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Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces

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Abstract

In this paper, new classes of rational Geraghty contractive mappings in the setup of b-metric spaces are introduced. Moreover, the existence of some fixed point for such mappings in ordered b-metric spaces are investigated. Also, some examples are provided to illustrate the results presented herein. Finally, an application of the main result is given.

MSC: 47H10; 54H25

Keywords: fixed point; complete metric space; b-metric space; contractive mappings

1 Introduction

Using different forms of contractive conditions in various generalized metric spaces, there is a large number of extensions of the Banach contraction principle [1]. Some of such generalizations are obtained via rational contractive conditions. Recently, Azam *et al.* [2] established some fixed point results for a pair of rational contractive mappings in complex valued metric spaces. Also, in [3], Nashine *et al.* proved some common fixed point theorems for a pair of mappings satisfying certain rational contractions in the framework of complex valued metric spaces. In [4], the authors proved some unique fixed point results for an operator T satisfying certain rational contractive condition in a partially ordered metric space. In fact, their results generalize the main result of Jaggi [5].

Ran and Reurings started the studying of fixed point results on partially ordered sets in [6], where they gave many useful results in matrix equations. Recently, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order and obtained many fixed point results in such spaces. For more details on fixed point results in ordered metric spaces we refer the reader to [7, 8] and [9].

Czerwik in [10] introduced the concept of a b-metric space. Since then, several papers dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces (see, *e.g.*, [11–16] and [17, 18]).

Definition 1 Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric if the following conditions are satisfied:

(b_1) $d(x, y) = 0$ iff $x = y$,

(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$

for all $x, y, z \in X$.

In this case, the pair (X,d) is called a b-metric space.

Definition 2 [19] Let (X,d) be a b-metric space.
(a) A sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$.
(b) $\{x_n\}$ in X is said to be b-Cauchy if and only if $d(x_n,x_m) \to 0$, as $n,m \to \infty$.
(c) The b-metric space (X,d) is called b-complete if every b-Cauchy sequence in X is b-convergent.

The following example (corrected from [20]) illustrates that a b-metric need not be a continuous function.

Example 1 Let $X = \mathbb{N} \cup \{\infty\}$ and $d : X \times X \to \mathbb{R}$ be defined by

$$d(m,n) = \begin{cases} 0, & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m,n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m,n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then $d(m,p) \leq \frac{5}{2}(d(m,n) + d(n,p))$ for all $m,n,p \in X$. Thus, (X,d) is a b-metric space (with $s = 5/2$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. So $d(2n,\infty) = \frac{1}{2n} \to 0$ as $n \to \infty$ that is, $x_n \to \infty$, but $d(x_n,1) = 2 \nrightarrow 5 = d(\infty,1)$ as $n \to \infty$.

Lemma 1 [21] *Let (X,d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y, respectively. Then*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$$

Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq s d(x,z).$$

Let $\mathfrak{S}$ denote the class of all real functions $\beta : [0,+\infty) \to [0,1)$ satisfying the condition

$$\beta(t_n) \to 1 \quad \text{implies that} \quad t_n \to 0, \quad \text{as } n \to \infty.$$

In order to generalize the Banach contraction principle, Geraghty proved the following.

Theorem 1 [22] *Let (X,d) be a complete metric space, and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \mathfrak{S}$ such that*

$$d(fx,fy) \leq \beta\big(d(x,y)\big) d(x,y)$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z.

Amini-Harandi and Emami [23] generalized the result of Geraghty to the framework of a partially ordered complete metric space as follows.

Theorem 2 *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $f : X \to X$ be an increasing self-map such that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exists $\beta \in \mathfrak{S}$ such that*

$$d(fx, fy) \leq \beta\big(d(x,y)\big)d(x,y)$$

holds for all $x, y \in X$ with $y \preceq x$. Assume that either f is continuous or X is such that if an increasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n. Then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ comparable with x and y, then the fixed point of f is unique.

In [24], some fixed point theorems for mappings satisfying Geraghty-type contractive conditions are proved in various generalized metric spaces. As in [24], we will consider the class $\mathcal{F}$ of functions $\beta : [0, \infty) \to [0, 1/s)$ such that

$$\beta(t_n) \to \frac{1}{s} \quad \text{implies that} \quad t_n \to 0, \quad \text{as } n \to \infty.$$

Theorem 3 [24] *Let $s > 1$, and let (X, D, s) be a complete metric type space. Suppose that a mapping $f : X \to X$ satisfies the condition*

$$D(fx, fy) \leq \beta\big(D(x,y)\big)D(x,y)$$

for all $x, y \in X$ and some $\beta \in \mathcal{F}$. Then f has a unique fixed point $z \in X$, and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z in (X, D, s).

Also, by unification of the recent results obtained by Zabihi and Razani [25] we have the following result.

Theorem 4 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space (with parameter $s > 1$). Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose there exists $\beta \in \mathcal{F}$ such that*

$$sd(fx, fy) \leq \beta\big(d(x,y)\big)M(x,y) + LN(x,y) \tag{1.1}$$

for all comparable elements $x, y \in X$, where $L \geq 0$,

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1 + d(fx,fy)}\right\}$$

and

$$N(x,y) = \min\big\{d(x,fx), d(x,fy), d(y,fx), d(y,fy)\big\}.$$

If f is continuous, or, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

The aim of this paper is to present some fixed point theorems for rational Geraghty contractive mappings in partially ordered b-metric spaces. Our results extend some existing results in the literature.

2 Main results

Let $\mathcal{F}$ denotes the class of all functions $\beta : [0,\infty) \to [0,\frac{1}{s})$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \quad \text{implies that} \quad t_n \to 0, \quad \text{as } n \to \infty.$$

Definition 3 Let $(X,d,\preceq)$ be a b-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type I if there exists $\beta \in \mathcal{F}$ such that

$$d(fx,fy) \le \beta\bigl(M(x,y)\bigr)M(x,y) \tag{2.1}$$

for all comparable elements $x,y \in X$, where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1 + d(x,y)}, \frac{d(x,fx)d(y,fy)}{1 + d(fx,fy)}\right\}.$$

Theorem 5 *Let $(X,\preceq)$ be a partially ordered set and suppose there exists a b-metric d on X such that (X,d) is a b-complete b-metric space (with parameter $s > 1$). Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose f is a rational Geraghty contraction of type I. If*

(I) *f is continuous, or,*

(II) *whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,*

then f has a fixed point.

Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Let $x_n = f^n(x_0)$ for all $n \ge 0$. Since $x_0 \preceq f(x_0)$ and f is increasing, we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots.$$

We do the proof in the following steps.

Step I: We show that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (2.1)

$$\begin{aligned} d(x_n,x_{n+1}) &= d(fx_{n-1},fx_n) \\ &\le \beta\bigl(M(x_{n-1},x_n)\bigr)M(x_{n-1},x_n), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_{n-1},x_n) &= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},fx_{n-1})d(x_n,fx_n)}{1+d(x_{n-1},x_n)}, \right. \\ &\quad \left. \frac{d(x_{n-1},fx_{n-1})d(x_n,fx_n)}{1+d(fx_{n-1},fx_n)}\right\} \\ &= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}\right\} \\ &\leq \max\bigl\{d(x_{n-1},x_n), d(x_n,x_{n+1})\bigr\}. \end{aligned}$$

If $\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, then from (2.2),

$$\begin{aligned} d(x_n,x_{n+1}) &\leq \beta\bigl(M(x_n,x_{n+1})\bigr)d(x_n,x_{n+1}) \\ &< \frac{1}{s}d(x_n,x_{n+1}) \\ &< d(x_n,x_{n+1}), \end{aligned} \tag{2.3}$$

which is a contradiction.

Hence, $\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_{n-1},x_n)$, so from (2.2),

$$d(x_n,x_{n+1}) \leq \beta\bigl(M(x_{n-1},x_n)\bigr)d(x_{n-1},x_n). \tag{2.4}$$

Since $\{d(x_n,x_{n+1})\}$ is a decreasing sequence, then there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We prove $r = 0$. Suppose on contrary that $r > 0$. Then, letting $n \to \infty$, from (2.4) we have

$$r \leq \lim_{n\to\infty} \beta\bigl(M(x_{n-1},x_n)\bigr)r,$$

which implies that $\frac{1}{s} \leq 1 \leq \lim_{n\to\infty} \beta(M(x_{n-1},x_n))$. Now, as $\beta \in \mathcal{F}$ we conclude that $M(x_{n-1},x_n) \to 0$, which yields $r = 0$, a contradiction. Hence, $r = 0$. That is,

$$\lim_{n\to\infty} d(x_{n-1},x_n) = 0. \tag{2.5}$$

Step II: Now, we prove that the sequence $\{x_n\}$ is a b-Cauchy sequence. Suppose the contrary, *i.e.*, $\{x_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d(x_{m_i},x_{n_i}) \geq \varepsilon. \tag{2.6}$$

This means that

$$d(x_{m_i},x_{n_i-1}) < \varepsilon. \tag{2.7}$$

From (2.5) and using the triangular inequality, we get

$$\varepsilon \leq d(x_{m_i},x_{n_i}) \leq sd(x_{m_i},x_{m_i+1}) + sd(x_{m_i+1},x_{n_i}).$$

By taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}). \tag{2.8}$$

The definition of $M(x, y)$ and (2.8) imply

$$\begin{aligned}
&\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) \\
&\quad = \limsup_{i\to\infty} \max\left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, fx_{m_i}) d(x_{n_i-1}, fx_{n_i-1})}{1 + d(x_{m_i}, x_{n_i-1})}, \right. \\
&\qquad \left. \frac{d(x_{m_i}, fx_{m_i}) d(x_{n_i-1}, fx_{n_i-1})}{1 + d(fx_{m_i}, fx_{n_i-1})} \right\} \\
&\quad = \limsup_{i\to\infty} \max\left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, x_{m_i+1}) d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i}, x_{n_i-1})}, \right. \\
&\qquad \left. \frac{d(x_{m_i}, x_{m_i+1}) d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i+1}, x_{n_i})} \right\} \\
&\quad \leq \varepsilon.
\end{aligned}$$

Now, from (2.1) and the above inequalities, we have

$$\begin{aligned}
\frac{\varepsilon}{s} &\leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}) \\
&\leq \limsup_{i\to\infty} \beta\bigl(M(x_{m_i}, x_{n_i-1})\bigr) \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) \\
&\leq \varepsilon \limsup_{i\to\infty} \beta\bigl(M(x_{m_i}, x_{n_i-1})\bigr),
\end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}))$. Now, as $\beta \in \mathcal{F}$ we conclude that $M(x_{m_i}, x_{n_i-1}) \to 0$, which yields $d(x_{m_i}, x_{n_i-1}) \to 0$. Consequently,

$$d(x_{m_i}, x_{n_i}) \leq s d(x_{m_i}, x_{n_i-1}) + s d(x_{n_i-1}, x_{n_i}) \to 0,$$

which is a contradiction to (2.6). Therefore, $\{x_n\}$ is a b-Cauchy sequence. b-Completeness of X shows that $\{x_n\}$ b-converges to a point $u \in X$.

Step III: u is a fixed point of f.

First, let f be continuous, so we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu.$$

Now, let (II) holds. Using the assumption on X we have $x_n \preceq u$. Now, we show that $u = fu$. By Lemma 1

$$\begin{aligned}
\frac{1}{s} d(u, fu) &\leq \limsup_{n\to\infty} d(x_{n+1}, fu) \\
&\leq \limsup_{n\to\infty} \beta\bigl(M(x_n, u)\bigr) \limsup_{n\to\infty} M(x_n, u),
\end{aligned}$$

where

$$\begin{aligned}\lim_{n\to\infty} M(x_n,u) &= \lim_{n\to\infty} \max\left\{d(x_n,u), \frac{d(x_n,fx_n)d(u,fu)}{1+d(x_n,u)}, \frac{d(x_n,fx_n)d(u,fu)}{1+d(fx_n,fu)}\right\} \\ &= \max\{0,0\} \\ &= 0.\end{aligned}$$

Therefore, from the above relations, we deduce that $d(u,fu) = 0$, so $u = fu$.

