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Common fixed point theorems for generalized expansive mappings in partial b-metric spaces and an application

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Abstract

In this paper, we first introduce the concepts of generalized $(\psi, f)_\lambda$-expansive mappings and generalized $(\phi, g, h)_\lambda$-weakly expansive mappings designed for three mappings. Then we establish some common fixed point results for such two new types of mappings in partial b-metric spaces. These results generalize and extend the main results of Karapınar *et al.* (J. Inequal. Appl. 2014:22, 2014), Nashine *et al.* (Fixed Point Theory Appl. 2013:203, 2013) and many comparable results from the current literature. Moreover, some examples and an application to a system of integral equations are given here to illustrate the usability of the obtained results.

MSC: 47H10; 54H25

Keywords: partial metric space; b-metric space; expansive mappings; weakly expansive mappings; common fixed point

1 Introduction and preliminaries

Fixed point theory in metric spaces is an important branch of nonlinear analysis, which is closely related to the existence and uniqueness of solutions of differential equations and integral equations.

There are many generalizations of the concept of metric spaces in the literature. In particular, Matthews [1] introduced the concept of a partial metric space as a part of the study of denotational data for networks and proved that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. After that, fixed point results in partial metric spaces have been studied by many authors (see [2–7]). On the other hand, the concept of a b-metric space was introduced and studied by Bakhtin [8] and Czerwik [9]. Since then, several papers have been published on the fixed point theory of the variational principle for single-valued and multi-valued operators in b-metric spaces (see [8–15] and the references therein). We begin with the definition of b-metric spaces.

Definition 1.1 ([8]) Let X be a nonempty set and $\lambda \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is said to be a b-metric on X if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x, y) = 0$ if and only if $x = y$,

(b_2) $d(y,x) = d(x,y)$,
(b_3) $d(x,z) \leq \lambda[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space.

Recently, Shukla [16] introduced the notion of a partial b-metric space as a generalization of partial metric spaces and b-metric spaces.

Definition 1.2 ([16]) Let X be a nonempty set and $\lambda \geq 1$ be a given real number. A mapping $p_b : X \times X \to R^+$ is said to be a partial b-metric on X if for all $x,y,z \in X$, the following conditions are satisfied:

(p_{b1}) $p_b(x,x) = p_b(y,y) = p_b(x,y)$ if and only if $x = y$,
(p_{b2}) $p_b(x,x) \leq p_b(x,y)$,
(p_{b3}) $p_b(x,y) = p_b(y,x)$,
(p_{b4}) $p_b(x,z) \leq \lambda[p_b(x,y) + p_b(y,z)] - p_b(y,y)$.

A partial b-metric space is a pair (X,p_b) such that X is a nonempty set and p_b is a partial b-metric on X. The number $\lambda \geq 1$ is called the coefficient of (X,p_b).

In [17], Mustafa *et al.* introduced a new concept of partial b-metric by modifying Definition 1.2 in order to guarantee that each partial b-metric p_b can induce a b-metric. The advantage of the new definition of partial b-metric is that by using it one can define a dependent b-metric which is called the b-metric associated with partial b-metric p_b. The new concept of partial b-metric is as follows.

Definition 1.3 ([17]) Let X be a nonempty set and $\lambda \geq 1$ be a given real number. A mapping $p_b : X \times X \to R^+$ is said to be a partial b-metric on X if for all $x,y,z \in X$, the following conditions are satisfied:

(p_{b1}) $p_b(x,x) = p_b(y,y) = p_b(x,y)$ if and only if $x = y$,
(p_{b2}) $p_b(x,x) \leq p_b(x,y)$,
(p_{b3}) $p_b(x,y) = p_b(y,x)$,
(p'_{b4}) $p_b(x,z) \leq \lambda[p_b(x,y) + p_b(y,z) - p_b(y,y)] + \frac{(1-\lambda)}{2}[p_b(x,x) + p_b(z,z)]$.

The pair (X,b) is called a partial b-metric space with coefficient $\lambda \geq 1$.

Since $\lambda \geq 1$, from (p'_{b4}), we have

$$p_b(x,z) \leq \lambda\bigl[p_b(x,y) + p_b(y,z) - p_b(y,y)\bigr] \leq \lambda\bigl[p_b(x,y) + p_b(y,z)\bigr] - p_b(y,y).$$

Hence, a partial b-metric in the sense of Definition 1.3 is also a partial b-metric in the sense of Definition 1.2.

In a partial b-metric space (X,p_b), if $p_b(x,y) = 0$, then (p_{b1}) and (p_{b2}) imply that $x = y$. But the converse does not hold always. It is clear that every partial metric space is a partial b-metric space with coefficient $\lambda = 1$ and every b-metric is a partial b-metric space with same coefficient and zero distance. However, the converse of these facts need not hold. The following example shows that a partial b-metric on X might be neither a partial metric, nor a b-metric on X.

Example 1.1 ([17]) Let $X = R$, $q > 1$ be a constant and $p_b : X \times X \to R^+$ be defined by

$$p_b(x,y) = |x-y|^q + 3,$$

for all $x, y \in X$. Then (X, p_b) is a partial b-metric space with the coefficient $\lambda = 2^{q-1} > 1$, but it is neither a b-metric nor a partial metric space.

Each partial b-metric p_b on X generates a topology τ_{p_b} on X, which has a subbase of the family of open p_b-balls $\{B_{p_b}(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_{p_b}(x,\varepsilon) = \{y \in X : p_b(x,y) < p_b(x,x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$. The topology space (X, p_b) is T_0, but does not need to be T_1. The topology τ_{p_b} on X is called a p_b-metric topology.

Definition 1.4 ([17]) A sequence $\{x_n\}$ in a partial b-metric space is said to be:

(1) p_b-convergent to a point $x \in X$ if $\lim_{n\to\infty} p_b(x,x_n) = p_b(x,x)$.

(2) a p_b-Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n,x_m)$ exists and is finite.

(3) A partial b-metric space (X, p_b) is said to be p_b-complete if every p_b-Cauchy sequence $\{x_n\}$ in X p_b-converges to a point $x \in X$ such that $\lim_{n\to\infty} p_b(x,x_n) = p_b(x,x)$.

It should be noted that the limit of a convergent sequence in a partial b-metric space may not be unique (see [16, Example 2]).

In [17], using Definition 1.3, Mustafa *et al.* proved the fact if p_b is a partial b-metric on X, then the function $p_b^s : X \times X \to R^+$ given by $p_b^s(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y)$ defines a b-metric on X. Using Definition 1.3, Mustafa *et al.* also obtained the following lemma which is the key to the proof of our theorems.

Lemma 1.1 ([17]) *Let (X, p_b) be a partial b-metric space. Then*:

(1) *A sequence $\{x_n\}$ in X is a p_b-Cauchy sequence in (X, p_b) if and only if it is a b-Cauchy sequence in b-metric space (X, p_b^s).*

(2) *A partial b-metric space (X, p_b) is p_b-complete if and only if the b-metric space (X, p_b^s) is b-complete. Moreover,* $\lim_{n\to\infty} p_b^s(x,x_n) = 0$ *if and only if* $\lim_{n,m\to\infty} p_b(x_n,x_m) = \lim_{n\to\infty} p_b(x,x_n) = p_b(x,x)$.

It should be noted that in general a partial b-metric function $p_b(x,y)$ for $\lambda > 1$ is not jointly continuous for all variables. The following example illustrates this fact.

Example 1.2 Let $X = N \cup \{\infty\}$, and let $p_b : X \times X \to R^+$ be defined by

$$p_b(m,n) = \begin{cases} 0, & \text{if } m = n, \\ 6, & \text{if one of } m, n \text{ is even and the other is even } (m \neq n) \text{ or } \infty, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 3, & \text{otherwise.} \end{cases}$$

Then considering all possible cases, it can be checked that, for all $m, n, p \in X$, we have

$$\begin{aligned} p_b(m,p) &\leq 2\bigl[p_b(m,n) + p_b(n,p)\bigr] \\ &= 2\bigl[p_b(m,n) + p_b(n,p) - p_b(n,n)\bigr] + \frac{1-2}{2}\bigl[p_b(m,m) + p_b(p,p)\bigr]. \end{aligned}$$

Thus, (X,p_b) is a partial b-metric space (with $\lambda = 2$). Let $x_n = 2n+1$ for each $n \in N$. Then $p_b(2n+1,\infty) = \frac{1}{2n+1} \to 0$ as $n \to \infty$, that is, $x_n \to \infty$, but $p_b(x_n,2) = 3 \nrightarrow 6 = p_b(\infty,2)$.

Since in general a partial b-metric is not continuous, we need the following simple lemma about the p_b-convergent sequences in the proof of our results.

