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Stability of the Young and Hölder inequalities

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Abstract
We give a simple proof of the Aldaz stability version of the Young and Hölder inequalities and further refinements of available stability versions of those inequalities.
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1 Introduction

In this paper, we study the Young and Hölder inequalities from the point of view of the deviation from equalities with better upper and lower bound estimates. Particularly, we give a further refinement of Aldaz stability type inequalities [1] as well as a simple proof based exclusively on an algebraic argument with the standard Young inequality.

Throughout this paper, the following remainder function [2] plays an important role:

$$R(\theta; a, b) = \theta a + (1-\theta)b - a^{\theta} b^{1-\theta}, \tag{1.1}$$

where $a, b > 0$ and $0 \le \theta \le 1$.

The standard Young inequality is described as

$$R(\theta; a, b) \ge 0, \tag{1.2}$$

which may be used without particular comments. The standard Hölder inequality follows from (1.2) and the equality

$$\int_\Omega |fg| \, d\mu = \|f\|_p \|g\|_{p'} \left(1 - \int_\Omega R\left(\frac{1}{p}; \frac{|f|^p}{\|f\|_p^p}, \frac{|g|^{p'}}{\|g\|_{p'}^{p'}}\right) d\mu\right) \tag{1.3}$$

for all $f \in L^p(\Omega, \mu) \setminus \{0\}$ and $g \in L^{p'}(\Omega, \mu) \setminus \{0\}$, where $L^q(\Omega, \mu)$ is the Banach space of qth integrable functions on a measure space (Ω, μ) with the norm $\| \cdot \|_q$, $1 < q < \infty$, and p' is the dual exponent of p defined by $1/p + 1/p' = 1$.

The purpose in this paper is to give a clear understanding of the standard Young and Hölder inequalities on the basis of upper and lower bound estimates on the remainder function $R(\theta; a, b)$. In Section 2, we reexamine the multiplication formula on $R(\theta; a, b)$ [2] and present its dual formula. As a corollary, we give an algebraic proof of Aldaz stability type inequalities for the Young and Hölder inequalities [1]. In Section 3, we compare upper and lower bound estimates on $R(\theta; a, b)$ in [1–4]. In Section 4, we give dyadic refinements of the multiplication formulae on $R(\theta; a, b)$ with their straightforward corollaries on (1.3) and discuss the associated dyadic refinements of the Hölder inequality.

There are many papers on the related subjects. We refer the reader to [1–7] and the references therein.

We close the introduction by giving some notation to be used in this paper. For $a, b \in \mathbb{R}$ we denote by $a \wedge b$ and $a \vee b$ their minimum and maximum, respectively.

2 Multiplication formulae

In this section, we revisit the original multiplication formula on $R(\theta; a, b)$ [2] in connection with Aldaz stability type inequalities [1]. First of all, we recall Kichenassamy's multiplication formula.

Proposition 2.1 (Kichenassamy [2]) *Let θ and σ satisfy $0 \le \theta, \sigma \le 1$. Then the equality*

$$R(\theta\sigma; a, b) = \theta R(\sigma; a, b) + b^{1-\sigma} R\bigl(\theta; a^{\sigma}, b^{\sigma}\bigr) \tag{2.1}$$

holds for all $a, b > 0$.

Proof The proposition follows from the equality

$$\begin{aligned} R(\sigma\theta, a, b) &= \sigma\theta a + (1 - \theta\sigma)b - a^{\sigma\theta} b^{1-\sigma\theta} \\ &= \theta\bigl(\sigma a + (1 - \sigma)b - a^{\sigma} b^{1-\sigma}\bigr) + \theta a^{\sigma} b^{1-\sigma} + (1 - \theta)b - a^{\sigma\theta} b^{1-\sigma\theta} \\ &= \theta R(\sigma, a, b) + b^{1-\sigma} \bigl(\theta a^{\sigma} + (1 - \theta)b^{\sigma} - a^{\sigma\theta} b^{\sigma(1-\theta)}\bigr). \end{aligned}$$

□

Corollary 2.2 *Let θ and σ satisfy $0 < \theta \le \sigma < 1$. Then the equality*

$$R(\theta; a, b) = \frac{\theta}{\sigma} R(\sigma; a, b) + b^{1-\sigma} R\left(\frac{\theta}{\sigma}; a^{\sigma}, b^{\sigma}\right) \tag{2.2}$$

holds for all $a, b > 0$.

Proposition 2.3 *Let θ and σ satisfy $0 \le \theta \le \sigma < 1$. Then the equality*

$$R(\sigma, a, b) = \frac{1 - \sigma}{1 - \theta} R(\theta, a, b) + a^{\theta} R\left(\frac{\sigma - \theta}{1 - \theta}, a^{1-\theta}, b^{1-\theta}\right) \tag{2.3}$$

holds for all $a, b > 0$.

Remark 2.1 Equality (2.3) is regarded as a dual formula for $R(\theta; a, b)$ in the sense that $\frac{1-\sigma}{1-\theta} + \frac{\sigma-\theta}{1-\theta} = 1$.

