of the operators defined in (12).

2 Properties of Steklov type functions

We start with the following auxiliary results.

Proposition 1 *If $f \in \mathcal{R}[0,1]$, $h \in (0,1)$, and the functions $L_h(f)$ and $M_h(f)$ are defined by* (10) *and* (11), *respectively, then the following assertions hold.*

(i) *The functions $L_h(f)$ and $M_h(f)$ are absolutely continuous. Moreover, if $\Psi_1(x) := L_h(f,x)$ and $\Psi_2(x) := M_h(f,x)$, then*

$$\begin{aligned}\Psi_j'(x) &= \frac{1-h}{h}\bigl(f\bigl((1-h)x+h\bigr) - f\bigl((1-h)x\bigr)\bigr) \\ &\quad + (-1)^j\frac{1-h}{h}\bigl(\omega\bigl(f,(1-h)x+h,h\bigr) - \omega\bigl(f,(1-h)x,h\bigr)\bigr), \quad j=1,2.\end{aligned} \tag{13}$$

(ii) *For each* $x \in [0,1]$,

$$L_h(f,x) \le f(x) \le M_h(f,x). \tag{14}$$

(iii) *For* $1 \le p < \infty$ *one has* $L_h(f)', M_h(f)' \in L_p[0,1]$

$$\max\{\|f - L_h(f)\|_p, \|f - M_h(f)\|_p\} \le \frac{2}{(1-h)^{1/p}} \tau(f,h)_p, \tag{15}$$

$$\|M_h(f) - L_h(f)\|_p \le \frac{2}{(1-h)^{1/p}} \tau(f,h)_p \tag{16}$$

and

$$\max\{\|L_h'(f)\|_p, \|M_h'(f)\|_p\} \le \frac{3}{h} \tau(f,h)_p. \tag{17}$$

Proof (i) Let $g \in L_1[0,1]$. Then the function

$$H(x) = \int_0^1 g\big((1-h)x + hs\big)\,ds,$$

is absolutely continuous with Radon-Nikodym derivative

$$H'(x) = \frac{1-h}{h}\Big(g\big((1-h)x + h\big) - g\big((1-h)x\big)\Big).$$

This, together with (10) and (11), shows (13).

(ii) Observe that

$$f(x) - M_h(f,x) = \frac{1}{h}\int_0^h \Big(f(x) - f\big((1-h)x + s\big) - \omega\big(f,(1-h)x + s, h\big)\Big)\,ds \le 0,$$

as follows from the definition of ω. Similarly, $L_h(f,x) \le f(x)$.

(iii) We present a proof for a fixed $1 < p < \infty$ (the case $p = 1$ follows analogously). As usual, take q such that $1/p + 1/q = 1$. Using (14) and Hölder inequality, we obtain

$$\begin{aligned}
\big(h\|M_h(f) - f\|_p\big)^p &\le \big(h\|M_h(f) - L_h(f)\|_p\big)^p \\
&= 2^p \int_0^1 \left(\int_0^h \omega\big(f,(1-h)x + s, h\big)\,ds\right)^p dx \\
&\le 2^p h^{p/q} \int_0^1 \int_0^h \omega^p\big(f,(1-h)x + s, h\big)\,ds\,dx \\
&= \frac{2^p h^{p/q}}{1-h} \int_0^h \int_s^{1-h+s} \omega^p(f,y,h)\,dy\,ds \\
&\le \frac{2^p h^{p/q}}{1-h} \int_0^h \int_0^1 \omega^p(f,y,h)\,dy\,ds \\
&= \frac{2^p h^{1+p/q}}{1-h} \tau^p(f,h)_p = \frac{2^p h^p}{1-h} \tau^p(f,h)_p.
\end{aligned}$$

For $\|f - L_h(f)\|_p$ the proof follows analogously.