Lemma 1.2 ([17]) *Let (X,p_b) be a partial b-metric space with the coefficient $\lambda \geq 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are p_b-convergent to x and y, respectively. Then we have*

$$\begin{aligned}\frac{p_b(x,y)}{\lambda^2} - \frac{p_b(x,x)}{\lambda} - p_b(y,y) &\leq \liminf_{n\to\infty} p_b(x_n,y_n) \leq \limsup_{n\to\infty} p_b(x_n,y_n) \\ &\leq \lambda p_b(x,x) + \lambda^2 p_b(y,y) + \lambda^2 p_b(x,y).\end{aligned}$$

In particular, if $p_b(x,y) = 0$, then we have $\lim_{n\to\infty} p_b(x_n,y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{p_b(x,z)}{\lambda} - p_b(x,x) \leq \liminf_{n\to\infty} p_b(x_n,z) \leq \limsup_{n\to\infty} p_b(x_n,z) \leq \lambda p_b(x,z) + \lambda p_b(x,x).$$

In particular, if $p_b(x,x) = 0$, then we have

$$\frac{p_b(x,z)}{\lambda} \leq \liminf_{n\to\infty} p_b(x_n,z) \leq \limsup_{n\to\infty} p_b(x_n,z) \leq \lambda p_b(x,z).$$

Jungck [18] introduced the concept of weakly compatible mappings as follows.

Definition 1.5 ([18]) Let X be a nonempty set, A and $T : X \to X$ be two self-maps. A and T are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, *i.e.*, if $Az = Tz$ for some $z \in X$, then $ATz = TAz$.

It is worth mentioning that most of the preceding references concerned with fixed point results of contractions in partial metric spaces and b-metric spaces, but we rarely see fixed point results of expansions in such two types of spaces. Recently, in [19], Karapınar *et al.* considered a generalized expansive mapping and proved the fixed point theorem in metric spaces. Nashine *et al.* [20] introduced ψ_S-contractive mappings and proved some fixed point theorems in ordered metric spaces. Here, we recall the relevant definition.

Definition 1.6 ([20]) Let $(X,d,\preceq)$ be an ordered metric space, and let $S,T : X \to X$. The mappings S, T are said to be ψ_S-contractive if

$$d(Sx,Ty) \leq \psi\bigl(d(x,y), d(x,Sx), d(y,Ty), d(x,Ty), d(y,Sx)\bigr),$$

for all $x \succeq y$, where $\psi : R^{+5} \to R^+$ is a strictly increasing and continuous function in each coordinate, and for all $t \in R^+\backslash\{0\}$, $\psi(t,t,t,0,2t) < t$, $\psi(t,t,t,2t,0) < t$, $\psi(0,0,t,t,0) < t$, $\psi(0,t,0,0,t) < t$ and $\psi(t,0,0,t,t) < t$.

Inspired by the notions of ψ_S-contractive mappings of [20], we first introduce the concepts of generalized $(\psi,f)_\lambda$-expansive mappings and generalized $(\phi,g,h)_\lambda$-weakly expansive mappings. Then we establish some common fixed point theorems for these classes of mappings in complete partial b-metric spaces. The obtained results generalize and extend the main results of [15–23]. We also provide some examples to show the generality of our results. Finally, an application is given to illustrate the usability of the obtained results.

2 Main results

The study of expansive mappings is a very interesting research area in fixed point theory (see [19, 21–23]). In this section, inspired by the notion of ψ_S-contractive mappings of [20], we first introduce the notions of generalized $(\psi,f)_\lambda$-expansive mappings and generalized $(\phi,g,h)_\lambda$-weakly expansive mappings in partial b-metric spaces.

For convenience, we denote by Ψ the class of functions $\psi : R^{+5} \to R^+$ satisfying the following conditions:

(i) ψ is a nondecreasing and continuous function in each coordinate;

(ii) for $t_i \in R^+$, $i = 1,2,\dots,5$, $\psi(t_1,t_2,t_3,t_4,t_5) > \min\{t_1, \frac{t_2+t_3}{2}\}$, where $\min\{t_1, \frac{t_2+t_3}{2}\} > 0$;

(iii) $\psi(0,0,0,0,0) = 0$ and $\psi(t_1,t_2,t_3,t_4,t_5) > \min\{t_1,t_5\}$, where $\min\{t_1,t_5\} > 0$.

The following are some easy examples of functions from class Ψ:

$\psi(t_1,t_2,t_3,t_4,t_5) = at_1$, for $a > 1$;

$\psi(t_1,t_2,t_3,t_4,t_5) = \max\{a\min\{t_1, \frac{t_2+t_3}{2}\}, c\min\{t_1,t_5\}\}$, for $a, c > 1$;

$\psi(t_1,t_2,t_3,t_4,t_5) = \max\{at_1 + b\frac{t_2+t_3}{2}, c\min\{t_1,t_5\}\}$, for $a,b \geq 0$, $a + b > 1$, and $c > 1$;

$\psi(t_1,t_2,t_3,t_4,t_5) = \max\{at_1 + b\frac{t_2+t_3}{2}, ct_1 + dt_5\}$, for $a,b,c,d \geq 0$, $a + b > 1$, and $c + d > 1$;

$\psi(t_1,t_2,t_3,t_4,t_5) = \max\{\min\{t_1, \frac{t_2+t_3}{2}\}, \min\{t_1,t_5\}\} + \phi(\max\{\min\{t_1, \frac{t_2+t_3}{2}\}, \min\{t_1,t_5\}\})$, where $\phi : R^+ \to R^+$ is a nondecreasing and continuous function, and $\phi(s) = 0$ if and only if $s = 0$.

Definition 2.1 Let (X,p_b) be a partial b-metric space with the coefficient $\lambda \geq 1$, A, S, and $T : X \to X$ be three mappings. Then A, S, and T are said to be generalized $(\psi,f)_\lambda$-expansive mappings if

$$f\left(\frac{p_b(Sx,Ty)}{\lambda^2}\right) \geq \psi\left(p_b(Ax,Ay), p_b(Ax,Sx), p_b(Ay,Ty), \frac{p_b(Ax,Ty)}{\lambda}, \frac{p_b(Ay,Sx)}{\lambda}\right), \quad (2.1)$$

for all $x,y \in X$, where $\psi \in \Psi$, $f : [0,\infty) \to [0,\infty)$ is a nondecreasing and continuous function, $f(0) = 0$, and for all $t > 0$, $\psi(t_1,t_2,t_3,t_4,t_5) > f(t)$, where $\min\{t_1, \frac{t_2+t_3}{2}\} = t$ or $\min\{t_1,t_5\} = t$.

Definition 2.2 Let (X,p_b) be a partial b-metric space with the coefficient $\lambda \geq 1$, A, S, and $T : X \to X$ be three mappings. Then A, S, and T are said to be generalized $(\phi,g,h)_\lambda$-weakly expansive mappings if

$$g\left(\frac{p_b(Sx,Ty)}{\lambda^2}\right) \geq h\bigl(M_\lambda(Ax,Ay)\bigr) + \phi\bigl(M_\lambda(Ax,Ay)\bigr) \quad (2.2)$$

for all $x,y \in X$, where $M_\lambda(Ax,Ay) = \max\{\min\{p_b(Ax,Ay), \frac{p_b(Ax,Sx)+p_b(Ay,Ty)}{2}\}, \min\{p_b(Ax,Ay), \frac{p_b(Ay,Sx)}{\lambda}\}\}$, $\phi, g, h : R^+ \to R^+$ are continuous and nondecreasing functions, $g(0) = h(0) = 0$, $\phi(s) = 0$ if and only if $s = 0$, and for all $t > 0$, $h(t) + \phi(t) > g(t)$.

It is easy to acquire the following example of generalized $(\psi,f)_\lambda$-expansive mappings or generalized $(\phi,g,h)_\lambda$-weakly expansive mappings.

Example 2.1 Let $X = R^{+2}$ be endowed with the partial b-metric $p_b : X \times X \to R^+$ given by

$$p_b(u,v) = \begin{cases} 0, & \text{if } x_1 = x_2 \text{ and } y_1 = y_2, \\ (x_1 + x_2)^2, & \text{if } x_1 \neq x_2 \text{ and } y_1 = y_2, \\ (y_1 + y_2)^2, & \text{if } x_1 = x_2 \text{ and } y_1 \neq y_2, \\ (x_1 + x_2)^2 + (y_1 + y_2)^2, & \text{otherwise,} \end{cases}$$

for $u = (x_1, y_1), v = (x_2, y_2) \in X$, where $\lambda = 2$. Let $\psi : R^{+5} \to R^+$ and $f : R^+ \to R^+$ be given by

$$\psi(t_1,t_2,t_3,t_4,t_5) = \max\left\{2\min\left\{t_1, \frac{t_2+t_3}{2}\right\}, 2\min\{t_1,t_5\}\right\}, \qquad f(t) = t,$$

for all $t_1, t_2, \ldots, t_5, t \in R^+$, and $A, S, T : X \to X$ be given by

$$S(x,y) = \left(3x, e^{2y} - 1 + y\right), \qquad T(x,y) = \left(3x, e^{2y} - 1 + y\right),$$
$$A(x,y) = \left(\frac{3}{4}x, \frac{3}{4}y\right), \quad \text{for all } (x,y) \in X.$$

Then A, S, and T are generalized $(\psi,f)_\lambda$-expansive mappings. In fact, if $\phi, g, h : R^+ \to R^+$ are defined by

$$g(t) = \frac{t}{2}, \qquad h(t) = \eta t, \qquad \phi(t) = (1-\eta)t,$$

for all $t \in R^+$, where $0 < \eta < 1$. Then A, S, and T are also generalized $(\phi,g,h)_\lambda$-weakly expansive mappings.