Proof of Proposition 2.3

$$\begin{aligned} &R(\sigma, a, b) \\ &\quad = \sigma a + (1 - \sigma)b - a^{\sigma} b^{1-\sigma} \\ &\quad = \frac{1 - \sigma}{1 - \theta}\bigl(\theta a + (1 - \theta)b - a^{\theta} b^{1-\theta}\bigr) + \frac{\sigma - \theta}{1 - \theta} a + \frac{1 - \sigma}{1 - \theta} a^{\theta} b^{1-\theta} - a^{\sigma} b^{1-\sigma} \\ &\quad = \frac{1 - \sigma}{1 - \theta} R(\theta, a, b) + a^{\theta} \left(\frac{\sigma - \theta}{1 - \theta} a^{1-\theta} + \frac{1 - \sigma}{1 - \theta} b^{1-\theta} - a^{\sigma-\theta} b^{(1-\theta)(1-\frac{\sigma-\theta}{1-\theta})}\right). \end{aligned}$$

□

Corollary 2.4 *Let θ and σ satisfy $0 \le \theta \le \sigma < 1$. Then the equality*

$$R(\theta;a,b) = \frac{1-\theta}{1-\sigma}R(\sigma;a,b) - \frac{1-\theta}{1-\sigma}a^{\theta}R\left(\frac{\sigma-\theta}{1-\theta};a^{1-\theta},b^{1-\theta}\right) \tag{2.4}$$

holds for all $a,b>0$.

Remark 2.2 Propositions 2.1 and 2.3 are equivalent. In fact, it follows from the reciprocal formula $R(\theta;a,b) = R(1-\theta,b,a)$ and Proposition 2.1 that

$$\begin{aligned} R(\sigma;a,b) &= R(1-\sigma;b,a) \\ &= \frac{1-\sigma}{1-\theta}R(1-\theta;b,a) + a^{1-(1-\theta)}R\left(\frac{1-\sigma}{1-\theta};b^{1-\theta},a^{1-\theta}\right) \\ &= \frac{1-\sigma}{1-\theta}R(\theta;a,b) + a^{\theta}R\left(\frac{\sigma-\theta}{1-\theta};a^{1-\theta},b^{1-\theta}\right), \end{aligned}$$

which is precisely (2.3). Conversely, given θ and σ with $0<\theta\le 1$, $0<\sigma\le 1$, we put $\theta' = 1-\theta\sigma$ and $\sigma' = 1-\sigma$. Then we have $0 \le \sigma' \le \theta' < 1$, $\sigma = 1-\sigma'$, $\theta = (1-\theta')/(1-\sigma')$, and $\theta\sigma = 1-\theta'$. By the reciprocal formula and Proposition 2.3, we have

$$\begin{aligned} R(\theta\sigma;a,b) &= R\bigl(1-\theta';a,b\bigr) = R\bigl(\theta';b,a\bigr) \\ &= \frac{1-\theta'}{1-\sigma'}R\bigl(\sigma';b,a\bigr) + b^{\sigma'}R\left(\frac{\theta'-\sigma'}{1-\sigma'};b^{1-\sigma'},a^{1-\sigma'}\right) \\ &= \frac{1-\theta'}{1-\sigma'}R\bigl(1-\sigma';a,b\bigr) + b^{\sigma'}R\left(\frac{1-\theta'}{1-\sigma'};a^{1-\sigma'},b^{1-\sigma'}\right) \\ &= \theta R(\sigma;a,b) + b^{1-\sigma}R\bigl(\theta;a^{\sigma},b^{\sigma}\bigr), \end{aligned}$$

which is precisely (2.1).

Proposition 2.5 (Aldaz [1], Kichenassamy [2]) *Let $0 \le \theta \le 1$. Then the inequalities*

$$\bigl(\theta \wedge (1-\theta)\bigr)\bigl(a^{1/2}-b^{1/2}\bigr)^2 \le R(\theta;a,b) \le \bigl(\theta \vee (1-\theta)\bigr)\bigl(a^{1/2}-b^{1/2}\bigr)^2 \tag{2.5}$$

hold for all $a,b>0$.

Proof Though the first inequality of (2.5) is shown in [2], we show the inequalities in (2.5) for completeness. In the case $0 \le \theta \le 1/2$, we use Corollaries 2.2 and 2.4 with $\sigma = 1/2$ to obtain

$$\begin{aligned} \theta\bigl(a^{1/2}-b^{1/2}\bigr)^2 &= 2\theta R(1/2;a,b) = R(\theta;a,b) - b^{1/2}R(2\theta;a,b) \\ &\le R(\theta;a,b) \\ &= 2(1-\theta)R(1/2;a,b) - 2(1-\theta)a^{\theta}R\left(\frac{1/2-\theta}{1-\theta};a^{1-\theta},b^{1-\theta}\right) \\ &\le 2(1-\theta)R(1/2;a,b) = (1-\theta)\bigl(a^{1/2}-b^{1/2}\bigr)^2. \end{aligned} \tag{2.6}$$

In the case $1/2 \leq \theta \leq 1$, we apply (2.6) with θ replaced by $1-\theta$ to obtain

$$2(1-\theta)R(1/2;b,a) \leq R(1-\theta;b,a) \leq 2\theta R(1/2;b,a),$$

which is precisely (2.5). □

Remark 2.3 An equivalent couple of inequalities in Proposition 2.5 were proved by Aldaz [1] by differential calculus. The proof above depends on algebraic identities with the standard Young inequality.