Finally, suppose that the set of fixed point of f is well ordered. Assume to the contrary that u and v are two fixed points of f such that $u \neq v$. Then by (2.1),

$$d(u,v) = d(fu,fv) \le \beta\big(M(u,v)\big)M(u,v) = \beta\big(d(u,v)\big)d(u,v) < \frac{1}{s}d(u,v), \tag{2.9}$$

because

$$M(u,v) = \max\left\{d(u,v), \frac{d(u,u)d(v,v)}{1+d(u,v)}\right\} = d(u,v).$$

So we get $d(u,v) < \frac{1}{s}d(u,v)$, a contradiction. Hence $u = v$, and f has a unique fixed point. Conversely, if f has a unique fixed point, then the set of fixed points of f is a singleton, and so it is well ordered. □

Definition 4 Let (X,d) be a b-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type II if there exists $\beta \in \mathcal{F}$ such that

$$d(fx,fy) \le \beta\big(M(x,y)\big)M(x,y) \tag{2.10}$$

for all comparable elements $x, y \in X$, where

$$\begin{aligned}M(x,y) = \max\biggl\{&d(x,y), \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+s[d(x,fx)+d(y,fy)]}, \\ &\frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)}\biggr\}.\end{aligned}$$

Theorem 6 *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X,d) is a b-complete b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose f is a rational Geraghty contractive mapping of type II. If*

(I) *f is continuous, or,*

(II) *whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,*

then f has a fixed point.

Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Set $x_n = f^n(x_0)$. Since $x_0 \preceq f(x_0)$ and f is increasing, we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots.$$

We do the proof in the following steps.

Step I: We show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (2.10)

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\le \beta\big(M(x_{n-1}, x_n)\big)M(x_{n-1}, x_n) \\ &\le \beta\big(d(x_{n-1}, x_n)\big)d(x_{n-1}, x_n) \\ &< \frac{1}{s}d(x_{n-1}, x_n) \\ &\le d(x_{n-1}, x_n), \end{aligned} \tag{2.11}$$

because

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_{n-1}, fx_n) + d(x_n, fx_n)d(x_n, fx_{n-1})}{1 + s[d(x_{n-1}, fx_{n-1}) + d(x_n, fx_n)]},\right. \\ &\qquad \left.\frac{d(x_{n-1}, fx_{n-1})d(x_{n-1}, fx_n) + d(x_n, fx_n)d(x_n, fx_{n-1})}{1 + d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})}\right\} \\ &= \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})d(x_n, x_n)}{1 + s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]},\right. \\ &\qquad \left.\frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})d(x_n, x_n)}{1 + d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}\right\} \\ &= d(x_{n-1}, x_n). \end{aligned}$$

Therefore, $\{d(x_n, x_{n+1})\}$ is decreasing. Then there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We will prove that $r = 0$. Suppose to the contrary that $r > 0$. Then, letting $n \to \infty$, from (2.11)

$$\frac{1}{s}r \le \lim_{n\to\infty} \beta\big(d(x_{n-1}, x_n)\big)r,$$

which implies that $d(x_{n-1}, x_n) \to 0$. Hence, $r = 0$, a contradiction. So,

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0 \tag{2.12}$$

holds true.

Step II: Now, we prove that the sequence $\{x_n\}$ is a b-Cauchy sequence. Suppose the contrary, *i.e.*, $\{x_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d(x_{m_i}, x_{n_i}) \ge \varepsilon. \tag{2.13}$$

This means that

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{2.14}$$

As in the proof of Theorem 5, we have

$$\frac{\varepsilon}{s} \le \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}). \tag{2.15}$$

From the definition of $M(x,y)$ and the above limits,

$$\begin{aligned}
&\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) \\
&\quad = \limsup_{i\to\infty} \max\left\{ d(x_{m_i}, x_{n_i-1}), \right. \\
&\qquad \frac{d(x_{m_i}, fx_{m_i}) d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{n_i-1}) d(x_{n_i-1}, fx_{m_i})}{1 + s[d(x_{m_i}, fx_{m_i}) + d(x_{n_i-1}, fx_{n_i-1})]}, \\
&\qquad \left. \frac{d(x_{m_i}, fx_{m_i}) d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{n_i-1}) d(x_{n_i-1}, fx_{m_i})}{1 + d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{m_i})} \right\} \\
&\quad = \limsup_{i\to\infty} \max\left\{ d(x_{m_i}, x_{n_i-1}), \right. \\
&\qquad \frac{d(x_{m_i}, x_{m_i+1}) d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{n_i}) d(x_{n_i-1}, x_{m_i+1})}{1 + s[d(x_{m_i}, x_{m_i+1}) + d(x_{n_i-1}, x_{n_i})]}, \\
&\qquad \left. \frac{d(x_{m_i}, x_{m_i+1}) d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{n_i}) d(x_{n_i-1}, x_{m_i+1})}{1 + d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{m_i+1})} \right\} \\
&\quad \leq \varepsilon.
\end{aligned}$$

Now, from (2.10) and the above inequalities, we have

$$\begin{aligned}
\frac{\varepsilon}{s} &\leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}) \leq \limsup_{i\to\infty} \beta\big(M(x_{m_i}, x_{n_i-1})\big) \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) \\
&\leq \varepsilon \limsup_{i\to\infty} \beta\big(M(x_{m_i}, x_{n_i-1})\big),
\end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}))$. Now, as $\beta \in \mathcal{F}$ we conclude that $\{x_n\}$ is a b-Cauchy sequence. b-Completeness of X shows that $\{x_n\}$ b-converges to a point $u \in X$.