Now, we first prove some fixed point results for generalized $(\psi,f)_\lambda$-expansive mappings in p_b-complete partial b-metric spaces.

Theorem 2.1 *Let (X,p_b) be a p_b-complete partial b-metric space, A, S, and $T : X \to X$ be three mappings satisfying the generalized $(\psi,f)_\lambda$-expansive condition* (2.1). *Suppose that the following conditions are satisfied*:

(i) *$A(X) \subset S(X)$, $A(X) \subset T(X)$, and $A(X)$ is a closed subset of (X, p_b^s)*;

(ii) *A is an injective and A and T are weakly compatible.*

Then A, S, and T have a unique common fixed point in X.

Proof Let $x_0 \in X$ be an arbitrary point in X. Since $A(X) \subset S(X)$, there exists an $x_1 \in X$ such that $Ax_0 = Sx_1$. Since $A(X) \subset T(X)$, there exists an $x_2 \in X$ such that $Ax_1 = Tx_2$. Continuing this process, we can construct a sequence $\{Ax_n\}$ in X such that

$$Ax_{2n} = Sx_{2n+1}, \qquad Ax_{2n+1} = Tx_{2n+2}, \quad \text{for } n = 0,1,2,\ldots.$$

We will complete the proof in three steps.

Step 1. We prove that

$$\lim_{n\to\infty} p_b(Ax_n, Ax_{n+1}) = 0. \tag{2.3}$$

Suppose that $p_b(Ax_n, Ax_{n+1}) = 0$ for some $n = n_0$. In the case that $n_0 = 2k$, $p_b(Ax_{2k}, Ax_{2k+1}) = 0$ gives $p_b(Ax_{2k+1}, Ax_{2k+2}) = 0$. Indeed, by (2.1), we have

$$
\begin{aligned}
0 &= f\left(\frac{p_b(Ax_{2k}, Ax_{2k+1})}{\lambda^2}\right) = f\left(\frac{p_b(Sx_{2k+1}, Tx_{2k+2})}{\lambda^2}\right) \\
&\geq \psi\Bigg(p_b(Ax_{2k+1}, Ax_{2k+2}), p_b(Ax_{2k+1}, Sx_{2k+1}), \\
&\quad p_b(Ax_{2k+2}, Tx_{2k+2}), \frac{p_b(Ax_{2k+1}, Tx_{2k+2})}{\lambda}, \frac{p_b(Ax_{2k+2}, Sx_{2k+1})}{\lambda}\Bigg) \\
&= \psi\left(p_b(Ax_{2k+1}, Ax_{2k+2}), 0, p_b(Ax_{2k+1}, Ax_{2k+2}), 0, \frac{p_b(Ax_{2k+2}, Ax_{2k})}{\lambda}\right),
\end{aligned}
$$

which implies that $\frac{p_b(Ax_{2k+1}, Ax_{2k+2})}{2} = 0$, that is, $p_b(Ax_{2k+1}, Ax_{2k+2}) = 0$. Similarly, if $n_0 = 2k+1$, then $p_b(Ax_{2k+2}, Ax_{2k+3}) = 0$. Consequently, $p_b(Ax_n, Ax_{n+1}) \equiv 0$ for $n \geq n_0$. Hence, $\lim_{n\to\infty} p_b(Ax_n, Ax_{n+1}) = 0$.

Now, suppose that $p_b(Ax_n, Ax_{n+1}) > 0$, for each n. By (2.1), we have

$$
\begin{aligned}
f\left(\frac{p_b(Ax_{2n}, Ax_{2n+1})}{\lambda^2}\right) &= f\left(p_b(Sx_{2n+1}, Tx_{2n+2})\right) \\
&\geq \psi\Bigg(p_b(Ax_{2n+1}, Ax_{2n+2}), p_b(Ax_{2n+1}, Sx_{2n+1}), \\
&\quad p_b(Ax_{2n+2}, Tx_{2n+2}), \frac{p_b(Ax_{2n+1}, Tx_{2n+2})}{\lambda}, \frac{p_b(Ax_{2n+2}, Sx_{2n+1})}{\lambda}\Bigg) \\
&= \psi\Bigg(p_b(Ax_{2n+1}, Ax_{2n+2}), p_b(Ax_{2n}, Ax_{2n+1}), p_b(Ax_{2n+1}, Ax_{2n+2}), \\
&\quad \frac{p_b(Ax_{2n+1}, Ax_{2n+1})}{\lambda}, \frac{p_b(Ax_{2n+2}, Ax_{2n})}{\lambda}\Bigg).
\end{aligned}
\tag{2.4}
$$

If $\min\{p_b(Ax_{2n+1}, Ax_{2n+2}), \frac{p_b(Ax_{2n}, Ax_{2n+1}) + p_b(Ax_{2n+1}, Ax_{2n+2})}{2}\} = \frac{p_b(Ax_{2n}, Ax_{2n+1}) + p_b(Ax_{2n+1}, Ax_{2n+2})}{2} > 0$, then we have $p_b(Ax_{2n}, Ax_{2n+1}) \leq p_b(Ax_{2n+1}, Ax_{2n+2})$. It follows from (2.4) and the properties of ψ and f that

$$
\begin{aligned}
f\left(p_b(Ax_{2n}, Ax_{2n+1})\right) \geq f\left(\frac{p_b(Ax_{2n}, Ax_{2n+1})}{\lambda^2}\right) &\geq \psi\Bigg(p_b(Ax_{2n+1}, Ax_{2n+2}), p_b(Ax_{2n}, Ax_{2n+1}), \\
&\quad p_b(Ax_{2n+1}, Ax_{2n+2}), \frac{p_b(Ax_{2n+1}, Ax_{2n+1})}{\lambda}, \frac{p_b(Ax_{2n+2}, Ax_{2n})}{\lambda}\Bigg) \\
&> f\left(\frac{p_b(Ax_{2n}, Ax_{2n+1}) + p_b(Ax_{2n+1}, Ax_{2n+2})}{2}\right).
\end{aligned}
$$

Since f is nondecreasing, we get $p_b(Ax_{2n}, Ax_{2n+1}) > p_b(Ax_{2n+1}, Ax_{2n+2})$, which is a contradiction. Thus,

$$
f\left(p_b(Ax_{2n}, Ax_{2n+1})\right) \geq f\left(\frac{p_b(Ax_{2n}, Ax_{2n+1})}{\lambda^2}\right) > f\left(p_b(Ax_{2n+1}, Ax_{2n+2})\right).
$$

Hence, we deduce that, for each $n \in N$, $p_b(Ax_{2n+1}, Ax_{2n+2}) < p_b(Ax_{2n}, Ax_{2n+1})$. Similarly, we can prove that $p_b(Ax_{2n}, Ax_{2n+1}) < p_b(Ax_{2n-1}, Ax_{2n})$, for all $n \geq 1$. Therefore, $\{p_b(Ax_n, Ax_{n+1})\}$

is a decreasing sequence of nonnegative real numbers. So, there exists $r \geq 0$ such that $\lim_{n\to\infty} p_b(Ax_n, Ax_{n+1}) = r$.