3 Upper and lower bounds of the remainder function

In this section, we collect and compare several bounds of the remainder function $R(\theta;a,b)$. For that purpose, we study the upper and lower bound estimates in terms of majorant $M(\theta;a,b)$ and minorant $m(\theta;a,b)$ in the form

$$m(\theta;a,b) \leq R(\theta;a,b) \leq M(\theta;a,b)$$

for all $a,b>0$. We introduce four couples of the bounds as follows:

$$\begin{aligned}
&[\mathrm{A}] \quad m_A(\theta;a,b) = \bigl(\theta \wedge (1-\theta)\bigr)\bigl(a^{1/2}-b^{1/2}\bigr)^2, \\
&\phantom{[\mathrm{A}]} \quad M_A(\theta;a,b) = \bigl(\theta \vee (1-\theta)\bigr)\bigl(a^{1/2}-b^{1/2}\bigr)^2, \\
&[\mathrm{K}] \quad m_K(\theta;a,b) = \frac{\theta(1-\theta)}{2}(a \wedge b)(\log a - \log b)^2, \\
&\phantom{[\mathrm{K}]} \quad M_K(\theta;a,b) = \frac{\theta(1-\theta)}{2}(a \vee b)(\log a - \log b)^2, \\
&[\mathrm{H}] \quad m_H(\theta;a,b) = \bigl(\theta \wedge (1-\theta)\bigr)\bigl|a^{\theta\wedge(1-\theta)} - b^{\theta\wedge(1-\theta)}\bigr|^{1/(\theta\wedge(1-\theta))}, \\
&\phantom{[\mathrm{H}]} \quad M_H(\theta;a,b) = \bigl(\theta \vee (1-\theta)\bigr)\bigl|a^{\theta\vee(1-\theta)} - b^{\theta\vee(1-\theta)}\bigr|^{1/(\theta\vee(1-\theta))}, \\
&[\mathrm{FO}] \quad m_{FO}(\theta;a,b) = \frac{\theta(1-\theta)}{2(a \vee b)}(a-b)^2, \\
&\phantom{[\mathrm{FO}]} \quad M_{FO}(\theta;a,b) = \frac{\theta(1-\theta)}{2(a \wedge b)}(a-b)^2.
\end{aligned}$$

Those couples are given respectively in [1, 2, 4], and [3].

Remark 3.1 By the monotonicity property suggested in [2], the remainder function with respect to $\theta \in [0,1]$ is approximated arbitrarily precisely by the remainder functions with respect to rationals which approximate θ. However, the approximation obtained by the monotonicity property is rather involved. Here, we focus only on lower and upper bounds with regard to a difference.

Simple relationships in those couples are summarized in the following.

Proposition 3.1 *Let* $0 \leq \theta \leq 1$. *Then the inequalities*

$$m_H(\theta;a,b) \leq m_A(\theta;a,b) \leq R(\theta,a,b) \leq M_A(\theta;a,b) \leq M_H(\theta,a,b), \tag{3.1}$$

$$m_K(\theta;a,b) \le m_{FO}(\theta;a,b) \le R(\theta,a,b) \le M_K(\theta;a,b) \le M_{FO}(\theta,a,b) \tag{3.2}$$

hold for all $a,b>0$.

Proof By homogeneity, (3.1) follows from the inequality

$$\left(x^\theta - 1\right)^{1/\theta} \le \left(x^\sigma - 1\right)^{1/\sigma} \tag{3.3}$$

for all $x \ge 1$ and any θ and σ with $0 \le \theta \le \sigma$. Inequality (3.3) follows from

$$x^\theta = \left(x^\sigma - 1 + 1\right)^{\theta/\sigma} \le \left(x^\sigma - 1\right)^{\theta/\sigma} + 1.$$

Although some inequalities in (3.2) are proved in [4, 8], we prove (3.2) for completeness. By the integral representations [4, 8]

$$\begin{aligned} R(\theta;a,b) &= \theta(1-\theta)\left[\int_0^1 \int_0^t \left(ta+(1-t)b\right)^{\theta-1}\left(sa+(1-s)b\right)^{-\theta}\,ds\,dt\right](a-b)^2 \\ &= \left[\int_0^1 \left(\left(\theta(1-t)\right)\wedge\left((1-\theta)t\right)\right)a^t b^{1-t}\,dt\right](\log a - \log b)^2, \end{aligned}$$

we have

$$\begin{aligned} &m_{FO}(\theta;a,b) \le R(\theta;a,b) \le M_{FO}(\theta;a,b), \\ &m_K(\theta;a,b) \le R(\theta;a,b) \le M_K(\theta;a,b). \end{aligned}$$

Then it suffices to prove that

$$\begin{aligned} &m_K(\theta;a,b) \le m_{FO}(\theta;a,b), \\ &M_K(\theta;a,b) \le M_{FO}(\theta;a,b). \end{aligned}$$