Step III: u is a fixed point of f.

First, let f be continuous, so we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu.$$

Now, let (II) hold. Using the assumption on X we have $x_n \preceq u$. Now, we show that $u = fu$. By Lemma 1

$$\begin{aligned}
\frac{1}{s} d(u, fu) &\leq \limsup_{n\to\infty} d(x_{n+1}, fu) \\
&\leq \limsup_{n\to\infty} \beta\big(M(x_n, u)\big) \limsup_{n\to\infty} M(x_n, u) \\
&= 0,
\end{aligned}$$

because

$$\begin{aligned}
\lim_{n\to\infty} M(x_n, u) = \lim_{n\to\infty} \max\left\{ d(x_n, u), \frac{d(x_n, fx_n) d(x_n, fu) + d(u, fu) d(u, fx_n)}{1 + s[d(x_n, fx_n) + d(u, fu)]}, \right. \\
\left. \frac{d(x_n, fx_n) d(x_n, fu) + d(u, fu) d(u, fx_n)}{1 + d(x_n, fu) + d(x_n, fu)} \right\}
\end{aligned}$$

$$= \max\{0, 0\}$$
$$= 0.$$

Therefore, $d(u, fu) = 0$, so $u = fu$. □

Definition 5 Let (X, d) be a b-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type *III* if there exists $\beta \in \mathcal{F}$ such that

$$d(fx, fy) \le \beta\big(M(x,y)\big)M(x,y) \tag{2.16}$$

for all comparable elements $x, y \in X$, where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1 + s[d(x,y) + d(x,fy) + d(y,fx)]}, \frac{d(x,fy)d(x,y)}{1 + sd(x,fx) + s^3[d(y,fx) + d(y,fy)]}\right\}.$$

Theorem 7 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose f is a rational Geraghty contractive mapping of type III. If*

(I) *f is continuous, or,*
(II) *whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,*

then f has a fixed point.

Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Set $x_n = f^n(x_0)$.

Step I: We show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (2.16)

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\le \beta\big(M(x_{n-1}, x_n)\big)M(x_{n-1}, x_n) \\ &\le \beta\big(d(x_{n-1}, x_n)\big)d(x_{n-1}, x_n) \\ &< \frac{1}{s}d(x_{n-1}, x_n) \\ &\le d(x_{n-1}, x_n), \end{aligned} \tag{2.17}$$

because

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1 + s[d(x_{n-1}, x_n) + d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})]}, \frac{d(x_{n-1}, fx_n)d(x_{n-1}, x_n)}{1 + sd(x_{n-1}, fx_{n-1}) + s^3[d(x_n, fx_{n-1}) + d(x_n, fx_n)]}\right\} \\ &= \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + s[d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]}, \right. \end{aligned}$$

$$\left.\frac{d(x_{n-1},x_{n+1})d(x_{n-1},x_n)}{1+sd(x_{n-1},x_n)+s^3[d(x_n,x_n)+d(x_n,x_{n+1})]}\right\}$$
$$\le \max\left\{d(x_{n-1},x_n),\frac{d(x_{n-1},x_n)s[d(x_n,x_{n-1})+d(x_{n-1},x_{n+1})]}{s[d(x_{n-1},x_n)+d(x_{n-1},x_{n+1})+d(x_n,x_n)]}\right\}$$
$$= d(x_{n-1},x_n).$$

Therefore, $\{d(x_n,x_{n+1})\}$ is decreasing. Similar to what we have done in Theorems 5 and 6, we have

$$\lim_{n\to\infty} d(x_{n-1},x_n)=0. \tag{2.18}$$

Step II: Now, we prove that the sequence $\{x_n\}$ is a b-Cauchy sequence. Suppose the contrary, *i.e.*, $\{x_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon>0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i>m_i>i \quad \text{and} \quad d(x_{m_i},x_{n_i})\ge\varepsilon. \tag{2.19}$$

This means that

$$d(x_{m_i},x_{n_i-1})<\varepsilon. \tag{2.20}$$

From (2.18) and using the triangular inequality, we get

$$\varepsilon\le d(x_{m_i},x_{n_i})\le sd(x_{m_i},x_{m_i+1})+sd(x_{m_i+1},x_{n_i}).$$

By taking the upper limit as $i\to\infty$, we get

$$\frac{\varepsilon}{s}\le\limsup_{i\to\infty} d(x_{m_i+1},x_{n_i}). \tag{2.21}$$

Using the triangular inequality, we have

$$d(x_{m_i},x_{n_i})\le sd(x_{m_i},x_{n_i-1})+sd(x_{n_i-1},x_{n_i}).$$

Taking the upper limit as $i\to\infty$ in the above inequality and using (2.20) we get

$$\limsup_{i\to\infty} d(x_{m_i},x_{n_i})\le\varepsilon s. \tag{2.22}$$

Again, using the triangular inequality, we have

$$d(x_{m_i},x_{n_i})\le sd(x_{m_i},x_{m_i+1})+s^2d(x_{m_i+1},x_{n_i-1})+s^2d(x_{n_i-1},x_{n_i}).$$

Taking the upper limit as $i\to\infty$ in the above inequality and using (2.20) we get

$$\limsup_{i\to\infty} d(x_{m_i+1},x_{n_i-1})\ge\frac{\varepsilon}{s^2}. \tag{2.23}$$