From Definition 1.3(p'_{b4}), we have

$$\begin{aligned} p_b(Ax_n, Ax_{n+2}) &\leq p_b(Ax_n, Ax_{n+2}) + p_b(Ax_{n+1}, Ax_{n+1}) \\ &\leq \lambda p_b(Ax_n, Ax_{n+1}) + \lambda p_b(Ax_{n+1}, Ax_{n+2}). \end{aligned} \tag{2.5}$$

It follows from (2.5) that $\{p_b(Ax_n, Ax_{n+2})\}$ and $\{p_b(Ax_{n+1}, Ax_{n+1})\}$ are two bounded sequences. Hence, the sequence $\{p_b(Ax_n, Ax_{n+2})\}$ has a subsequence $\{p_b(Ax_{n_k}, Ax_{n_k+2})\}$ which converges to a real number $\alpha \leq 2\lambda r$, and the sequence $\{p_b(Ax_{n_k+1}, Ax_{n_k+1})\}$ has a subsequence $\{p_b(Ax_{n_{k(i)}+1}, Ax_{n_{k(i)}+1})\}$ which converges to a real number $\beta \leq 2\lambda r$. By (2.4), we have

$$\begin{aligned} f\left(\frac{p_b(Ax_{2n_{k(i)}}, Ax_{2n_{k(i)}+1})}{\lambda^2}\right) &= f\bigl(p_b(Sx_{2n_{k(i)}+1}, Tx_{2n_{k(i)}+2})\bigr) \\ &\geq \psi\Bigl(p_b(Ax_{2n_{k(i)}+1}, Ax_{2n_{k(i)}+2}), p_b(Ax_{2n_{k(i)}}, Ax_{2n_{k(i)}+1}), \\ &\quad p_b(Ax_{2n_{k(i)}+1}, Ax_{2n_{k(i)}+2}), \\ &\quad \frac{p_b(Ax_{2n_{k(i)}+1}, Ax_{2n_{k(i)}+1})}{\lambda}, \frac{p_b(Ax_{2n_{k(i)}+2}, Ax_{2n_{k(i)}})}{\lambda}\Bigr). \end{aligned}$$

Letting $n_{k(i)} \to \infty$ in the above inequality, by the properties of ψ and f, we have $f(r) \geq f(\frac{r}{\lambda^2}) \geq \psi(r, r, r, \frac{\beta}{\lambda}, \frac{\alpha}{\lambda})$, which implies that $r = 0$. Hence, $\lim_{n\to\infty} p(Ax_n, Ax_{n+1}) = 0$.

Step 2. We show that $\{Ax_n\}$ is a p_b-Cauchy sequence.

Indeed, we first prove that $\lim_{m,n\to\infty} p_b(Ax_n, Ax_m) = 0$. Because of (2.3), it is sufficient to show that $\lim_{m,n\to\infty} p_b(Ax_{2n}, Ax_{2m}) = 0$. Suppose on the contrary, then there exists $\varepsilon > 0$ for which we can find two subsequences $\{Ax_{2n(k)}\}$ and $\{Ax_{2m(k)}\}$ of $\{Ax_{2n}\}$ such that $m(k)$ is the smallest index for which

$$m(k) > n(k) > k, \quad p_b(Ax_{2m(k)}, A_{2n(k)}) \geq \varepsilon, \quad \text{for every } k. \tag{2.6}$$

This means that

$$p_b(Ax_{2m(k)-2}, A_{2n(k)}) < \varepsilon. \tag{2.7}$$

From (2.6), using the triangular inequality, we can see that

$$0 < \varepsilon \leq p_b(Ax_{2m(k)}, Ax_{2n(k)}) \leq \lambda p_b(Ax_{2m(k)}, Ax_{2n(k)+1}) + \lambda p_b(Ax_{2n(k)+1}, Ax_{2n(k)})$$

and

$$\begin{aligned} 0 < \varepsilon \leq p_b(Ax_{2m(k)}, Ax_{2n(k)}) &\leq \lambda p_b(Ax_{2m(k)}, Ax_{2m(k)-1}) + \lambda p_b(Ax_{2m(k)-1}, Ax_{2n(k)}) \\ &\leq \lambda p_b(Ax_{2m(k)}, Ax_{2m(k)-1}) + \lambda^2 p_b(Ax_{2m(k)-1}, Ax_{2n(k)+1}) + \lambda^2 p_b(Ax_{2n(k)+1}, Ax_{2n(k)}). \end{aligned}$$

By means of (2.3), taking the lower limit as $k \to \infty$ in the above inequality, we get

$$\varepsilon \le \liminf_{k\to\infty} p_b(Ax_{2m(k)}, Ax_{2n(k)}) \le \limsup_{k\to\infty} p_b(Ax_{2m(k)}, Ax_{2n(k)}). \tag{2.8}$$

$$\frac{\varepsilon}{\lambda} \le \liminf_{k\to\infty} p_b(Ax_{2m(k)}, Ax_{2n(k)+1}) \le \limsup_{k\to\infty} p_b(Ax_{2m(k)}, Ax_{2n(k)+1}), \tag{2.9}$$

$$\frac{\varepsilon}{\lambda^2} \le \liminf_{k\to\infty} p_b(Ax_{2m(k)-1}, Ax_{2n(k)+1}) \le \limsup_{k\to\infty} p_b(Ax_{2m(k)-1}, Ax_{2n(k)+1}). \tag{2.10}$$

On the other hand, we have

$$p_b(Ax_{2n(k)}, Ax_{2m(k)-1}) \le \lambda p_b(Ax_{2n(k)}, Ax_{2m(k)-2}) + \lambda p_b(Ax_{2m(k)-2}, Ax_{2m(k)-1}).$$

With the help of (2.3) and (2.7) and taking the upper limit as $k \to \infty$ in the above inequality, it is not difficult to see that

$$\limsup_{k\to\infty} p_b(Ax_{2n(k)}, Ax_{2m(k)-1}) \le \lambda\varepsilon. \tag{2.11}$$

From (2.1), we have

$$\begin{aligned}
f\left(\frac{p_b(Ax_{2n(k)}, Ax_{2m(k)-1})}{\lambda^2}\right)
&= f\left(\frac{p_b(Sx_{2n(k)+1}, Tx_{2m(k)})}{\lambda^2}\right) \\
&\ge \psi\Bigg(p_b(Ax_{2n(k)+1}, Ax_{2m(k)}), p_b(Ax_{2n(k)+1}, Sx_{2n(k)+1}), \\
&\qquad p_b(Ax_{2m(k)}, Tx_{2m(k)}), \frac{p_b(Ax_{2n(k)+1}, Tx_{2m(k)})}{\lambda}, \frac{p_b(Ax_{2m(k)}, Sx_{2n(k)+1})}{\lambda}\Bigg) \\
&= \psi\Bigg(p_b(Ax_{2n(k)+1}, Ax_{2m(k)}), p_b(Ax_{2n(k)+1}, Ax_{2n(k)}), \\
&\qquad p_b(Ax_{2m(k)}, Ax_{2m(k)-1}), \frac{p_b(Ax_{2n(k)+1}, Ax_{2m(k)-1})}{\lambda}, \frac{p_b(Ax_{2m(k)}, Ax_{2n(k)})}{\lambda}\Bigg).
\end{aligned}$$

Now, taking upper limit as $k \to \infty$ in the above inequality, the properties of ψ, f, and (2.8)-(2.11) guarantee that

$$\begin{aligned}
f\left(\frac{\varepsilon}{\lambda}\right) &\ge f\left(\frac{\limsup_{k\to\infty} p_b(Ax_{2n(k)}, Ax_{2m(k)-1})}{\lambda^2}\right) \\
&= f\left(\frac{\limsup_{k\to\infty} p_b(Sx_{2n(k)+1}, Tx_{2m(k)})}{\lambda^2}\right) \\
&\ge \psi\Bigg(\liminf_{k\to\infty} p_b(Ax_{2n(k)+1}, Ax_{2m(k)}), \liminf_{k\to\infty} p_b(Ax_{2n(k)+1}, Ax_{2n(k)}), \\
&\qquad \liminf_{k\to\infty} p_b(Ax_{2m(k)}, Ax_{2m(k)-1}), \\
&\qquad \frac{\liminf_{k\to\infty} p_b(Ax_{2n(k)+1}, Ax_{2m(k)-1})}{\lambda}, \frac{\liminf_{k\to\infty} p_b(Ax_{2m(k)}, Ax_{2n(k)})}{\lambda}\Bigg),
\end{aligned}$$

which implies that $f(\frac{\varepsilon}{\lambda}) \geq \psi(\frac{\varepsilon}{\lambda}, 0, 0, \frac{\varepsilon}{\lambda^3}, \frac{\varepsilon}{\lambda})$. Thus, $\varepsilon = 0$, a contradiction. Hence, $\lim_{m,n\to\infty} p_b(Ax_n, Ax_m) = 0$, that is, $\{Ax_n\}$ is a p_b-Cauchy sequence.

Step 3. We will show that A, S, and T have a unique common fixed point.

Since $\{Ax_n\}$is a p_b-Cauchy sequence in (X, p_b), and thus it is also b-Cauchy sequence in the b-metric space (X, p_b^s) by Lemma 1.1. Since (X, p_b) is p_b-complete, from Lemma 1.1, (X, p_b^s) is also b-complete, so the sequence $\{Ax_n\}$ is b-convergent in the b-metric space (X, p_b^s). Therefore, there exists $x^* \in X$ such that $\lim_{n\to\infty} p_b^s(Ax_n, x^*) = 0$. Then $\lim_{m,n\to\infty} p_b(Ax_n, Ax_m) = \lim_{n\to\infty} p_b(Ax_n, x^*) = p_b(x^*, x^*) = 0$.