The last two inequalities are equivalent and follow from

$$x(\log x)^2 \le (x-1)^2$$

for all $x>0$. □

Proposition 3.2 *Let* $0<\theta<1$ *and let*

$$t_0(\theta) = \left(\sqrt{\frac{2}{(\theta \vee (1-\theta))}} - 1\right)^2.$$

Then the following inequalities hold for all $a,b>0$:

$$m_A(\theta,a,b) \le m_{FO}(\theta,a,b) \quad \textit{if } (a\vee b)\,t_0(\theta) \le a \wedge b, \tag{3.4}$$

$$m_A(\theta,a,b) \ge m_{FO}(\theta,a,b) \quad \textit{if } 0 < a\wedge b \le (a\vee b)\,t_0(\theta). \tag{3.5}$$

Remark 3.2 Since $0 < \theta \wedge (1-\theta) \leq 1/2 \leq \theta \vee (1-\theta) < 1$, $t_0(\theta)$ satisfies

$$(\sqrt{2}-1)^2 < t_0(\theta) \leq 1$$

for all θ with $0 \leq \theta \leq 1$. Proposition 3.2 shows that $m_{FO}(\theta; a, b)$ is better than $m_A(\theta; a, b)$ in a neighborhood of the diagonal $a = b$ in the quarter plane $(0, \infty) \times (0, \infty)$.

Proof of Proposition 3.2 It is sufficient to show inequalities (3.4) and (3.5) with $0 < a < b$. We have

$$\lim_{a \to 0} m_A(\theta, a, b) = \bigl(\theta \wedge (1-\theta)\bigr) b \geq \lim_{a \to 0} m_{FO}(\theta, a, b) = \frac{\theta(1-\theta)}{2} b,$$
$$\lim_{a \to b} \frac{m_A(\theta, a, b)}{(a^{1/2} - b^{1/2})^2} = \theta \wedge (1-\theta) \leq \lim_{a \to b} \frac{m_{FO}(\theta, a, b)}{(a^{1/2} - b^{1/2})^2} = 2\theta(1-\theta).$$

Moreover, $m_A(\theta, a, b) = m_{FO}(\theta, a, b)$ is equivalent to the equation

$$\bigl((a/b)^{1/2} + 1\bigr)^2 = \frac{2(\theta \wedge (1-\theta))}{\theta(1-\theta)}. \tag{3.6}$$

Since the ratio of a/b satisfying (3.6) with given θ is uniquely determined, inequalities (3.4) and (3.5) follow. □

To compare M_A and M_K, we prepare Lambert's W function, which is defined as the inverse function of $[-1, \infty) \ni x \mapsto xe^{1/x} \in [-1/e, \infty)$. For details, see [8].

Proposition 3.3 *Let* $0 \leq \theta \leq 1$ *and let*

$$t_1(\theta) = -\sqrt{2\bigl(\theta \wedge (1-\theta)\bigr)}\, W\left(\frac{-1}{\sqrt{2(\theta \wedge (1-\theta))}} \exp\left(\frac{-1}{\sqrt{2(\theta \wedge (1-\theta))}}\right)\right),$$

where $t_1(0)$ *and* $t_1(1)$ *are understood to be*

$$\lim_{\theta \downarrow 0} t_1(\theta) = \lim_{\theta \uparrow 1} t_1(\theta) = 1.$$

Then the following inequalities hold for any $a, b > 0$:

$$M_A(\theta; a, b) \leq M_K(\theta; a, b) \quad \textit{if } (a \wedge b) \leq t_1(\theta)(a \vee b), \tag{3.7}$$
$$M_A(\theta; a, b) \geq M_K(\theta; a, b) \quad \textit{if } (a \wedge b) \geq t_1(\theta)(a \vee b). \tag{3.8}$$

Remark 3.3 Since $0 \leq \theta \wedge (1-\theta) \leq 1/2$, $t_1(\theta)$ satisfies $0 \leq t_1(\theta) \leq 1$ for $0 \leq \theta \leq 1$. In the proof below, we see that $0 < t_1(\theta) < 1$ if $0 < \theta < 1$. Proposition 3.3 shows that $M_K(\theta; a, b)$ is better than $M_A(\theta)$ in a neighborhood of the diagonal $a = b$ in the quarter plane $(0, \infty) \times (0, \infty)$.