From the definition of $M(x,y)$ and the above limits,

$$\begin{aligned}
&\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) \\
&\quad = \limsup_{i\to\infty} \max\left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, fx_{m_i}) d(x_{n_i-1}, fx_{n_i-1})}{1 + s[d(x_{m_i}, x_{n_i-1}) + d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{m_i})]}, \right. \\
&\qquad \left. \frac{d(x_{m_i}, fx_{n_i-1}) d(x_{m_i}, x_{n_i-1})}{1 + sd(x_{m_i}, fx_{m_i}) + s^3[d(x_{n_i-1}, fx_{m_i}) + d(x_{n_i-1}, fx_{n_i-1})]} \right\} \\
&\quad = \limsup_{i\to\infty} \max\left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, x_{m_i+1}) d(x_{n_i-1}, x_{n_i})}{1 + s[d(x_{m_i}, x_{n_i-1}) + d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{m_i+1})]}, \right. \\
&\qquad \left. \frac{d(x_{m_i}, x_{n_i}) d(x_{m_i}, x_{n_i-1})}{1 + sd(x_{m_i}, x_{m_i+1}) + s^3[d(x_{n_i-1}, x_{m_i+1}) + d(x_{n_i-1}, x_{n_i})]} \right\} \\
&\quad \leq \varepsilon.
\end{aligned}$$

Now, from (2.16) and the above inequalities, we have

$$\begin{aligned}
\frac{\varepsilon}{s} &\leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}) \\
&\leq \limsup_{i\to\infty} \beta\bigl(M(x_{m_i}, x_{n_i-1})\bigr) \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) \\
&\leq \varepsilon \limsup_{i\to\infty} \beta\bigl(M(x_{m_i}, x_{n_i-1})\bigr),
\end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}))$. Now, as $\beta \in \mathcal{F}$ we conclude that $\{x_n\}$ is a b-Cauchy sequence. b-Completeness of X shows that $\{x_n\}$ b-converges to a point $u \in X$.

Step III: u is a fixed point of f.

When f is continuous, the proof is straightforward.

Now, let (II) hold. By Lemma 1

$$\begin{aligned}
\frac{1}{s} d(u, fu) &\leq \limsup_{n\to\infty} d(x_{n+1}, fu) \\
&\leq \limsup_{n\to\infty} \beta\bigl(M(x_n, u)\bigr) \limsup_{n\to\infty} M(x_n, u),
\end{aligned}$$

where

$$\begin{aligned}
\lim_{n\to\infty} M(x_n, u) &= \lim_{n\to\infty} \max\left\{ d(x_n, u), \frac{d(x_n, fx_n) d(u, fu)}{1 + s[d(x_n, u) + d(x_n, fu) + d(u, fx_n)]}, \right. \\
&\qquad \left. \frac{d(x_n, fu) d(x_n, u)}{1 + sd(x_n, fx_n) + s^3[d(u, fu) + d(u, fx_n)]} \right\} \\
&= \max\{0, 0\} \\
&= 0.
\end{aligned}$$

Therefore, from the above relations, we deduce that $d(u, fu) = 0$, so $u = fu$. □

If in the above theorems we take $\beta(t) = r$, where $0 \leq r < \frac{1}{s}$, then we have the following corollary.

Corollary 1 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$d(fx, fy) \leq rM(x, y)$$

for all comparable elements $x, y \in X$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(x, y)}, \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}\right\}$$

or

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + s[d(x, fx) + d(y, fy)]}, \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + d(x, fy) + d(y, fx)}\right\},$$

or

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + s[d(x, y) + d(x, fy) + d(y, fx)]}, \frac{d(x, fy)d(x, y)}{1 + sd(x, fx) + s^3[d(y, fx) + d(y, fy)]}\right\}.$$

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in N$, then f has a fixed point.

Corollary 2 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose*

$$d(fx, fy) \leq ad(x, y) + b\frac{d(x, fx)d(y, fy)}{1 + d(x, y)} + c\frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}$$

or

$$d(fx, fy) \leq ad(x, y) + b\frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + s[d(x, fx) + d(y, fy)]} + c\frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + d(x, fy) + d(y, fx)},$$

or

$$d(fx, fy) \leq ad(x, y) + b\frac{d(x, fx)d(y, fy)}{1 + s[d(x, y) + d(x, fy) + d(y, fx)]} + c\frac{d(x, fy)d(x, y)}{1 + sd(x, fx) + s^3[d(y, fx) + d(y, fy)]}$$

for all comparable elements $x, y \in X$, where $a, b, c \geq 0$ and $0 \leq a + b + c < \frac{1}{s}$.

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Corollary 3 *Let $(X, \preceq, d)$ be an ordered b-complete b-metric space, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f^m(x_0)$ and*

$$d\bigl(f^m x, f^m y\bigr) \le \beta\bigl(M(x,y)\bigr)M(x,y)$$

for all comparable elements $x, y \in X$, where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,f^m x)d(y,f^m y)}{1 + d(x,y)}, \frac{d(x,f^m x)d(y,f^m y)}{1 + d(f^m x, f^m y)}\right\}$$

or

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,f^m x)d(x,f^m y) + d(y,f^m y)d(y,f^m x)}{1 + s[d(x,f^m x) + d(y,f^m y)]}, \frac{d(x,f^m x)d(x,f^m y) + d(y,f^m y)d(y,f^m x)}{1 + d(x,f^m y) + d(y,f^m x)}\right\},$$

or

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,f^m x)d(y,f^m y)}{1 + s[d(x,y) + d(x,f^m y) + d(y,f^m x)]}, \frac{d(x,f^m y)d(x,y)}{1 + sd(x,f^m x) + s^3[d(y,f^m x) + d(y,f^m y)]}\right\}$$

for some positive integer m.

If f^m is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Let Ψ be the family of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{n\to\infty} \psi^n(t) = 0$$

for all $t > 0$.

Lemma 2 *If $\psi \in \Psi$, then the following are satisfied.*

(a) $\psi(t) < t$ *for all* $t > 0$;

(b) $\psi(0) = 0$.

As an example $\psi_1(t) = kt$, for all $t \ge 0$, where $k \in [0,1)$, and $\psi_2(t) = \ln(t+1)$, for all $t \ge 0$, are in Ψ.

Theorem 8 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X,d) is a b-complete b-metric space, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$sd(fx,fy) \le \psi\bigl(M(x,y)\bigr), \tag{2.24}$$

where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(x,y)}, \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\}$$

for all comparable elements $x, y \in X$. If f is continuous, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Since $x_0 \preceq f(x_0)$ and f is increasing, we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots.$$

Putting $x_n = f^n(x_0)$, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then $x_{n_0} = fx_{n_0}$ and so we have nothing to prove. Hence, we assume that $d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$.