Since $A(X)$ is a closed set of (X, p_b^s), $A(X) \subset T(X)$, and $\lim_{n\to\infty} p_b^s(Ax_n, x^*) = 0$, we get $x^* \in A(X) \subset T(X)$. Hence, there exists $z_1 \in X$ such that $Tz_1 = x^*$. This together with (2.1) ensures that

$$\begin{aligned} f\left(\frac{p_b(Ax_{2n}, Tz_1)}{\lambda^2}\right) &= f\left(\frac{p_b(Sx_{2n+1}, Tz_1)}{\lambda^2}\right) \\ &\geq \psi\biggl(p_b(Ax_{2n+1}, Az_1), p_b(Ax_{2n+1}, Sx_{2n+1}), \\ &\qquad p_b(Az_1, Tz_1), \frac{p_b(Ax_{2n+1}, Tz_1)}{\lambda}, \frac{p_b(Az_1, Sx_{2n+1})}{\lambda}\biggr) \\ &= \psi\biggl(p_b(Ax_{2n+1}, Az_1), p_b(Ax_{2n+1}, Ax_{2n}), \\ &\qquad p_b(Az_1, Tz_1), \frac{p_b(Ax_{2n+1}, Tz_1)}{\lambda}, \frac{p_b(Az_1, Ax_{2n})}{\lambda}\biggr). \end{aligned} \tag{2.12}$$

Since $\lim_{n\to\infty} p_b(Ax_n, x^*) = p_b(x^*, x^*) = 0$ and $Tz_1 = x^*$, we can find by Lemma 1.1 that

$$\limsup_{n\to\infty} p_b(Ax_{2n}, Tz_1) = 0, \qquad \liminf_{n\to\infty} p_b(Ax_{2n+1}, Az_1) \geq \frac{p_b(Tz_1, Az_1)}{\lambda}, \tag{2.13}$$

$$\liminf_{n\to\infty} p_b(Ax_{2n+1}, Tz_1) = 0, \qquad \liminf_{n\to\infty} p_b(Az_1, Ax_{2n}) \geq \frac{p_b(Az_1, Tz_1)}{\lambda}. \tag{2.14}$$

Taking the upper limit as $n \to \infty$ in (2.12), using the properties of ψ and f, (2.13), and (2.14), it is clear that

$$\begin{aligned} 0 = f(0) &= f\Bigl(\limsup_{n\to\infty} p_b(Ax_{2n}, Tz_1)\Bigr) \\ &\geq \psi\left(\frac{p_b(Az_1, Tz_1)}{\lambda}, 0, p_b(Az_1, Tz_1), 0, \frac{p_b(Az_1, Tz_1)}{\lambda^2}\right), \end{aligned}$$

which implies that $p_b(Az_1, Tz_1) = 0$. Hence, $Az_1 = Tz_1 = x^*$. Similarly, since $x^* \in A(X) \subset S(X)$, there exists $z_2 \in X$ such that $Sz_2 = x^*$, we have $Az_2 = Sz_2 = x^*$. Hence, $Az_1 = Az_2 = x^*$. Since A is an injective, we get $z_1 = z_2$.

Let $z = z_1 = z_2$. Then $Az = Sz = Tz = x^*$. Since A and T are weakly compatible, it is obvious that $Ax^* = AAz = ATz = TAz = Tx^*$. By (2.1), we get

$$\begin{aligned} f\left(\frac{p_b(Az, Ax^*)}{\lambda}\right) \geq f\left(\frac{p_b(Az, Ax^*)}{\lambda^2}\right) &= f\left(\frac{p_b(Sz, Tx^*)}{\lambda^2}\right) \\ &\geq \psi\Bigl(p_b(Az, Ax^*), p_b(Az, Sz), p_b\bigl(Ax^*, Tx^*\bigr), \end{aligned}$$

$$\frac{p_b(Az, Tx^*)}{\lambda}, \frac{p_b(Ax^*, Sz)}{\lambda}\Bigg)$$
$$= \psi\left(p_b(Az, Ax^*), p_b(Az, Az), p_b(Ax^*, Ax^*),\right.$$
$$\left.\frac{p_b(Az, Ax^*)}{\lambda}, \frac{p_b(Ax^*, Az)}{\lambda}\right),$$

which implies that $p_b(Az, Ax^*) = 0$. Thus, $Az = Ax^*$. Since A is an injective, we get $z = x^*$. Hence, $Az = Sz = Tz = z$, that is, z is a common fixed point of A, S, and T.

Now, we prove the uniqueness of common fixed points of A, S, and T. Suppose that $x^*, y^* \in X$ such that $Ax^* = Sx^* = Tx^* = x^*$ and $Ay^* = Sy^* = Ty^* = y^*$. By means of (2.1), we have

$$f\left(\frac{p_b(x^*, y^*)}{\lambda}\right) \geq f\left(\frac{p_b(x^*, y^*)}{\lambda^2}\right) = f\left(\frac{p_b(Sx^*, Ty^*)}{\lambda^2}\right)$$
$$\geq \psi\left(p_b(Ax^*, Ay^*), p_b(Ax^*, Sx^*), p_b(Ay^*, Ty^*),\right.$$
$$\left.\frac{p_b(Ax^*, Ty^*)}{\lambda}, \frac{p_b(Ay^*, Sx^*)}{\lambda}\right)$$
$$= \psi\left(p_b(x^*, y^*), p_b(x^*, x^*), p_b(x^*, y^*), \frac{p_b(x^*, y^*)}{\lambda}, \frac{p_b(y^*, x^*)}{\lambda}\right),$$

which implies that $p_b(x^*, y^*) = 0$. Hence, $x^* = y^*$. This completes the proof. □

Remark 2.1 Let I be the identity mappings on X. Taking $A = I, f(t) = t$, for all $t \in R^+$ in Theorem 2.1, we have the following corollary, which extends and generalizes Theorem 2.1 in [19] and Theorem 2 in [20].

Corollary 2.1 *Let (X, p_b) be a p_b-complete partial b-metric space, S and $T : X \to X$ be two bijective mappings. Suppose that*

$$\frac{p_b(Sx, Ty)}{\lambda^2} \geq \psi\left(p_b(x, y), p_b(x, Sx), p_b(y, Ty), \frac{p_b(x, Ty)}{\lambda}, \frac{p_b(y, Sx)}{\lambda}\right),$$

for all $x, y \in X$, where $\psi \in \Psi$. Then S and T have a unique common fixed point in X.

Now, in order to support the usability of our results, we present the following example.

Example 2.2 Let $C[0,1]$ be the set of all real continuous functions defined on $[0,1]$ and $X = \{x \geq \theta : x \in C[0,1]\}$. Define a partial b-metric $p_b : X \times X \to R^+$ by

$$p_b(x, y) = \begin{cases} \max_{t\in[0,1]} |x(t)|^2, & \text{if } x = y, \\ \max_{t\in[0,1]} (x(t) + y(t))^2, & \text{otherwise.} \end{cases}$$

It is easy to see that (X, p_b) is a p_b-complete partial b-metric space with $\lambda = 3$. Let $A, S, T : X \to X$ be defined by

$$(Ax)(t) = \frac{2}{3}\int_0^t x(s)\,ds, \qquad (Sx)(t) = 4\int_0^t x(s)\,ds,$$

$$(Tx)(t) = 7\int_0^t x(s)\,ds, \quad \text{for all } x \in X.$$

Then it is easy to show that all the conditions (i)-(ii) of Theorem 2.1 are satisfied. Define $\psi : R^{+5} \to R^+$ and $f : R^+ \to R^+$ by $\psi(t_1, t_2, t_3, t_4, t_5) = \max\{\frac{6}{5}\min\{t_1, \frac{t_1+t_2}{2}\}, \frac{6}{5}\min\{t_1, t_5\}\}, f(t) = t$.

Now, we consider following cases:

Case 1. If $x = y = \theta$, then $\frac{p_b(Sx,Ty)}{9} = 0 = \frac{6}{5}p_b(Ax, Ay)$.

Case 2. If $x = y \neq \theta$, then

$$\begin{aligned}\frac{p_b(Sx, Ty)}{9} &= \frac{\max_{t\in[0,1]}(4\int_0^t x(s)\,ds + 7\int_0^t y(s)\,ds)^2}{9} \\ &= \frac{121}{9}\max_{t\in[0,1]}\left(\int_0^t x(s)\,ds\right)^2 \geq \frac{8}{15}\max_{t\in[0,1]}\left(\int_0^t x(s)\,ds\right)^2 = \frac{6}{5}p_b(Ax, Ay).\end{aligned}$$

Case 3. If $x \neq y$ and $4x = 7y$, then

$$\begin{aligned}\frac{p_b(Sx, Ty)}{9} &= \frac{16}{9}\max_{t\in[0,1]}\left(\int_0^t x(s)\,ds\right)^2 \geq \frac{8}{15}\max_{t\in[0,1]}\left(\int_0^t x(s)\,ds + \frac{4}{7}\int_0^t x(s)\,ds\right)^2 \\ &= \frac{6}{5}p_b(Ax, Ay).\end{aligned}$$

Case 4. If $x \neq y$ and $4x \neq 7y$, then

$$\begin{aligned}\frac{p_b(Sx, Ty)}{9} &= \frac{\max_{t\in[0,1]}(4\int_0^t x(s)\,ds + 7\int_0^t y(s)\,ds)^2}{9} \\ &\geq \frac{16}{9}\max_{t\in[0,1]}\left(\int_0^t x(s)\,ds + \int_0^t y(s)\,ds\right)^2 \\ &\geq \frac{8}{15}\max_{t\in[0,1]}\left(\int_0^t x(s)\,ds + \int_0^t y(s)\,ds\right)^2 = \frac{6}{5}p_b(Ax, Ay).\end{aligned}$$

That is,

$$\begin{aligned}\frac{p_b(Sx, Ty)}{\lambda^2} &\geq \frac{6}{5}p_b(Ax, Ay) \\ &\geq \psi\left(p_b(Ax, Ay), p_b(Ax, Sx), p_b(Ay, Ty), \frac{p_b(Ax, Ty)}{\lambda}, \frac{p_b(Ay, Sx)}{\lambda}\right),\end{aligned}$$

for all $x, y \in X$. Thus, all conditions of Theorem 2.1 are satisfied. Hence, A, S, and T have a unique common fixed point $x = \theta$.