Proof of Proposition 3.3 Let $t > 0$ satisfy $t^2 = (a \wedge b)/(a \vee b)$. The magnitude correlation of $M_A(\theta, a, b)$ and $M_K(\theta; a, b)$ coincides with that of

$$\sqrt{(a \vee b)^{-1} M_A(\theta, a, b)} = 1 - t$$

Table 1 The signs at the important values of *x*

t	0	$\cdots$	$t_1(\theta)$	$\cdots$	$\sqrt{2(\theta \wedge (\theta - 1))}$	$\cdots$	1
$f'(t)$	∞	$+$	$+$	$+$	0	$-$	$-$
$f(t)$	$-\infty$	$\nearrow$	0	$\nearrow$	$+$	$\searrow$	0

and

$$\sqrt{(a \vee b)^{-1} M_A(\theta, a, b)} = -\sqrt{2\bigl(\theta \wedge (1-\theta)\bigr)} \log t.$$

Let $f(t) = 1 - t + \sqrt{2(\theta \wedge (1-\theta))} \log(t)$. We have $f(t_1(\theta)) = 0$ since

$$\begin{aligned} &\frac{-t_1(\theta)}{\sqrt{2(\theta \wedge (1-\theta))}} \exp\left(\frac{-t_1(\theta)}{\sqrt{2(\theta \wedge (1-\theta))}}\right) \\ &\quad = \frac{-1}{\sqrt{2(\theta \wedge (1-\theta))}} \exp\left(\frac{-1}{\sqrt{2(\theta \wedge (1-\theta))}}\right), \end{aligned}$$

which is rewritten as

$$\exp\left(\frac{1 - t_1(\theta)}{\sqrt{2(\theta \wedge (1-\theta))}}\right) = t_1(\theta)^{-1},$$

and, moreover,

$$1 - t_1(\theta) = -\sqrt{2\bigl(\theta \wedge (1-\theta)\bigr)} \log\bigl(t_1(\theta)\bigr).$$

In addition,

$$f'(t) = -1 + \sqrt{2\bigl(\theta \wedge (1-\theta)\bigr)}/t.$$

Then inequalities (3.7) and (3.8) follow from the Table 1. □

4 Dyadic refinements of multiplication formulae and their applications

In this section, we give dyadic refinements of the multiplication and dual multiplication formulae on the remainder function $R(\theta; a, b)$ and their applications. By the reciprocal formula $R(\theta; a, b) = R(1-\theta; b, a)$, it is important to describe the formation of the remainder function as $\theta \to 0$ and $\theta \to 1/2$ with the principal terms $2\theta R(1/2; a, b)$ and $2(1-\theta) R(1/2; a, b)$. For that purpose, we utilize dyadic decomposition.

Proposition 4.1 *Let* θ *satisfy* $0 < \theta \le 2^{-n}$ *with an integer* $n \ge 1$. *Then the equality*

$$R(\theta, a, b) = \theta \sum_{j=1}^{n} 2^{j-1} b^{1-2^{1-j}} \bigl(a^{2^{-j}} - b^{2^{-j}}\bigr)^2 + b^{1-2^{-n}} R\bigl(2^n \theta, a^{2^{-n}}, b^{2^{-n}}\bigr) \tag{4.1}$$

holds for all $a, b > 0$.

Proof We apply Corollary 2.2 with $\sigma = 1/2$ to obtain

$$R(\theta; a, b) = \theta\bigl(a^{1/2} - b^{1/2}\bigr) + b^{1/2} R\bigl(2\theta; a^{1/2}, b^{1/2}\bigr),$$

$$
\begin{aligned}
&b^{1-2^{-j}} R\bigl(2^j\theta; a^{2^{-j}}, b^{2^{-j}}\bigr) \\
&\quad = b^{1-2^{-j}} \bigl(2^{j+1}\theta R\bigl(1/2; a^{2^{-j}}, b^{2^{-j}}\bigr) + b^{2^{-j-1}} R\bigl(2^{j+1}\theta; a^{2^{-j-1}}, b^{2^{-j-1}}\bigr)\bigr) \\
&\quad = 2^j\theta b^{1-2^{-j}} \bigl(a^{2^{-j}} - b^{2^{-j}}\bigr)^2 + b^{1-2^{-j-1}} R\bigl(2^{j+1}\theta; a^{2^{-j-1}}, b^{2^{-j-1}}\bigr)
\end{aligned}
$$

for any j with $1 \le j \le n$. Then (4.1) follows immediately. □

Proposition 4.2 *Let θ satisfy $(2^{m-1}-1)/(2^m-1) \le \theta \le 1/2$ with an integer $m \ge 1$. Then the equality*

$$
\begin{aligned}
R(\theta; a, b) = {} & (1-\theta)\bigl(a^{1/2} - b^{1/2}\bigr)^2 \\
& - (1-2\theta) \sum_{j=1}^{m} 2^{j-1} a^\theta b^{(1-\theta)(1-2^{1-j})} \bigl(a^{(1-\theta)2^{-j}} - b^{(1-\theta)2^{-j}}\bigr)^2 \\
& - 2(1-\theta) a^\theta b^{(1-\theta)(1-2^{-m})} R\left(2^m \cdot \frac{1/2-\theta}{1-\theta}; a^{(1-\theta)2^{-m}}, b^{(1-\theta)2^{-m}}\right)
\end{aligned}
\tag{4.2}
$$

holds for all $a, b > 0$.