Finally, in order to estimate $\|L_h(f)'\|_p$ and $\|M_h(f)'\|_p$, we use the representation of the derivative given in (13). Recall that the usual modulus of continuity for $f \in L_p[0,1]$ is defined as follows:

$$\omega(f,h)_p = \sup_{0<t\le h}\left(\int_0^{1-t}\left|f(x+t)-f(x)\right|^p\,dx\right)^{1/p}.$$

It can be proved (see the proof of Lemma 4 in [2]) that, for any $f \in \mathcal{R}[0,1]$,

$$\omega(f,h)_p \le \tau(f,h)_p.$$

Therefore

$$\begin{aligned}&\left(\int_0^1\left|f\bigl((1-h)x+h\bigr)-f\bigl((1-h)x\bigr)\right|^p\,dx\right)^{1/p}\\&\quad=\left(\frac{1}{1-h}\int_0^{1-h}\left|f(y+h)-f(y)\right|^p\,dy\right)^{1/p}\le\frac{\omega(f,h)_p}{(1-h)^{1/p}}\le\frac{\tau(f,h)_p}{(1-h)^{1/p}}.\end{aligned}\tag{18}$$

On the other hand

$$\begin{aligned}&\left(\int_0^1\left|\omega\bigl(f,(1-h)x+h,h\bigr)-\omega\bigl(f,(1-h)x,h\bigr)\right|^p\,dx\right)^{1/p}\\&\quad\le\left(\int_0^1\omega^p\bigl(f,(1-h)x+h,h\bigr)\,dx\right)^{1/p}+\left(\int_0^1\omega^p\bigl(f,(1-h)x,h\bigr)\,dx\right)^{1/p}\\&\quad\le\frac{1}{(1-h)^{1/p}}\left[\left(\int_h^1\omega^p(f,y,h)\,dy\right)^{1/p}+\left(\int_0^{1-h}\omega^p(f,y,h)\,dy\right)^{1/p}\right]\\&\quad\le\frac{2}{(1-h)^{1/p}}\tau(f,h)_p.\end{aligned}\tag{19}$$

From (13), (18), and (19), we obtain (17). The proof is complete. □

3 Approximation of Riemann integrable functions

Theorem 1 *Fix $p \in [1,\infty)$, $n \in \mathbb{N}$, $h \in (0,1)$, and $f \in \mathcal{R}[0,1]$. Let $A_{n,h}(f)$ and $B_{n,h}(f)$ be as in* (12) *and let α_n be as in* (9). *Then $A_{n,h}(f), B_{n,h}(f) \in \mathbb{P}_n$,*

$$A_{n,h}(f,x) \le f(x) \le B_{n,h}(f,x), \quad x \in [0,1],$$

$$\max\bigl\{\|f-A_{n,h}(f)\|_p, \|f-B_{n,h}(f)\|_p\bigr\} \le \left(\frac{2}{1-h}+\frac{3\alpha_n}{h}\right)\tau(f,h)_p \tag{20}$$

and

$$\|B_{n,h}(f)-A_{n,h}(f)\|_p \le \left(\frac{4}{1-h}+\frac{6\alpha_n}{h}\right)\tau(f,h)_p. \tag{21}$$

Proof Let $L_h(f)$ and $M_h(f)$ be as in (10) and (11), respectively. We know that $A_{n,h}(f)$, $B_{n,h}(f) \in \mathbb{P}_n$. Moreover, from (7) and (14) we have

$$A_{n,h}(f) = \lambda_n\bigl(L_h(f)\bigr) \le L_h(f) \le f \le M_h(f) \le \Lambda_n\bigl(M_h(f)\bigr) = B_{n,h}(f).$$

On the other hand, from (8), (15), and (17) one has

$$\begin{aligned}\left\|f - A_{n,h}(f)\right\|_p &\le \left\|f - L_h(f)\right\|_p + \left\|L_h(f) - A_{n,h}(f)\right\|_p \\ &\le \frac{2}{(1-h)^{1/p}}\tau(f,h)_p + \alpha_n\left\|(L_h f)'\right\|_p \\ &\le \left(\frac{2}{(1-h)^{1/p}} + \frac{3\alpha_n}{h}\right)\tau(f,h)_p \\ &\le \left(\frac{2}{1-h} + \frac{3\alpha_n}{h}\right)\tau(f,h)_p.\end{aligned}$$

The estimate for $\|f - B_{n,h}(f)\|_p$ follows analogously. Finally, (21) follows immediately from (20). □

For $n \in \mathbb{N}$ the Fejér-Korovkin kernel is defined by

$$K_n(t) = \frac{2\sin^2(\pi/(n+2))}{n+2}\left(\frac{\cos((n+2)t/2)}{\cos(\pi/(n+2)) - \cos t}\right)^2$$

for $t \neq \pm\pi/(n+2)$ and $K_n(t) = (n+2)/2$ for $t = \pm\pi/(n+2)$.