Now, we state and prove some fixed point results for generalized $(\phi, g, h)_\lambda$-weakly expansive mappings in partial b-metric spaces.

Theorem 2.2 *Let (X, p_b) be a p_b-complete partial b-metric space, A, S, and $T : X \to X$ be three mappings satisfying the generalized $(\phi, g, h)_\lambda$-weakly expansive condition* (2.2). *Suppose that the following conditions are satisfied*:

(i) *$A(X) \subset S(X)$, $A(X) \subset T(X)$, and $A(X)$ is a closed subset of (X, p_b^s)*;

(ii) *A is an injective and A and T are weakly compatible.*

Then A, S, and T have a unique common fixed point in X.

Proof Let $x_0 \in X$. Repeating the proof of Theorem 2.1, we know that there exists a sequence $\{Ax_n\}$ in X such that $Ax_{2n} = Sx_{2n+1}$ and $Ax_{2n+1} = Tx_{2n+2}$, for $n = 0, 1, 2, \ldots$.

We will complete the proof in three steps.

Step 1. We prove that $\lim_{n\to\infty} p_b(Ax_n, Ax_{n+1}) = 0$.

Suppose first that $p_b(Ax_n, Ax_{n+1}) = 0$ for some $n = n_0$. Then $Ax_{n_0} = Ax_{n_0+1}$. In the case that $n_0 = 2k$, then $p_b(Ax_{2k}, Ax_{2k+1}) = 0$ gives $p_b(Ax_{2k+1}, Ax_{2k+2}) = 0$. Indeed, by (2.2), we have

$$\begin{aligned} 0 = g\left(\frac{p_b(Ax_{2k}, Ax_{2k+1})}{\lambda^2}\right) &= g\left(\frac{p_b(Sx_{2k+1}, Tx_{2k+2})}{\lambda^2}\right) \\ &\geq h\bigl(M_\lambda(Ax_{2k+1}, Ax_{2k+2})\bigr) + \phi\bigl(M_\lambda(Ax_{2k+1}, Ax_{2k+2})\bigr), \end{aligned}$$

where

$$\begin{aligned} M_\lambda(Ax_{2k+1}, Ax_{2k+2}) = \max\biggl\{ &\min\biggl\{p_b(Ax_{2k+1}, Ax_{2k+2}), \frac{0 + p_b(Ax_{2k+1}, Ax_{2k+2})}{2}\biggr\}, \\ &\min\biggl\{p_b(Ax_{2k+1}, Ax_{2k+2}), \frac{p_b(Ax_{2k}, Ax_{2k+2})}{\lambda}\biggr\}\biggr\}. \end{aligned}$$

Thus, $\phi(M_\lambda(Ax_{2k+1}, Ax_{2k+2})) = 0$, implies that $p_b(Ax_{2k+1}, Ax_{2k+2}) = 0$. Similarly, if $n_0 = 2k+1$, then $p_b(Ax_{2k+2}, Ax_{2k+3}) = 0$. Consequently, $p_b(Ax_n, Ax_{n+1}) \equiv 0$ for $n \geq n_0$. Hence, $\lim_{n\to\infty} p_b(Ax_n, Ax_{n+1}) = 0$.

Now, suppose that $p_b(Ax_n, Ax_{n+1}) > 0$, for each n. By (2.2), we have

$$\begin{aligned} g\left(\frac{p_b(Ax_{2n}, Ax_{2n+1})}{\lambda^2}\right) &= g\left(\frac{p_b(Sx_{2n+1}, Tx_{2n+2})}{\lambda^2}\right) \\ &\geq h\bigl(M_\lambda(Ax_{2n+1}, Ax_{2n+2})\bigr) + \phi\bigl(M_\lambda(Ax_{2n+1}, Ax_{2n+2})\bigr), \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} M_\lambda(Ax_{2n+1}, Ax_{2n+2}) = \max\biggl\{ &\min\biggl\{p_b(Ax_{2n+1}, Ax_{2n+2}), \\ &\frac{p_b(Ax_{2n}, Ax_{2n+1}) + p_b(Ax_{2n+1}, Ax_{2n+2})}{2}\biggr\}, \\ &\min\biggl\{p_b(Ax_{2n+1}, Ax_{2n+2}), \frac{p_b(Ax_{2n+2}, Ax_{2n})}{\lambda}\biggr\}\biggr\} > 0. \end{aligned}$$

If $\min\{p_b(Ax_{2n+1}, Ax_{2n+2}), \frac{p_b(Ax_{2n}, Ax_{2n+1}) + p_b(Ax_{2n+1}, Ax_{2n+2})}{2}\} = \frac{p_b(Ax_{2n}, Ax_{2n+1}) + p_b(Ax_{2n+1}, Ax_{2n+2})}{2} > 0$, then we have $p_b(Ax_{2n}, Ax_{2n+1}) \leq p_b(Ax_{2n+1}, Ax_{2n+2})$. It follows from (2.15) and the properties of ϕ, g, h that

$$\begin{aligned} g\bigl(p_b(Ax_{2n}, Ax_{2n+1})\bigr) &\geq g\left(\frac{p_b(Ax_{2n}, Ax_{2n+1})}{\lambda^2}\right) \\ &\geq h\bigl(M_\lambda(Ax_{2n+1}, Ax_{2n+2})\bigr) + \phi\bigl(M_\lambda(Ax_{2n+1}, Ax_{2n+2})\bigr) \\ &> g\bigl(M_\lambda(Ax_{2n+1}, Ax_{2n+2})\bigr) \\ &\geq g\left(\frac{p_b(Ax_{2n}, Ax_{2n+1}) + p_b(Ax_{2n+1}, Ax_{2n+2})}{2}\right). \end{aligned}$$

Since g is nondecreasing, we get $p_b(Ax_{2n},Ax_{2n+1}) > p_b(Ax_{2n+1},Ax_{2n+2})$, which is a contradiction. Thus, $M_\lambda(Ax_{2n+1},Ax_{2n+2}) \geq p_b(Ax_{2n+1},Ax_{2n+2})$. From (2.15), using the properties of ϕ, g, h, we have

$$g\bigl(p_b(Ax_{2n},Ax_{2n+1})\bigr) \geq g\left(\frac{p_b(Ax_{2n},Ax_{2n+1})}{\lambda^2}\right) > g\bigl(p_b(Ax_{2n+1},Ax_{2n+2})\bigr).$$

Hence, we deduce that, for each $n \in N$, $p_b(Ax_{2n+1},Ax_{2n+2}) < p_b(Ax_{2n},Ax_{2n+1})$. Similarly, we can prove that $p_b(Ax_{2n},Ax_{2n+1}) < p_b(Ax_{2n-1},Ax_{2n})$, for all $n \geq 1$. Therefore, $\{p_b(Ax_n,Ax_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. So, there exists $r \geq 0$ such that $\lim_{n\to\infty} p_b(Ax_n,Ax_{n+1}) = r$. Following the proof of Theorem 2.1, we know that the sequence $\{p_b(Ax_{2n},Ax_{2n+2})\}$ has a subsequence $\{p_b(Ax_{n_k},Ax_{n_k+2})\}$ which converges to a real number $\alpha \leq 2\lambda r$. By (2.15), we get

$$\begin{aligned} g\left(\frac{p_b(Ax_{2n_k},Ax_{2n_k+1})}{\lambda^2}\right) &= g\left(\frac{p_b(Sx_{2n_k+1},Tx_{2n_k+2})}{\lambda^2}\right) \\ &\geq h\bigl(M_\lambda(Ax_{2n_k+1},Ax_{2n_k+2})\bigr) + \phi\bigl(M_\lambda(Ax_{2n_k+1},Ax_{2n_k+2})\bigr). \end{aligned}$$

Letting $n_k \to \infty$ in the above inequality, using the properties of ϕ, g, h, we can see that

$$\begin{aligned} g(r) \geq g\left(\frac{r}{\lambda^2}\right) &\geq h\left(\max\left\{\min\{r,r\},\min\left\{r,\frac{\alpha}{\lambda}\right\}\right\}\right) + \phi\left(\max\left\{\min\{r,r\},\min\left\{r,\frac{\alpha}{\lambda}\right\}\right\}\right) \\ &\geq h(r) + \phi(r), \end{aligned}$$

which implies that $r = 0$. Therefore, $\lim_{n\to\infty} p(Ax_n,Ax_{n+1}) = 0$.