Proof We apply Corollary 2.4 with $\sigma = 1/2$ to obtain

$$
R(\theta; a, b) = (1-\theta)\bigl(a^{1/2} - b^{1/2}\bigr)^2 - 2(1-2\theta) a^\theta R\left(\frac{1/2-\theta}{1-\theta}; a^{1-\theta}, b^{1-\theta}\right). \tag{4.3}
$$

Then (4.2) follows by applying Proposition 4.1 to the last term on the right-hand side of (4.3) with $0 \le (1/2-\theta)/(1-\theta) \le 2^{-m}$. □

Corollary 4.3 *Let $0 \le \theta \le 1/2$. Then the inequalities*

$$
\begin{aligned}
&\theta\bigl(a^{1/2} - b^{1/2}\bigr)^2 + \bigl(2\theta \wedge (1-2\theta)\bigr) b^{1/2} \bigl(a^{1/4} - b^{1/4}\bigr)^2 \\
&\quad \le R(\theta; a, b) \\
&\quad \le (1-\theta)\bigl(a^{1/2} - b^{1/2}\bigr)^2 - (1-2\theta) a^\theta \bigl(a^{(1-\theta)/2} - b^{(1-\theta)/2}\bigr)^2 \\
&\qquad - 2\bigl(\theta \wedge (1-2\theta)\bigr) a^\theta b^{(1-\theta)/2} \bigl(a^{(1-\theta)/4} - b^{(1-\theta)/4}\bigr)^2
\end{aligned}
\tag{4.4}
$$

hold for all $a, b > 0$.

Corollary 4.4 *Let $1/2 \le \theta \le 1$. Then the inequalities*

$$
\begin{aligned}
&(1-\theta)\bigl(a^{1/2} - b^{1/2}\bigr)^2 + \bigl(\bigl(2(1-\theta)\bigr) \wedge (2\theta-1)\bigr) a^{1/2} \bigl(a^{1/4} - b^{1/4}\bigr)^2 \\
&\quad \le R(\theta; a, b) \\
&\quad \le \theta\bigl(a^{1/2} - b^{1/2}\bigr)^2 - (2\theta-1) b^{1-\theta} \bigl(a^{\theta/2} - b^{\theta/2}\bigr)^2 \\
&\qquad - 2\bigl((1-\theta) \wedge (2\theta-1)\bigr) a^{(1-\theta)/2} b^\theta \bigl(a^{\theta/4} - b^{\theta/4}\bigr)^2
\end{aligned}
\tag{4.5}
$$

hold for all $a, b > 0$.

Remark 4.1 Some of the lower bounds in Corollaries 4.3 and 4.4 may be found already in [2], Section 3.2.

Remark 4.2 Inequalities (4.4) and (4.5) improve (2.5). Inequalities (2.5) become an equality when $\theta = 1/2$, while (4.4) become an equality when $\theta = 0, 1/2$ and (4.5) become an equality when $\theta = 1/2, 1$.

We are now in a position to apply the equalities above to Hölder type inequalities.

Theorem 4.5 *Let p satisfy $2 \le p < \infty$ and let m and n be unique integers satisfying*

$$\begin{cases} 2^n \le p < 2^{n+1}, \quad n \ge 1, \\ (2^{m+1}-1)/(2^m-1) \le p < (2^m-1)/(2^{m-1}-1), \quad m \ge 1. \end{cases}$$

Then the equalities

$$\begin{aligned} &\|f\|_p\|g\|_{p'}\Biggl(1 - \frac{1}{p'}\int_\Omega \biggl(\frac{|f|^{p/2}}{\|f\|_p^{p/2}} - \frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\biggr)^2\,d\mu \\ &\quad + \biggl(\frac{1}{p'} - \frac{1}{p}\biggr)\sum_{j=1}^m 2^{j-1}\int_\Omega \frac{|f|}{\|f\|_p}\frac{|g|^{1-2^{1-j}}}{\|g\|_{p'}^{1-2^{1-j}}}\biggl(\frac{|f|^{(p-1)2^{-j}}}{\|f\|_p^{(p-1)2^{-j}}} - \frac{|g|^{2^{-j}}}{\|g\|_{p'}^{2^{-j}}}\biggr)^2\,d\mu \\ &\quad + \frac{2}{p'}\int_\Omega \frac{|f|}{\|f\|_p}\frac{|g|^{(1-2^{-m})}}{\|g\|_{p'}^{(1-2^{-m})}} R\biggl(2^{m-1}\frac{p-2}{p-1}, \frac{|f|^{(p-1)2^{-m}}}{\|f\|_p^{(p-1)2^{-m}}}, \frac{|g|^{2^{-m}}}{\|g\|_{p'}^{2^{-m}}}\biggr)\,d\mu\Biggr) \\ &= \|fg\|_1 \\ &= \|f\|_p\|g\|_{p'}\Biggl(1 - \frac{1}{p}\sum_{j=1}^n 2^{j-1}\int_\Omega \frac{|g|^{p'(1-2^{1-j})}}{\|g\|_{p'}^{p'(1-2^{1-j})}}\biggl(\frac{|f|^{p2^{-j}}}{\|f\|_p^{p2^{-j}}} - \frac{|g|^{p'2^{-j}}}{\|g\|_{p'}^{p'2^{-j}}}\biggr)^2\,d\mu \\ &\quad - \int_\Omega \frac{|g|^{p'(1-2^{-n})}}{\|g\|_{p'}^{p'(1-2^{-n})}} R\biggl(\frac{2^n}{p}, \frac{|f|^{p2^{-n}}}{\|f\|_p^{p2^{-n}}}, \frac{|g|^{p'2^{-n}}}{\|g\|_{p'}^{p'2^{-n}}}\biggr)\,d\mu\Biggr) \end{aligned} \tag{4.6}$$

hold for all $f \in L^p(\Omega,\mu)\backslash\{0\}$ and $g \in L^{p'}(\Omega,\mu)\backslash\{0\}$.