Let $F : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function such that

$$F(x) = \begin{cases} 1, & \text{if } x \in [-\pi/2, \pi/2], \\ 0, & \text{if } x \in [-\pi, \pi] \setminus [-\pi/2, \pi/2], \end{cases} \tag{22}$$

and, for $x, u, t \in [-\pi, \pi]$, set

$$U(F, x, t, u) = \omega\bigl(F, x+t, |u|\bigr) + \omega\bigl(F, x+u, |t|\bigr).$$

For $n \in \mathbb{N}$ define

$$T_n^-(x) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\bigl[F(x+t) - U(F,x,t,u)\bigr]K_n(t)\,dt\right)K_n(u)\,du$$

and

$$T_n^+(x) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\bigl[F(x+t) + U(F,x,t,u)\bigr]K_n(t)\,dt\right)K_n(u)\,du.$$

The following result was proved in [4].

Proposition 2 *Let G be given by* (3). *For $n \in \mathbb{N}$ and $x \in [-1,1]$, define*

$$P_n(x) = T_n^-(\arccos x) \quad \textit{and} \quad Q_n(x) = T_n^+(\arccos x).$$

Then $P_n, Q_n \in \mathbb{P}_n$,

$$P_n(x) \le G(x) \le Q_n(x), \quad x \in [-1,1]$$

and

$$\|Q_n - P_n\|_{1,[-1,1]} \leq \frac{4\pi^2}{n+2}. \tag{23}$$

From Theorem 1 and Proposition 2 we can state our main results.

Theorem 2 *Fix $p \in [1,\infty)$. For $n \in \mathbb{N}$, let P_n, Q_n be the sequences of polynomials constructed as in Proposition* 2. *For $f \in \mathcal{R}[0,1]$ and $n \geq 2$, set*

$$\mathcal{A}_n(f) = A_{n,\frac{1}{n}}(f), \qquad \mathcal{B}_n(f) = B_{n,\frac{1}{n}}(f),$$

where $A_{n,h}$ and $B_{n,h}$ are given by (12). *Then*

$$\mathcal{A}_n(f), \mathcal{B}_n(f) \in \mathbb{P}_n,$$

$$\mathcal{A}_n(f,x) \leq f(x) \leq \mathcal{B}_n(f,x), \quad x \in [0,1],$$

$$\max\left\{\left\|f - \mathcal{A}_n(f)\right\|_p, \left\|f - \mathcal{B}_n(f)\right\|_p\right\} \leq 2\left(1 + 6\pi^2\right)\tau\left(f, \frac{1}{n}\right)_p \tag{24}$$

and

$$\left\|\mathcal{B}_n(f) - \mathcal{A}_n(f)\right\|_p \leq 4\left(1 + 6\pi^2\right)\tau\left(f, \frac{1}{n}\right)_p.$$

Proof The first two assertions follow from Proposition 2 with $h = 1/n$. So, in order to prove the theorem it remains to verify (24). Taking into account (9) and (23), we have $\alpha_n \leq 4\pi^2/(n+2)$. Then, from (20) with $h = 1/n$ and $n \geq 2$, we obtain

$$\begin{aligned}\max\left\{\left\|f - \mathcal{A}_n(f)\right\|_p, \left\|f - \mathcal{B}_n(f)\right\|_p\right\} &\leq \left(\frac{2n}{n-1} + \frac{12\pi^2 n}{n+2}\right)\tau\left(f, \frac{1}{n}\right)_p \\ &\leq 2\left(1 + 6\pi^2\right)\tau\left(f, \frac{1}{n}\right)_p.\end{aligned}$$

This completes the proof. □

Finally, from Theorem 2 we have immediately the following.

Corollary 1 *Fix $p > 1$ and $n \in \mathbb{N}$, $n \geq 2$. For any $f \in \mathcal{R}[0,1]$ we have*

$$\widetilde{E}_n(f)_p \leq 4\left(1 + 6\pi^2\right)\tau\left(f, \frac{1}{n}\right)_p,$$

where $\widetilde{E}_n(f)_p$ is the best one-sided approximation defined in (1).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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