Step 2. We now show that $\{Ax_n\}$ is a p_b-Cauchy sequence.

Indeed, we first prove that $\lim_{m,n\to\infty} p_b(Ax_n,Ax_m) = 0$. Since $\lim_{n\to\infty} p_b(Ax_n,Ax_{n+1}) = 0$, it is sufficient to show that $\lim_{m,n\to\infty} p_b(Ax_{2n},Ax_{2m}) = 0$. Suppose on the contrary, then there exists $\varepsilon > 0$ for which we can find two subsequences $\{Ax_{2n(k)}\}$ and $\{Ax_{2m(k)}\}$ of $\{Ax_{2n}\}$ such that $m(k)$ is the smallest index, for which $m(k) > n(k) > k$, $p_b(Ax_{2m(k)},A_{2n(k)}) \geq \varepsilon$, for every k. This means that $p_b(Ax_{2m(k)-2},A_{2n(k)}) < \varepsilon$.

Repeating to the proof of Theorem 2.1, we also have (2.8)-(2.11). By means of (2.2), we get

$$\begin{aligned} g\left(\frac{p_b(Ax_{2n(k)},Ax_{2m(k)-1})}{\lambda^2}\right) &= g\left(\frac{p_b(Sx_{2n(k)+1},Tx_{2m(k)})}{\lambda^2}\right) \\ &\geq h\bigl(M_\lambda(Ax_{2n(k)+1},Ax_{2m(k)})\bigr) + \phi\bigl(M_\lambda(Ax_{2n(k)+1},Ax_{2m(k)})\bigr), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} M_\lambda(Ax_{2n(k)+1},Ax_{2m(k)}) = \max\Biggl\{&\min\Biggl\{p_b(Ax_{2n(k)+1},Ax_{2m(k)}), \\ &\frac{p_b(Ax_{2n(k)+1},Ax_{2n(k)}) + p_b(Ax_{2m(k)},Ax_{2m(k)-1})}{2}\Biggr\}, \\ &\min\left\{p_b(Ax_{2n(k)+1},Ax_{2m(k)}),\frac{p_b(Ax_{2m(k)},Ax_{2n(k)})}{\lambda}\right\}\Biggr\}. \end{aligned}$$

Taking the lower limit as $k \to \infty$, using (2.8), (2.9), and (2.10), it is clear that

$$\liminf_{k\to\infty} M_\lambda(Ax_{2n(k)+1}, Ax_{2m(k)}) \geq \max\left\{\min\left\{\frac{\varepsilon}{\lambda}, 0\right\}, \min\left\{\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}\right\}\right\} = \frac{\varepsilon}{\lambda}. \tag{2.17}$$

Taking the upper limit as $k \to \infty$ in (2.16), using the properties of ϕ, g, h, (2.11), and (2.17), we obtain

$$g\left(\frac{\varepsilon}{\lambda}\right) \geq g\left(\frac{\limsup_{n\to\infty} p_b(Ax_{2n(k)}, Ax_{2m(k)-1})}{\lambda^2}\right) \geq h\left(\frac{\varepsilon}{\lambda}\right) + \phi\left(\frac{\varepsilon}{\lambda}\right),$$

which implies that $\varepsilon = 0$, a contradiction. Hence, we obtain $\lim_{m,n\to\infty} p_b(Ax_n, Ax_m) = 0$, that is, $\{Ax_n\}$ is a p_b-Cauchy sequence.

Step 3. We will show that A, S, and T have a unique common fixed point.

Since $\{Ax_n\}$ is a p_b-Cauchy sequence in (X, p_b). Similar to the proof of Theorem 2.1, we know that there exists $x^* \in X$ such that $\lim_{n\to\infty} p_b^s(Ax_n, x^*) = 0$.

Since $A(X)$ is a closed set of (X, p_b^s), $A(X) \subset T(X)$, and $\lim_{n\to\infty} p_b^s(Ax_n, x^*) = 0$, we get $x^* \in A(X) \subset T(X)$. Hence, there exists $z_1 \in X$ such that $Tz_1 = x^*$. This together with (2.2) ensures that

$$\begin{aligned} g\left(\frac{p_b(Ax_{2n}, Tz_1)}{\lambda^2}\right) &= g\left(\frac{p_b(Sx_{2n+1}, Tz_1)}{\lambda^2}\right) \\ &\geq h\bigl(M_\lambda(Ax_{2n+1}, Az_1)\bigr) + \phi\bigl(M_\lambda(Ax_{2n+1}, Az_1)\bigr), \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} M_\lambda(Ax_{2n+1}, Az_1) = {}& \max\left\{\min\left\{p_b(Ax_{2n+1}, Az_1), \frac{p_b(Ax_{2n+1}, Ax_{2n}) + p_b(Az_1, Tz_1)}{2}\right\},\right. \\ &\left.\min\left\{p_b(Ax_{2n+1}, Az_1), \frac{p_b(Az_1, Ax_{2n})}{\lambda}\right\}\right\}. \end{aligned}$$

Taking the upper limit as $n \to \infty$ in (2.18), using the properties of ϕ, g, h, (2.13), and (2.14), we get

$$\begin{aligned} 0 &= g(0) \\ &\geq h\left(\max\left\{\min\left\{\frac{p_b(Az_1, Tz_1)}{\lambda}, \frac{p_b(Az_1, Tz_1)}{2}\right\}, \min\left\{\frac{p_b(Az_1, Tz_1)}{\lambda}, \frac{p_b(Az_1, Tz_1)}{\lambda^2}\right\}\right\}\right) \\ &\quad + \phi\left(\max\left\{\min\left\{\frac{p_b(Az_1, Tz_1)}{\lambda}, \frac{p_b(Az_1, Tz_1)}{2}\right\}, \min\left\{\frac{p_b(Az_1, Tz_1)}{\lambda}, \frac{p_b(Az_1, Tz_1)}{\lambda^2}\right\}\right\}\right), \end{aligned}$$

which implies that $p(Az_1, Tz_1) = 0$. Hence, $Az_1 = Tz_1 = x^*$. Similarly, since $x^* \in A(X) \subset S(X)$, there exists $z_2 \in X$ such that $Sz_2 = x^*$, we have $Az_2 = Sz_2 = x^*$. Hence, $Az_1 = Az_2 = x^*$. Since A is an injective, we get $z_1 = z_2$.

Let $z = z_1 = z_2$. Then $Az = Sz = Tz = x^*$. Since A and T are weakly compatible, it is obvious that $Ax^* = AAz = ATz = TAz = Tx^*$. Then we can find by (2.2) that

$$g\left(\frac{p_b(Az, Ax^*)}{\lambda^2}\right) = g\left(\frac{p_b(Sz, Tx^*)}{\lambda^2}\right) \geq h\bigl(M_\lambda(Az, Ax^*)\bigr) + \phi\bigl(M_\lambda(Az, Ax^*)\bigr),$$

where

$$M_\lambda(Az, Ax^*) = \max\left\{\min\left\{p_b(Az, Ax^*), \frac{p_b(Ax^*, Ax^*)}{2}\right\}, \min\left\{p_b(Az, Ax^*), \frac{p_b(Ax^*, Az)}{\lambda}\right\}\right\} \geq \frac{p_b(Ax^*, Az)}{\lambda}.$$

Thus,

$$g\left(\frac{p_b(Az, Ax^*)}{\lambda}\right) \geq g\left(\frac{p_b(Az, Ax^*)}{\lambda^2}\right) \geq h\left(\frac{p_b(Ax^*, Az)}{\lambda}\right) + \phi\left(\frac{p_b(Ax^*, Az)}{\lambda}\right),$$

which implies that $p_b(Az, Ax^*) = 0$. Thus, $Az = Ax^*$. Since A is an injective, we get $z = x^*$. Thus, $Az = Sz = Tz = z$ and z is a common fixed point of A, S, and T.