Proof The theorem follows from (1.3) and Propositions 4.1 and 4.2 with $\theta = 1/p$, $a = |f|^p/\|f\|_p^p$, $b = |g|^{p'}/\|g\|_{p'}^{p'}$. □

Corollary 4.6 *Let p, m, n be as in Theorem 4.5. Then the inequalities*

$$\begin{aligned} &\|f\|_p\|g\|_{p'}\Biggl(1 - \frac{1}{p'}\int_\Omega \biggl(\frac{|f|^{p/2}}{\|f\|_p^{p/2}} - \frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\biggr)^2\,d\mu \\ &\quad + \biggl(\frac{1}{p'} - \frac{1}{p}\biggr)\sum_{j=1}^m 2^{j-1}\int_\Omega \frac{|f|}{\|f\|_p}\frac{|g|^{1-2^{1-j}}}{\|g\|_{p'}^{1-2^{1-j}}}\biggl(\frac{|f|^{(p-1)2^{-j}}}{\|f\|_p^{(p-1)2^{-j}}} - \frac{|g|^{2^{-j}}}{\|g\|_{p'}^{2^{-j}}}\biggr)^2\,d\mu \\ &\quad + \frac{2}{p'}\biggl(1 - 2^{m-1}\frac{p-2}{p-1}\biggr)\int_\Omega \frac{|f|}{\|f\|_p}\frac{|g|^{(1-2^{-m})}}{\|g\|_{p'}^{(1-2^{-m})}}\biggl(\frac{|f|^{(p-1)2^{-m-1}}}{\|f\|_p^{(p-1)2^{-m-1}}} - \frac{|g|^{2^{-m-1}}}{\|g\|_{p'}^{2^{-m-1}}}\biggr)^2\,d\mu\Biggr) \end{aligned}$$

$$\begin{aligned}
&\le \|fg\|_1 \\
&\le \|f\|_p\|g\|_{p'}\Biggl(1-\frac{1}{p}\sum_{j=1}^{n}2^{j-1}\int_\Omega \frac{|g|^{p'(1-2^{1-j})}}{\|g\|_{p'}^{p'(1-2^{1-j})}}\biggl(\frac{|f|^{p2^{-j}}}{\|f\|_p^{p2^{-j}}}-\frac{|g|^{p'2^{-j}}}{\|g\|_{p'}^{p'2^{-j}}}\biggr)^2\,d\mu \\
&\quad -\frac{p-2^n}{p}\int_\Omega \frac{|g|^{p'(1-2^{-n})}}{\|g\|_{p'}^{p'(1-2^{-n})}}\biggl(\frac{|f|^{p2^{-n-1}}}{\|f\|_p^{p2^{-n-1}}}-\frac{|g|^{p'2^{-n-1}}}{\|g\|_{p'}^{p'2^{-n-1}}}\biggr)^2\,d\mu\Biggr)
\end{aligned} \tag{4.7}$$

hold for all $f \in L^p(\Omega,\mu)\backslash\{0\}$ *and* $g \in L^{p'}(\Omega,\mu)\backslash\{0\}$.

Proof The required inequalities follow from Theorem 4.5 and Proposition 2.5. □

Corollary 4.7 *Let* $p \ge 2^n$ *with a positive integer* n. *Then the inequalities*

$$\begin{aligned}
&\|f\|_p\|g\|_{p'}\Biggl(1-\frac{1}{p'}\biggl\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\biggr\|_2^2 \\
&\quad +\biggl(\frac{1}{p'}-\frac{1}{p}\biggr)\biggl\|\frac{|f|^{1/2}}{\|f\|_p^{1/2}}\biggl(\frac{|f|^{(p-1)/2}}{\|f\|_p^{(p-1)/2}}-\frac{|g|^{1/2}}{\|g\|_{p'}^{1/2}}\biggr)\biggr\|_2^2\Biggr) \\
&\le \|fg\|_1 \\
&\le \|f\|_p\|g\|_{p'}\Biggl(1-\frac{1}{p}\sum_{j=1}^{n}2^{j-1}\biggl\|\frac{|g|^{p'(1/2-2^{-j})}}{\|g\|_{p'}^{p'(1/2-2^{-j})}}\biggl(\frac{|f|^{p2^{-j}}}{\|f\|_p^{p2^{-j}}}-\frac{|g|^{p'2^{-j}}}{\|g\|_{p'}^{p'2^{-j}}}\biggr)\biggr\|_2^2 \\
&\quad -\frac{p-2^n}{p}\biggl\|\frac{|g|^{p'(1/2-2^{-n-1})}}{\|g\|_{p'}^{p'(1/2-2^{-n-1})}}\biggl(\frac{|f|^{p2^{-n-1}}}{\|f\|_p^{p2^{-n-1}}}-\frac{|g|^{p'2^{-n-1}}}{\|g\|_{p'}^{p'2^{-n-1}}}\biggr)\biggr\|_2^2\Biggr)
\end{aligned} \tag{4.8}$$

hold for all $f \in L^p(\Omega,\mu)\backslash\{0\}$ *and* $g \in L^{p'}(\Omega,\mu)\backslash\{0\}$.