In the following steps, we will complete the proof.

Step I: We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Using condition (2.24), we obtain

$$d(x_{n+1}, x_n) \le sd(x_{n+1}, x_n) = sd(fx_n, fx_{n-1}) \le \psi\big(M(x_n, x_{n-1})\big),$$

because

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\bigg\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1+d(x_{n-1}, x_n)}, \\ &\qquad \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1+d(fx_{n-1}, fx_n)}\bigg\} \\ &= \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1+d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1+d(x_n, x_{n+1})}\right\} \\ &\le \max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\}. \end{aligned}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\le sd(x_n, x_{n+1}) = sd(fx_{n-1}, x_n) \\ &\le \psi\big(M(x_{n-1}, x_n)\big) < M(x_{n-1}, x_n) \le d(x_n, x_{n+1}), \end{aligned} \tag{2.25}$$

which is a contradiction. Hence, $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$, so from (2.25),

$$\begin{aligned} d(x_n, x_{n+1}) &\le sd(x_n, x_{n+1}) = sd(fx_{n-1}, x_n) \\ &\le \psi\big(M(x_{n-1}, x_n)\big) < M(x_{n-1}, x_n) \le d(x_{n-1}, x_n). \end{aligned} \tag{2.26}$$

Hence,

$$d(x_n, x_{n+1}) \leq sd(x_n, x_{n+1}) \leq \psi\bigl(d(x_{n-1}, x_n)\bigr).$$

By induction,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \psi\bigl(d(x_n, x_{n-1})\bigr) \leq \psi^2\bigl(d(x_{n-1}, x_{n-2})\bigr) \\ &\leq \cdots \leq \psi^n\bigl(d(x_1, x_0)\bigr). \end{aligned} \tag{2.27}$$

As $\psi \in \Psi$, we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.28}$$

Step II: Now, we prove that the sequence $\{x_n\}$ is a b-Cauchy sequence. Suppose the contrary, *i.e.*, $\{x_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d(x_{m_i}, x_{n_i}) \geq \varepsilon. \tag{2.29}$$

This means that

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{2.30}$$

From (2.29) and using the triangular inequality, we get

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i}).$$

Taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}). \tag{2.31}$$

From the definition of $M(x, y)$ and the above limits,

$$\begin{aligned} &\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) \\ &\quad = \limsup_{i\to\infty} \max\biggl\{d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, fx_{m_i})d(x_{n_i-1}, fx_{n_i-1})}{1 + d(x_{m_i}, x_{n_i-1})}, \\ &\qquad \frac{d(x_{m_i}, fx_{m_i})d(x_{n_i-1}, fx_{n_i-1})}{1 + d(fx_{m_i}, fx_{n_i-1})}\biggr\} \\ &\quad = \limsup_{i\to\infty} \max\biggl\{d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i}, x_{n_i-1})}, \\ &\qquad \frac{d(x_{m_i}, x_{m_i+1})d(x_{n_i-1}, x_{n_i})}{1 + d(x_{m_i+1}, x_{n_i})}\biggr\} \\ &\quad \leq \varepsilon. \end{aligned}$$

Now, from (2.24) and the above inequalities, we have

$$\begin{aligned} \varepsilon = s \cdot \frac{\varepsilon}{s} &\leq s \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}) \\ &\leq \limsup_{i\to\infty} \psi\bigl(M(x_{m_i}, x_{n_i-1})\bigr) \\ &\leq \psi(\varepsilon) < \varepsilon, \end{aligned}$$

which is a contradiction. Consequently, $\{x_n\}$ is a b-Cauchy sequence. b-Completeness of X shows that $\{x_n\}$ b-converges to a point $u \in X$.

Step III: Now we show that u is a fixed point of f,

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu,$$

as f is continuous. □

Theorem 9 *Under the same hypotheses as Theorem* 8, *without the continuity assumption of f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Proof By repeating the proof of Theorem 8, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$. Using the assumption on X we have $x_n \preceq u$. Now we show that $u = fu$. By (2.24) we have

$$d(fu, x_n) = d(fu, fx_{n-1}) \leq \psi\bigl(M(u, x_{n-1})\bigr), \tag{2.32}$$

where

$$\begin{aligned} M(u, x_{n-1}) &= \max\left\{ d(u, x_{n-1}), \frac{d(u,fu)d(x_{n-1},fx_{n-1})}{1 + d(fu,fx_{n-1})}, \frac{d(u,fu)d(x_{n-1},fx_{n-1})}{1 + d(u,x_{n-1})} \right\} \\ &= \max\left\{ d(u, x_{n-1}), \frac{d(u,fu)d(x_{n-1},x_n)}{1 + d(fu,x_n)}, \frac{d(u,fu)d(x_{n-1},x_n)}{1 + d(u,x_{n-1})} \right\}. \end{aligned}$$

Letting $n \to \infty$,

$$\limsup_{n\to\infty} M(u, x_{n-1}) = 0. \tag{2.33}$$

Again, taking the upper limit as $n \to \infty$ in (2.32) and using Lemma 1 and (2.33),

$$\begin{aligned} \frac{1}{s} d(fu, u) &\leq \limsup_{n\to\infty} d(fu, x_n) \\ &\leq \limsup_{n\to\infty} \psi\bigl(M(u, x_{n-1})\bigr) \\ &= 0. \end{aligned}$$

So we get $d(fu, u) = 0$, *i.e.*, $fu = u$. □

Remark 1 In Theorems 8 and 9, we can replace $M(x,y)$ by the following:

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(x,fy) + d(y,fy)d(y,fx)}{1 + s[d(x,fx) + d(y,fy)]}, \frac{d(x,fx)d(x,fy) + d(y,fy)d(y,fx)}{1 + d(x,fy) + d(y,fx)}\right\}$$

or

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1 + s[d(x,y) + d(x,fy) + d(y,fx)]}, \frac{d(x,fy)d(x,y)}{1 + sd(x,fx) + s^3[d(y,fx) + d(y,fy)]}\right\}.$$

Example 2 Let $X = \{0,1,3\}$ and define the partial order $\preceq$ on X by

$$\preceq := \{(0,0),(1,1),(3,3),(0,3),(3,1),(0,1)\}.$$

Consider the function $f : X \to X$ given as

$$\mathbf{f} = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \end{pmatrix},$$

which is increasing with respect to $\preceq$. Let $x_0 = 0$. Hence, $f(x_0) = 3$, so $x_0 \preceq fx_0$. Define first the b-metric d on X by $d(0,1) = 6$, $d(0,3) = 9$, $d(1,3) = \frac{1}{2}$, and $d(x,x) = 0$. Then (X,d) is a b-complete b-metric space with $s = \frac{18}{13}$. Let $\beta \in \mathcal{F}$ is given by

$$\beta(t) = \frac{13}{18}e^{\frac{-t}{9}}, \quad t \geq 0$$

and $\beta(0) \in [0, \frac{13}{18})$. Then

$$d(f0,f3) = d(3,1) = \frac{1}{2} \leq \beta\bigl(M(0,3)\bigr)M(0,3) = 9\beta(9).$$

This is because

$$\begin{aligned} M(0,3) &= \max\left\{d(0,3), \frac{d(0,f0)d(3,f3)}{1 + d(f0,f3)}, \frac{d(0,f0)d(3,f3)}{1 + d(0,3)}\right\} \\ &= \max\left\{d(0,3), \frac{d(0,3)d(3,1)}{1 + d(3,1)}, \frac{d(0,3)d(3,1)}{1 + d(0,3)}\right\} = 9. \end{aligned}$$

Also,

$$d(f0,f1) = d(3,1) = \frac{1}{2} \leq \beta\bigl(M(0,1)\bigr)M(0,1) = 6\beta(6),$$

because

$$\begin{aligned} M(0,1) &= \max\left\{d(0,1), \frac{d(0,f0)d(1,f1)}{1 + d(f0,f1)}, \frac{d(0,f0)d(1,f1)}{1 + d(0,1)}\right\} \\ &= \max\left\{d(0,1), \frac{d(0,3)d(1,1)}{1 + d(3,1)}, \frac{d(0,3)d(1,1)}{1 + d(0,1)}\right\} = 6. \end{aligned}$$

Also,

$$d(f1,f3) = d(1,1) = 0 \leq \beta\big(M(1,3)\big)M(1,3).$$

Hence, f satisfies all the assumptions of Theorem 5 and thus it has a fixed point (which is $u = 1$).

Example 3 Let $X = [0,1]$ be equipped with the usual order and b-complete b-metric given by $d(x,y) = |x-y|^2$ with $s = 2$. Consider the mapping $f : X \to X$ defined by $f(x) = \frac{1}{16}x^2e^{-x^2}$ and the function β given by $\beta(t) = \frac{1}{4}$. It is easy to see that f is an increasing function and $0 \leq f(0) = 0$. For all comparable elements $x, y \in X$, by the mean value theorem, we have

$$\begin{aligned} d(fx,fy) &= \left|\frac{1}{16}x^2e^{-x^2} - \frac{1}{16}y^2e^{-y^2}\right|^2 \\ &\leq \frac{1}{8}\big|x^2e^{-x^2} - y^2e^{-y^2}\big|^2 \\ &\leq \frac{1}{8}|x-y|^2 \leq \frac{1}{4}d(x,y) = \beta\big(d(x,y)\big)d(x,y) \\ &\leq \beta\big(M(x,y)\big)M(x,y). \end{aligned}$$

So, from Theorem 5, f has a fixed point.

Example 4 Let $X = [0,1]$ be equipped with the usual order and b-complete b-metric d be given by $d(x,y) = |x-y|^2$ with $s = 2$. Consider the mapping $f : X \to X$ defined by $f(x) = \frac{1}{4}\ln(x^2+1)$ and the function $\psi \in \Psi$ given by $\psi(t) = \frac{1}{4}t$, $t \geq 0$. It is easy to see that f is increasing and $0 \leq f(0) = 0$. For all comparable elements $x, y \in X$, using the mean value problem, we have

$$\begin{aligned} d(fx,fy) &= \left|\frac{1}{4}\ln\big(x^2+1\big) - \frac{1}{4}\ln\big(y^2+1\big)\right|^2 \\ &\leq \frac{1}{4}|x-y|^2 \\ &= \frac{1}{4}d(x,y) = \psi\big(d(x,y)\big) \leq \psi\big(M(x,y)\big), \end{aligned}$$

so, using Theorem 8, f has a fixed point.

3 Application

In this section, we present an application where Theorem 8 can be applied. This application is inspired by [9] (also, see [26] and [27]).

Let $X = C([0,T])$ be the set of all real continuous functions on $[0,T]$. We first endow X with the b-metric

$$d(u,v) = \max_{t\in[0,T]}\big(\big|u(t)-v(t)\big|\big)^p$$

for all $u, v \in X$ where $p > 1$. Clearly, (X,d) is a complete b-metric space with parameter $s = 2^{p-1}$. Secondly, $C([0,T])$ can also be equipped with a partial order given by

$$x \preceq y \quad \text{iff} \quad x(t) \leq y(t) \quad \text{for all } t \in [0,T].$$

Moreover, as in [9] it is proved that $(C([0,T]),\preceq)$ is regular, that is, whenever $\{x_n\}$ in X is an increasing sequence such that $x_n \to x$ as $n \to \infty$, we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Consider the first-order periodic boundary value problem

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x(T), \end{cases} \tag{3.1}$$

where $t \in I = [0,T]$ with $T > 0$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

A lower solution for (3.1) is a function $\alpha \in C^1[0,T]$ such that

$$\begin{cases} \alpha'(t) \leq f(t, \alpha(t)), \\ \alpha(0) \leq \alpha(T), \end{cases} \tag{3.2}$$

where $t \in I = [0,T]$.