Now, we prove the uniqueness of common fixed points of A, S, and T. Suppose that $x^*, y^* \in X$ such that $Ax^* = Sx^* = Tx^* = x^*$ and $Ay^* = Sy^* = Ty^* = y^*$. By means of (2.2), we have

$$g\left(\frac{p_b(x^*, y^*)}{\lambda^2}\right) = g\left(\frac{p_b(Sx^*, Ty^*)}{\lambda^2}\right) \geq h\big(M_\lambda(Ax^*, Ay^*)\big) + \phi\big(Ax^*, Ay^*\big),$$

where

$$M_\lambda(Ax^*, Ay^*) = \max\left\{\min\left\{p_b(x^*, y^*), \frac{p_b(x^*, x^*) + p_b(y^*, y^*)}{2}\right\}, \min\left\{p_b(x^*, y^*), \frac{p_b(x^*, y^*)}{\lambda}\right\}\right\} \geq \frac{p_b(x^*, y^*)}{\lambda}.$$

Hence,

$$g\left(\frac{p_b(x^*, y^*)}{\lambda}\right) \geq g\left(\frac{p_b(x^*, y^*)}{\lambda^2}\right) \geq h\left(\frac{p_b(x^*, y^*)}{\lambda}\right) + \phi\left(\frac{p_b(x^*, y^*)}{\lambda}\right),$$

which implies that $p_b(x^*, y^*) = 0$. Hence, $x^* = y^*$. This completes the proof. □

Remark 2.2 Taking $g \equiv h$ in Theorem 2.2, we have the following corollary, which extends and generalizes Theorem 2.1 in [15] and Theorem 1 in [17].

Corollary 2.2 *Let (X, p_b) be a p_b-complete partial b-metric space, A, S, and $T : X \to X$ be three mappings. Suppose that the following conditions are satisfied*:

(i) *$A(X) \subset S(X)$, $A(X) \subset T(X)$, and $A(X)$ is a closed subset of (X, p_b^s)*;

(ii) *A is an injective and A and T are weakly compatible*;

(iii) *for all $x, y \in X$, we have*

$$g\left(\frac{p_b(Sx, Ty)}{\lambda^2}\right) \geq g\big(M_\lambda(Ax, Ay)\big) + \phi\big(M_\lambda(Ax, Ay)\big),$$

where $M_\lambda(Ax, Ay) = \max\{\min\{p_b(Ax, Ay), \frac{p_b(Ax,Sx)+p_b(Ay,Ty)}{2}\}, \min\{p_b(Ax, Ay), \frac{p_b(Ay,Sx)}{\lambda}\}\}$, ϕ, g *are the same as in Definition* 2.2.

Then A, S, and T have a unique common fixed point in X.

In the sequel, we will take an example to support our results of Theorem 2.2.

Example 2.3 Let $X = R^+$. Define a partial b-metric $p_b : X \times X \to R^+$ by

$$p_b(x,y) = \bigl(\max\{x,y\}\bigr)^2, \quad \text{for all } x,y \in X.$$

It is easy to see that (X,p_b) is a p_b-complete partial b-metric space with $\lambda = 2$. Let $A,S,T : X \to X$ be defined by

$$Ax = x, \qquad Sx = Tx = \frac{5}{2}x\sqrt{1 + \frac{1}{1+x^2}}, \quad \text{for all } x \in X.$$

Then it is easy to show that all the conditions (i)-(ii) of Theorem 2.2 are satisfied. Define $\phi,g,h : R^+ \to R^+$ by $g(t) = h(t) = t$, $\phi(t) = \frac{t}{1+t}$, for all $t \in R^+$. Without loss of generality, we assume that $x \le y$. Then

$$\begin{aligned} \frac{p_b(Sx,Ty)}{4} &= \frac{(\max\{\frac{5}{2}x\sqrt{1+\frac{1}{1+x^2}}, \frac{5}{2}y\sqrt{1+\frac{1}{1+y^2}}\})^2}{4} = \frac{(\frac{5}{2}y\sqrt{1+\frac{1}{1+y^2}})^2}{4} \\ &= \frac{25}{16}\left(y^2 + \frac{y^2}{1+y^2}\right) \ge y^2 + \frac{y^2}{1+y^2} = p_b(Ax,Ay) + \phi\bigl(p_b(Ax,Ay)\bigr). \end{aligned}$$

That is,

$$g\left(\frac{p_b(Sx,Ty)}{\lambda^2}\right) \ge h\bigl(p_b(Ax,Ay)\bigr) + \phi\bigl(p_b(Ax,Ay)\bigr) \ge h\bigl(M_\lambda(Ax,Ay)\bigr) + \phi\bigl(M_\lambda(Ax,Ay)\bigr),$$

for all $x,y \in X$, where $M_\lambda(Ax,Ay) = \max\{\min\{p_b(Ax,Ay), \frac{p_b(Ax,Sx)+p_b(Ay,Ty)}{2}\}, \min\{p_b(Ax,Ay), \frac{p_b(Ay,Sx)}{\lambda}\}\}$. Thus, all conditions of Theorem 2.2 are satisfied. Hence, A, S, and T have a unique common fixed point $x = 0$.

3 An application

In this section, we establish the existence theorem for the solutions of a class of system of integral equations.

Consider the system of integral equations

$$\begin{cases} x(t) = \int_0^T K(t,s)f_1(t,s,x(s))\,ds + x_0(t); \\ x(t) = \int_0^T K(t,s)f_2(t,s,x(s))\,ds + x_0(t), \end{cases} \tag{3.1}$$

for $t \in I = [0,T]$, where $T > 0$, $K : I^2 \to R^+$ is a continuous function and $f_1, f_2 : I^2 \times R \to R$ are also continuous functions.

Let $X = C(I,R)$ be the set of all real continuous functions defined on I. We endowed X with the partial b-metric

$$p_b(x,y) = \max_{t\in I}\bigl|x(t) - y(t)\bigr|^q + a,$$

for all $x,y \in X$, where $a \in R^+$ and $q \ge 1$. It is not difficult to prove that (X,p_b) is a p_b-complete partial b-metric space with coefficient $\lambda = 2^{q-1}$.

Now, we define S and $T : X \to X$ by

$$Sx(t) = \int_0^T K(t,s)f_1\bigl(t,s,x(s)\bigr)\,ds + x_0(t), \qquad Tx(t) = \int_0^T K(t,s)f_2\bigl(t,s,x(s)\bigr)\,ds + x_0(t),$$

for all $x \in X$. Then x is a solution of (3.1) if and only if it is a common fixed point of S and T.

We shall prove the existence of common fixed point of S and T under certain conditions.

Theorem 3.1 *Suppose that the following hypotheses hold*:

(i) *there exist a continuous function $G : I^2 \to R^+$ and $\psi \in \Psi$ such that*

$$\begin{aligned}&\frac{K(t,s)[f_1(t,s,x(s)) - f_2(t,s,y(s))] + \frac{a}{T}}{4^{q-1}} \\ &\quad\geq G(t,s)\psi\left(p_b(x,y), p_b(x,Sx), p_b(y,Ty), \frac{p_b(x,Ty)}{2^{q-1}}, \frac{p_b(y,Sx)}{2^{q-1}}\right),\end{aligned}$$

for all $t,s \in I$, where $K(t,s)f_1(t,s,x(s)) + \frac{a}{T} \geq K(t,s)f_2(t,s,y(s))$, for all $t,s \in I$.

(ii) $\inf_{t\in I}\int_0^T G(t,s)\,ds \geq 1$.

Then the system of integral equations (3.1) *has a solution $x^* \in X$.*

Proof Let $\lambda = 2^{q-1}$. From the conditions (i) and (ii), we have

$$\begin{aligned}&\frac{\max_{t\in I}|Sx(t) - Ty(t)| + a}{\lambda^2} \\ &\quad= \frac{\max_{t\in I}|\int_0^T K(t,s)f_1(t,s,x(s))\,ds - \int_0^T K(t,s)f_2(t,s,y(s))\,ds| + a}{\lambda^2} \\ &\quad\geq \frac{\max_{t\in I}|\int_0^T [K(t,s)(f_1(t,s,x(s)) - f_2(t,s,y(s))) + \frac{a}{T}]\,ds|}{\lambda^2} \\ &\quad\geq \frac{\int_0^T [K(t,s)(f_1(t,s,x(s)) - f_2(t,s,y(s))) + \frac{a}{T}]\,ds}{\lambda^2} \\ &\quad\geq \int_0^T G(t,s)\psi\left(p_b(x,y), p_b(x,Sx), p_b(y,Ty), \frac{p_b(x,Ty)}{\lambda}, \frac{p_b(y,Sx)}{\lambda}\right) ds \\ &\quad= \psi\left(p_b(x,y), p_b(x,Sx), p_b(y,Ty), \frac{p_b(x,Ty)}{\lambda}, \frac{p_b(y,Sx)}{\lambda}\right)\int_0^T G(t,s)\,ds \\ &\quad\geq \psi\left(p_b(x,y), p_b(x,Sx), p_b(y,Ty), \frac{p_b(x,Ty)}{\lambda}, \frac{p_b(y,Sx)}{\lambda}\right),\end{aligned}$$

for all $x,y \in X$. Thus, for any $x,y \in X$, we get the inequality of Corollary 2.1. Hence, all the hypotheses of Corollary 2.1 are satisfied. Then S and T have a common fixed point $x^* \in X$, that is, x^* is a solution of the system of integral equations (3.1). □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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