Remark 4.3 In the case where $n = 1$ in Corollary 4.7, the coefficients of the upper and lower bounds of $\|fg\|_1$ are symmetric as follows:

$$\begin{aligned}
&\|f\|_p\|g\|_{p'}\Biggl(1-\frac{1}{p'}\biggl\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\biggr\|_2^2 \\
&\quad +\biggl(\frac{1}{p'}-\frac{1}{p}\biggr)\biggl\|\frac{|f|^{1/2}}{\|f\|_p^{1/2}}\biggl(\frac{|f|^{(p-1)/2}}{\|f\|_p^{(p-1)/2}}-\frac{|g|^{1/2}}{\|g\|_{p'}^{1/2}}\biggr)\biggr\|_2^2\Biggr) \\
&\le \|fg\|_1 \\
&\le \|f\|_p\|g\|_{p'}\Biggl(1-\frac{1}{p}\biggl\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\biggr\|_2^2 \\
&\quad -\biggl(\frac{1}{p'}-\frac{1}{p}\biggr)\biggl\|\frac{|g|^{p'/4}}{\|g\|_{p'}^{p'/4}}\biggl(\frac{|f|^{p/4}}{\|f\|_p^{p/4}}-\frac{|g|^{p'/4}}{\|g\|_{p'}^{p'/4}}\biggr)\biggr\|_2^2\Biggr).
\end{aligned}$$

Remark 4.4 Inequalities (4.8) improve the Aldaz stability version of the Hölder inequality [1]

$$\begin{aligned}&\|f\|_p\|g\|_{p'}\left(1-\frac{1}{p'}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right\|_2^2\right)\\&\quad\le\|fg\|_1\le\|f\|_p\|g\|_{p'}\left(1-\frac{1}{p}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right\|_2^2\right).\end{aligned}\tag{4.9}$$

As Aldaz observed, (4.9) become

$$\|f\|_p\|g\|_{p'}\left(1-\frac{2}{p'}\right)\le\|fg\|_1=0\le\|f\|_p\|g\|_{p'}\left(1-\frac{2}{p}\right)$$

if $\operatorname{supp} f\cap\operatorname{supp} g=\emptyset$. In this respect, Corollary 4.7 is sharp since both sides of the inequalities in (4.8) vanish as follows:

$$\begin{aligned}&\|f\|_p\|g\|_{p'}\left(1-\frac{1}{p'}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{p'/2}}{\|g\|_{p'}^{p'/2}}\right\|_2^2\right.\\&\qquad\left.+\left(\frac{1}{p'}-\frac{1}{p}\right)\left\|\frac{|f|^{1/2}}{\|f\|_p^{1/2}}\left(\frac{|f|^{(p-1)/2}}{\|f\|_p^{(p-1)/2}}-\frac{|g|^{1/2}}{\|g\|_{p'}^{1/2}}\right)\right\|_2^2\right)\\&\quad=\|f\|_p\|g\|_{p'}\left(1-\frac{2}{p'}+\frac{1}{p'}-\frac{1}{p}\right)=0,\\&\|f\|_p\|g\|_{p'}\left(1-\frac{1}{p}\sum_{j=1}^{n}2^{j-1}\left\|\frac{|g|^{p'(1/2-2^{-j})}}{\|g\|_{p'}^{p'(1/2-2^{-j})}}\left(\frac{|f|^{p2^{-j}}}{\|f\|_p^{p2^{-j}}}-\frac{|g|^{p'2^{-j}}}{\|g\|_{p'}^{p'2^{-j}}}\right)\right\|_2^2\right.\\&\qquad\left.-\frac{p-2^n}{p}\left\|\frac{|g|^{p'(1/2-2^{-n-1})}}{\|g\|_{p'}^{p'(1/2-2^{-n-1})}}\left(\frac{|f|^{p2^{-n-1}}}{\|f\|_p^{p2^{-n-1}}}-\frac{|g|^{p'2^{-n-1}}}{\|g\|_{p'}^{p'2^{-n-1}}}\right)\right\|_2^2\right)\\&\quad=\|f\|_p\|g\|_{p'}\left(1-\frac{2}{p}-\frac{1}{p}\sum_{j=2}^{n}2^{j-1}-\frac{p-2^n}{p}\right)=0.\end{aligned}$$

In addition, (4.8) coincides with the polarization identity

$$\big(|f|,|g|\big)=\|f\|_2\|g\|_2\left(1-\frac{1}{2}\left\|\frac{|f|}{\|f\|_2}-\frac{|g|}{\|g\|_2}\right\|_2^2\right)$$

when $p=2$, where $(\cdot,\cdot)$ is the standard L^2 inner product.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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