Assume that there exists $\lambda > 0$ such that for all $x, y \in X$ we have

$$\bigl|f\bigl(t,x(t)\bigr) + \lambda x(t) - f\bigl(t,y(t)\bigr) - \lambda y(t)\bigr| \leq \frac{\lambda}{2^{p-1}} \sqrt[p]{\ln\bigl(\bigl|x(t) - y(t)\bigr|^p + 1\bigr)}. \tag{3.3}$$

Then the existence of a lower solution for (3.1) provides the existence of an unique solution of (3.1).

Problem (3.1) can be rewritten as

$$\begin{cases} x'(t) + \lambda x(t) = f(t, x(t)) + \lambda x(t), \\ x(0) = x(T). \end{cases}$$

Consider

$$\begin{cases} x'(t) + \lambda x(t) = \delta(t) = F(t, x(t)), \\ x(0) = x(T), \end{cases}$$

where $t \in I$.

Using the variation of parameters formula, we get

$$x(t) = x(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \delta(s)\, ds, \tag{3.4}$$

which yields

$$x(T) = x(0)e^{-\lambda T} + \int_0^T e^{-\lambda(T-s)} \delta(s)\, ds.$$

Since $x(0) = x(T)$, we get

$$x(0)\bigl[1 - e^{-\lambda T}\bigr] = e^{-\lambda T} \int_0^T e^{\lambda(s)} \delta(s)\, ds$$

or

$$x(0) = \frac{1}{e^{\lambda T} - 1} \int_0^T e^{\lambda s} \delta(s)\, ds.$$

Substituting the value of $x(0)$ in (3.4) we arrive at

$$x(t) = \int_0^T G(t,s)\delta(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \le s \le t \le T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \le t \le s \le T. \end{cases}$$

Now define the operator $S : C[0,T] \to C[0,T]$ by

$$Sx(t) = \int_0^T G(t,s)F\bigl(s,x(s)\bigr)\,ds.$$

The mapping S is nondecreasing [26]. Note that if $u \in C[0,T]$ is a fixed point of S then $u \in C^1[0,T]$ is a solution of (3.1).

Let $x, y \in X$. Then we have

$$\begin{aligned} 2^{p-1}\bigl|Sx(t) - Sy(t)\bigr| &= 2^{p-1}\left|\int_0^T G(t,s)F\bigl(s,x(s)\bigr)\,ds - \int_0^T G(t,s)F\bigl(s,y(s)\bigr)\,ds\right| \\ &\le 2^{p-1}\int_0^T \bigl|G(t,s)\bigr|\bigl[\bigl|F\bigl(s,x(s)\bigr) - F\bigl(s,y(s)\bigr)\bigr|\bigr]\,ds \\ &\le 2^{p-1}\int_0^T \bigl|G(t,s)\bigr|\frac{\lambda}{2^{p-1}}\sqrt[p]{\ln\bigl(\bigl|x(t)-y(t)\bigr|^p + 1\bigr)}\,ds \\ &\le \lambda\sqrt[p]{\ln\bigl(d(x,y)+1\bigr)}\left[\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}\,ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}\,ds\right] \\ &= \lambda\sqrt[p]{\ln\bigl(d(x,y)+1\bigr)}\left[\frac{1}{\lambda(e^{\lambda T}-1)}\bigl(e^{\lambda(T+s-t)}|_0^t + e^{\lambda(s-t)}|_t^T\bigr)\right] \\ &= \lambda\sqrt[p]{\ln\bigl(d(x,y)+1\bigr)}\left[\frac{1}{\lambda(e^{\lambda T}-1)}\bigl(e^{\lambda T} - e^{\lambda(T-t)} + e^{\lambda(T-t)} - 1\bigr)\right] \\ &= \sqrt[p]{\ln\bigl(d(x,y)+1\bigr)} \\ &\le \sqrt[p]{\ln\bigl(M(x,y)+1\bigr)}, \end{aligned}$$

or, equivalently,

$$2^{p-1}\bigl(\bigl|Sx(t) - Sy(t)\bigr|\bigr)^p \le \ln\bigl(M(x,y)+1\bigr),$$

which shows that

$$2^{p-1}d(Sx,Sy) \le \ln\bigl(M(x,y)+1\bigr),$$

where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)d(y,Sy)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Sy)}{1+d(Sx,Sy)}\right\}$$

or

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)d(x,Sy) + d(y,Sy)d(y,Sx)}{1 + 2^{p-1}[d(x,Sx) + d(y,Sy)]}, \frac{d(x,Sx)d(x,Sy) + d(y,Sy)d(y,Sx)}{1 + d(x,Sy) + d(y,Sx)}\right\},$$

or

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)d(y,Sy)}{1 + 2^{p-1}[d(x,y) + d(x,Sy) + d(y,Sx)]}, \frac{d(x,Sy)d(x,y)}{1 + 2^{p-1}d(x,Sx) + 2^{3p-3}[d(y,Sx) + d(y,Sy)]}\right\}.$$

Finally, let α be a lower solution for (3.1). In [26] it was shown that $\alpha \preceq S(\alpha)$.

Hence, the hypotheses of Theorem 8 are satisfied with $\psi(t) = \ln(t+1)$. Therefore, there exists a fixed point $\hat{x} \in C[0,T]$ such that $S\hat{x} = \hat{x}$.

Remark 2 In the above theorem, we can replace (3.3) by the following inequality:

$$\left|f\bigl(t,x(t)\bigr) + \lambda x(t) - f\bigl(t,y(t)\bigr) - \lambda y(t)\right| \leq \frac{\lambda}{2^{\frac{p^2-1}{p}}} \sqrt[p]{e^{-M(x,y)}M(x,y)} \qquad (3.5)$$

for all $x \neq y \in X$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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